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Stretching and folding processes in the 3D Euler and Navier-Stokes equations

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Abstract

Stretching and folding dynamics in the incompressible, stratified 3D Euler and Navier-Stokes equations are reviewed in the context of the vector $\boldsymbol{B} = \nabla q \times \nabla\theta$ where, in atmospheric physics, θ is a temperature, $q = \boldsymbol{\omega}\cdot\nabla\theta$ is the potential vorticity, and $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$ is the vorticity. These ideas are then discussed in the context of the full compressible Navier-Stokes equations where q is taken in the form $q = \boldsymbol{\omega}\cdot\nabla f(\rho)$. In the two cases $f = \rho$ and $f = \ln\rho$, q is shown to satisfy the quasi-conservative relation $\partial_t q + \text{div}\,\mathbf{J} = 0$.

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1. Introduction

1.1. Stretching and folding in the incompressible 3D Euler equations

The Euler fluid equations for three-dimensional incompressible flow may be written as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p\,, \qquad \text{div}\,\boldsymbol{u} = 0\,, \tag{1}$$

which can also be expressed as

$$\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega} = -\nabla\left(p + \tfrac{1}{2}u^2\right). \tag{2}$$

Here, $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity field and $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$ is the vorticity of the fluid. The material derivative is defined by

$$\frac{D}{Dt} = \partial_t + \boldsymbol{u}\cdot\nabla\,. \tag{3}$$

Following equation (2) the vorticity field $\boldsymbol{\omega}$ satisfies

$$\partial_t\boldsymbol{\omega} - \text{curl}(\boldsymbol{u}\times\boldsymbol{\omega}) = 0\,, \tag{4}$$

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which can also be written in the familiar vortex stretching format

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \boldsymbol{u} \equiv \mathbf{S}\,\omega\,, \tag{5}$$

where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the rate of strain matrix. It is generally acknowledged that the stretching and folding processes caused by the rapid alignment or anti-alignment of ω with positive or negative eigenvectors of $\mathbf{S}$ can roughen smooth initial data very quickly. In fact, the 3D incompressible Euler equations have an array of very weak solutions [1, 2, 3, 4, 5, 7, 6, 8], but the Leray-type weak solutions associated with the Navier-Stokes equations are unknown [9]. Our lack of knowledge forces us to make some assumptions about the existence of solutions of both the incompressible Euler and Navier-Stokes equations in order to perform formal manipulations [10, 11, 12]. Likewise in later sections we also discuss the compressible case in the same spirit. This paper aims to marry the ideas on stretching and folding processes in incompressible flows developed by the authors in [13, 14] with their work on compressible flows [15].

The phenomenon of stretching and folding requires some explanation. In fact it occurs naturally in the incompressible Euler equations through the dynamics displayed in (5). The incompressibility condition insists that the eigenvalues of $\mathbf{S}$ are constrained by $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Thus at any given point in the flow the vorticity ω will stretch exponentially (or worse) when it aligns or anti-aligns close to an eigenvector of $\mathbf{S}$ whose corresponding eigenvalue is positive or, likewise, collapse if the eigenvalue is negative. Rapid changes in the signs of the eigenvalues will thus cause a sequence of fine-scale growth and collapse events that lead to the roughening of an initially smooth field, akin to the crinkling of a piece of paper which has undergone repeated folding.

We begin by observing that the familiar vortex stretching format expressed in (5) also appears in a different context. Let $\boldsymbol{u}$ be a divergence-free Euler flow and let θ and q be two arbitrary (for now) passive scalars riding on this flow

$$\frac{D\theta}{Dt} = 0 \qquad \text{and} \qquad \frac{Dq}{Dt} = 0\,. \tag{6}$$

Then it has been shown that the vector [16, 17, 18, 19] (see also [13, 14])

$$\boldsymbol{B} = \nabla q \times \nabla \theta \tag{7}$$

satisfies

$$\partial_t \boldsymbol{B} - \text{curl}\,(\boldsymbol{u} \times \boldsymbol{B}) = 0, \qquad \Longrightarrow \qquad \frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B} \cdot \nabla \boldsymbol{u}\,, \tag{8}$$

which is exactly (5). The vector $\boldsymbol{B}$ is not a Clebsch decomposition of the vorticity field (see Lamb [20] page 200), but is a construction from the gradients of two passive scalars. The $\boldsymbol{B}$-field, however, formally shares the same stretching and folding properties of the ω-field, in the same manner as a magnetic field in ideal MHD [21, 22].

The context of this result can be seen by considering the Euler equations with constant rotation $2\boldsymbol{\Omega}$ and buoyancy (proportional to the temperature θ), written in the dimensionless form

$$\frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + a_0 \theta\,\hat{\boldsymbol{k}} = -\nabla p\,, \qquad \frac{D\theta}{Dt} = 0\,, \tag{9}$$

where a_0 is a dimensionless constant [13, 14] and $\hat{\boldsymbol{k}}$ is the vertical unit vector. The equation for the vorticity $\omega_{rot} = \omega + 2\boldsymbol{\Omega}$ is

$$\frac{D\omega_{rot}}{Dt} + a_0 \nabla^{\perp}\theta = \omega_{rot} \cdot \nabla \boldsymbol{u} \tag{10}$$

where $\nabla^{\perp} = (\partial_y, -\partial_x, 0)$. With q taken as a potential vorticity $q = \omega_{rot} \cdot \nabla\theta$ it is easily seen that

$$\frac{Dq}{Dt} = \left(\frac{D\omega_{rot}}{Dt} - \omega_{rot} \cdot \nabla \boldsymbol{u}\right) \cdot \nabla\theta + \omega_{rot} \cdot \nabla \left(\frac{D\theta}{Dt}\right)\,, \tag{11}$$

which is a geometric relation reminiscent of Ertel's Theorem [23]. Because $\nabla^{\perp}\theta \cdot \nabla\theta = 0$ it is clear that

$$\frac{Dq}{Dt} = 0\,, \tag{12}$$

and so (6) is satisfied. It can now be seen that the stretching and folding properties of $\boldsymbol{B} = \nabla q \times \nabla\theta$ in equation (8) have been lifted to higher gradients of ω because $\boldsymbol{B}$ contains a gradient of ω in projection and two gradients of θ.

1.2. Stretching and folding in the incompressible 3D Navier-Stokes equations

The stratified Navier-Stokes equations are the viscous counterpart of (9)

$$\frac{D\boldsymbol{u}}{Dt} + a_0\theta\,\hat{\boldsymbol{k}} = Re^{-1}\Delta\boldsymbol{u} - \nabla p\,, \tag{13}$$

$$\frac{D\theta}{Dt} = \left(\sigma Re\right)^{-1}\Delta\theta\,, \tag{14}$$

where Re is the Reynolds number. Here, the potential vorticity $q = \omega\cdot\nabla\theta$ is no longer a material constant (the 'rot' suffix has been dropped) but, instead, evolves according to

$$\begin{aligned}\frac{Dq}{Dt} &= \left(\frac{D\omega}{Dt} - \omega\cdot\nabla\boldsymbol{u}\right)\cdot\nabla\theta + \omega\cdot\nabla\left(\frac{D\theta}{Dt}\right)\\ &= \left(Re^{-1}\Delta\omega - a_0\nabla^{\perp}\theta\right)\cdot\nabla\theta + \omega\cdot\nabla\left[(\sigma Re)^{-1}\Delta\theta\right]\\ &= \mathrm{div}\left\{Re^{-1}\Delta\boldsymbol{u}\times\nabla\theta + (\sigma Re)^{-1}\omega\,\Delta\theta\right\}.\end{aligned} \tag{15}$$

The material advection property no longer holds but the introduction of a *modified velocity field* $\boldsymbol{U}_q$ transforms (15) into a continuity equation

$$\partial_t q + \mathrm{div}\left(q\boldsymbol{U}_q\right) = 0\,, \tag{16}$$

thus making q a potential vorticity density, and where $\boldsymbol{U}_q$ is defined through

$$q\left(\boldsymbol{U}_q - \boldsymbol{u}\right) = -Re^{-1}\left(\Delta\boldsymbol{u}\times\nabla\theta + \sigma^{-1}\omega\Delta\theta\right). \tag{17}$$

Moreover, θ evolves according to

$$\begin{aligned}\partial_t\theta + \boldsymbol{U}_q\cdot\nabla\theta &= \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta - Re^{-1}q^{-1}\left\{\Delta\boldsymbol{u}\times\nabla\theta + \sigma^{-1}\omega\Delta\theta\right\}\cdot\nabla\theta\\ &= \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta - \left(\sigma Re\right)^{-1}\Delta\theta\\ &= 0\,.\end{aligned} \tag{18}$$

The formal result for the stratified Navier-Stokes equation is :

Theorem 1 *The scalar quantities q and θ satisfy*

$$\partial_t q + div\left(q\boldsymbol{U}_q\right) = 0\,, \qquad \partial_t\theta + \boldsymbol{U}_q\cdot\nabla\theta = 0\,, \tag{19}$$

and $\boldsymbol{B} = \nabla q\times\nabla\theta$ satisfies the stretching and folding relation

$$\partial_t\boldsymbol{B} - curl\left(\boldsymbol{U}_q\times\boldsymbol{B}\right) = \boldsymbol{D}_q\,, \tag{20}$$

where the divergence-less vector $\boldsymbol{D}_q$ is given by

$$\boldsymbol{D}_q = -\nabla(q\,div\,\boldsymbol{U}_q)\times\nabla\theta\,, \tag{21}$$

and the modified velocity $\boldsymbol{U}_q$ is defined as in (17). Moreover, for any surface $\boldsymbol{S}(\boldsymbol{U}_q)$ moving with the flow $\boldsymbol{U}_q$ we have

$$\frac{d}{dt}\int_{\boldsymbol{S}(\boldsymbol{U}_q)}\boldsymbol{B}\cdot d\boldsymbol{S} = \int_{\boldsymbol{S}(\boldsymbol{U}_q)}\boldsymbol{D}_q\cdot d\boldsymbol{S}\,. \tag{22}$$

The introduction of a modified velocity field $\boldsymbol{U}_q$ that 'hides' the dissipation is based originally on an idea due to Haynes & McIntyre in an atmospheric context [25, 26]. The two obvious drawbacks are that firstly the divergence-free property is lost ($\mathrm{div}\,\boldsymbol{U}_q = O\left(Re\right)^{-1}$) and secondly the transformation (17) fails at zeros of q where a change of topology of vortex lines could occur. Moreover, the stretching and folding relation (20) now has a non-zero right hand side $\boldsymbol{D}_q$ defined in (21), that drives and modifies the process. Numerical studies on reconnection (Herring, Kerr & Rotunno [27]) suggest that in the early or intermediate stages of a flow this divergence may be small.

2. The compressible Navier-Stokes equations

The aim of this section is to now develop the ideas of §1 in the context of compressible flows [15]. In this context the mass density ρ plays the role of θ even though it is not a passive quantity. The 3D compressible Navier-Stokes equations are expressed as [28]

$$\rho \frac{D\boldsymbol{u}}{Dt} = \mu\Delta\boldsymbol{u} - \nabla\varpi\,, \qquad \varpi = p - (\mu/3+\mu^{v})\mathrm{div}\,\boldsymbol{u}\,, \tag{23}$$

where ρ and the temperature θ satisfy

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\boldsymbol{u} = 0\,, \qquad c_v\frac{D\theta}{Dt} = \frac{p}{\rho}\,\mathrm{div}\,\boldsymbol{u} + Q\,. \tag{24}$$

μ is the shear viscosity and μ^v is the volume viscosity, both of which are taken as constitutive constants of the fluid. An ideal gas equation of state $p = R\rho\theta$ has been chosen to relate pressure, p, temperature, θ, and mass density, ρ. In addition, R is the gas constant, c_v the specific heat is constant and Q is the heating rate, which is assumed to be known. However, the geometric considerations that follow are universal for any viscous compressible fluid flow because the dynamics of the temperature and the choice of equation of state will not affect our considerations of the transport dynamics of the two choices of the projection q discussed in §2.1 and §2.2. It is well known that compressible flows have weak shock solutions which have subtle properties [29, 30, 31]. In the following our manipulations are based on the assumption that the necessary differentiations are allowed although there well may be some situations in which they are not.

In the next two subsections q as a projection of ω onto $\nabla\rho$ will be considered in the generalized form $q = \omega\cdot\nabla f(\rho)$: two particular cases arise, $f(\rho) = \rho$ and $f(\rho) = \ln\rho$, which will be dealt with in turn[1].

2.1. 1st projection

The first choice for q as a projection of ω onto $\nabla\rho$ is

$$q = \omega\cdot\nabla\rho\,. \tag{25}$$

According to equations (23) and (24) the vorticity ω evolves according to

$$\partial_t\omega - \mathrm{curl}\,(\boldsymbol{u}\times\omega) = \mu\rho^{-1}\Delta\omega + \nabla\left(\rho^{-1}\right)\times\left[\mu\Delta\boldsymbol{u} - \nabla\left(\varpi + \tfrac{1}{2}u^2\right)\right] \tag{26}$$

Using (11) with θ replaced by ρ, q now satisfies

$$\partial_t q + \mathrm{div}\,q\boldsymbol{u} + \mathrm{div}\big(\omega\rho\,\mathrm{div}\,\boldsymbol{u} - \mu\Delta\boldsymbol{u}\times\nabla(\ln\rho)\big) = 0\,. \tag{27}$$

Note that the pressure terms have disappeared with no approximation. Now we apply an observation of Haynes and McIntyre [25, 26] that first arose in atmospheric physics and which leads to the definition of a current density $\mathbf{J}$

$$\mathbf{J} = q\boldsymbol{u} + \omega\rho\,\mathrm{div}\,\boldsymbol{u} - \mu\Delta\boldsymbol{u}\times\nabla(\ln\rho) + \nabla\phi\times\nabla\psi(\rho)\,, \tag{28}$$

where ϕ is an undetermined gauge potential and ψ is an arbitrary differentiable function of ρ. Using the definition of $\mathbf{J}$ in (28), equations (27) and (33) can be rewritten in the *quasi-conservative* form

$$\partial_t q + \mathrm{div}\,\mathbf{J} = 0 \quad \text{and} \quad q\,\partial_t\rho + \mathbf{J}\cdot\nabla\rho = 0\,. \tag{29}$$

The relation $\mathbf{J}\cdot\nabla\rho = q\,\mathrm{div}\,\rho\boldsymbol{u}$ allows zero projection, $q = 0$, by the second equation in (29). Thus, the projection q may vanish anywhere in the flow, but it cannot be maintained, because $\mathrm{div}\,\mathbf{J}\neq 0$. Together, the equations in (29) imply a family of conserved quantities, since

$$\partial_t(q\Phi'(\rho)) + \mathrm{div}\,(\mathbf{J}\Phi'(\rho)) = 0\,, \tag{30}$$

for any function $\Phi'(\rho) = d\Phi/d\rho$. The conserved densities $q\Phi'(\rho) = \mathrm{div}(\Phi(\rho)\omega)$ in (30) possess quite different flow properties from those of mass, energy and momentum.

[1] By this method, we have been unable to find any other form of $f(\rho)$ that leads to the existence of a current density $\mathbf{J}$.

2.2. 2nd projection

There is a second choice for q as a projection of ω onto $\nabla\rho$ which is

$$q = \omega\cdot\nabla(\ln\rho)\,, \tag{31}$$

in which case

$$\frac{Dq}{Dt} = \left(-\omega\,\mathrm{div}\,\boldsymbol{u} + \mu\rho^{-1}\Delta\omega\right)\cdot\nabla\left(\rho^{-1}\right) - \omega\cdot\nabla\left(\mathrm{div}\,\boldsymbol{u}\right). \tag{32}$$

Equivalent to (27) we have

$$\partial_t q + \mathrm{div}\,(q\boldsymbol{u}) + \mathrm{div}\,\left\{\mu\Delta\boldsymbol{u}\times\nabla\left(\rho^{-1}\right) + \omega\,\mathrm{div}\,\boldsymbol{u}\right\} = 0\,. \tag{33}$$

Therefore a second definition of the current density $\mathbf{J}$ is

$$\mathbf{J} = q\boldsymbol{u} + \omega\,\mathrm{div}\,\boldsymbol{u} + \mu\Delta\boldsymbol{u}\times\nabla\left(\rho^{-1}\right) + \nabla\phi\times\nabla\psi(\rho)\,, \tag{34}$$

and similar conclusions to those in (29) and (30) follow.

2.3. Physical interpretation?

Equations (29) are purely kinematic, because the projection taken against $\nabla\rho$ has removed any dependence on the temperature and the choice of equation of state. Nonetheless, equations (29) have been derived without approximation from the Navier-Stokes fluid equations for compressible motion and mass transport. Moreover, their analogues also occur and have been found useful in other areas of fluid dynamics, particularly in the dynamics of the ocean and atmosphere [25, 26]. There (29) was locally interpreted as a type of *impermeability theorem* for the *quasi-Lagrangian* transport of the projection q and the mass density ρ by a modified velocity $\mathbf{U}_q$ defined by

$$q\boldsymbol{U}_q = \mathbf{J}\,. \tag{35}$$

Namely, the projections within $q = \omega\cdot\nabla\rho$ and $q = \omega\cdot\nabla(\ln\rho)$ cannot be transported[2] by the modified velocity $\boldsymbol{U}_q$ across level sets of the mass density, ρ. These equations may also be useful in the study of stretching and folding in compressible fluid flows, as investigated in the atmospheric context in [13, 14, 32].

3. Conclusion

Although the usual concerns of compressible flows focus on shock formation [29, 30, 31], some final remarks are in order about how the gradient of the projection q participates in stretching and folding in compressible Navier-Stokes fluid flows as in Theorem 1. In preparation, we re-write equations (24) and (29) in terms of the modified velocity $\boldsymbol{U}_q$ defined in (35) as

$$\partial_t q + \mathrm{div}\,\left(q\boldsymbol{U}_q\right) = 0 \qquad\qquad \partial_t\rho + \boldsymbol{U}_q\cdot\nabla\rho = 0. \tag{36}$$

Note however that $\mathrm{div}\,\boldsymbol{U}_q \neq 0$. With the definition

$$\boldsymbol{B} = \nabla q\times\nabla\rho\,, \tag{37}$$

a direct computation shows that $\boldsymbol{B}$ satisfies

$$\partial_t\boldsymbol{B} - \mathrm{curl}\,(\boldsymbol{U}_q\times\boldsymbol{B}) = \boldsymbol{D}_q \tag{38}$$

where

$$\boldsymbol{D}_q = -\nabla\left(q\,\mathrm{div}\,\boldsymbol{U}_q\right)\times\nabla\rho \tag{39}$$

[2] By the definition of $\mathbf{J}$ the modified velocity $\boldsymbol{U}_q = \mathbf{J}/q$ is determined only up to the addition of the curl of a vector proportional to $\nabla\rho$.

as in Theorem 1.

Based on the modified velocity defined in (35), the left hand side of (38) makes it clear that the vector $\boldsymbol{B}$ undergoes the same type of stretching and folding processes driven by the $\boldsymbol{D}_q$-vector on the right hand side as occurs in the vorticity equation (26). This is remarkable, because $\boldsymbol{B} = \nabla q \times \nabla \rho$ contains information not only about $\nabla \omega$ but also about $\nabla \rho$ and even $\nabla^2 \rho$. Thus, the stretching and folding in the original vorticity equation (26) compounds itself in the same form in the $\boldsymbol{B}$-equation (38), but with higher spatial derivatives.

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