Some Classes of Integral Matrices

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ABSTRACT

Integral matrices A for which the system $Ax = b$ will have an integral basic solution are considered. Results for these matrices are presented which parallel results concerning other matrices for which the system has integral solutions.

1. INTRODUCTION

Consider systems of equations $Ax = b$ where A and b are integral. Those matrices A for which integral solutions exist, as well as those for which all basic solutions are integral, have been characterized [2, 5, 9, 12]. The case for basic solutions is important in linear programming, and is related to the unimodularity of A. An excellent summary of these results is given by Stephen Barnett in [1, Chapter 7]. In this paper we introduce three new classes of matrices for which consistent systems $Ax = b$ will have at least one integral basic solution. We establish relationships between these classes and four classes of integral matrices which have been previously studied. The classes considered include unimodular matrices, matrices whose columns form a unimodular set, and matrices which have an integral {1}-inverse. Summaries of previous results are given along with new results in order to exhibit parallels between the various classes.

2. DEFINITIONS

Let A denote an $m \times n$ rank-r integral matrix, and let b denote an integral column vector. Unless stated otherwise this notation will be used throughout the paper. It is known [2, Theorem 4] that $Ax = b$ has an integral solution for all integral b in the column space of A if and only if A has an integral

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{1}-inverse, that is, an integral matrix X such that $AXA = A$. It is also known that A will have an integral {1}-inverse if and only if the gcd of all $r \times r$ subdeterminants of A is 1 [3, Theorem 2], or if and only if the Smith canonical form of A contains the $r \times r$ identity submatrix I_r [2, Theorem 3]. This last characterization was first established for the case where A has full row rank by Smith [9].

Even when the system $Ax = b$ has an integral solution, most solutions may not be integral, since the null space of A may contain nonintegral vectors. There are circumstances where all basic solutions are integral. A basis for A is an $m \times r$ submatrix of A consisting of r linearly independent columns. If $Ax = b$, then x is a basic solution if there is a basis B for A such that the nonzero entries of x correspond to columns of B.

The term unimodular was first applied to nonsingular integral matrices whose determinants are ± 1. The term was then extended to $m \times n$ rank m matrices all of whose $m \times m$ subdeterminants are 0 or ± 1. In general an $m \times n$ rank-r integral matrix is said to be unimodular if for every basis B of A, the gcd of all $r \times r$ subdeterminants of B is 1. Let $Z(A)$ denote the set of all integral b in the column space of A. Then in any of the above cases, A is unimodular if and only if all basic solutions of $Ax = b$ are integral for all $b \in Z(A)$ [10, 12].

This concept is generalized by the Dantzig property. An $m \times n$ rank r integral matrix A is said to have the Dantzig property (or the columns of A form a unimodular set) if in each submatrix A_0 of r rows of A, the nonzero $r \times r$ subdeterminants of A_0 have the same absolute value [6, (4.4)]. Let $Z_0(A)$ denote the set of all integral linear combinations of the columns of A. Then A has the Dantzig property if and only if all basic solutions of $Ax = b$ are integral for all $b \in Z_0(A)$ [6, (3.9)].

Let UNM denote the collection of unimodular matrices, let DAN denote the collection of matrices with the Dantzig property, and let INT denote the collection of all integral matrices which have an integral {1}-inverse. Consider the question of when a matrix with the Dantzig property will also be unimodular. Now if $A \in \mathrm{INT}$, then $Z_0(A) = Z(A)$, and one is led to the conclusion that $\mathrm{INT} \cap \mathrm{DAN} = \mathrm{UNM}$. That is, a Dantzig matrix will also be unimodular if and only if it has an integral {1}-inverse. A slightly different characterization of the Dantzig property is as follows: $A \in \mathrm{DAN}$ if and only if all basic solutions of $Ax = b$ are integral whenever b is such that there is at least one integral basic solution [5, (1.5)]. In light of this characterization of DAN, the above question leads one to consider matrices which have the property that the system $Ax = b$ has an integral basic solution for all $b \in Z(A)$. We will say that such matrices satisfy the *basic solution property* and denote the collection of all these matrices by BSP. Related to this class, we define the class WBSP of matrices which satisfy the *weak basic solution property*,

TABLE 1

	there is an integral solution to $Ax = b$.	there is an integral basic solution to $Ax = b$.	all basic solutions of $Ax = b$ are integral.
For all $b \in Z_0(A)$:	$M(Z)$	WBSP	DAN
For all $b \in Z(A)$:	INT	BSP	UNM

namely, the system $Ax = b$ has an integral basic solution for all $b \in Z_0(A)$. The above characterizations can be summarized as in Table 1 where $M(Z)$ denotes the class of all integral matrices. To use Table 1 combine the phrase at the beginning of a row with that at the top of a column to obtain the characterization of the class appearing in that row and column.

The classes INT, DAN, and UNM have been characterized by subdeterminants. For an $m \times n$ rank m integral matrix, $A \in$ DAN if and only if all nonzero $m \times m$ subdeterminants have the same absolute value, and $A \in$ UNM if and only if all nonzero $m \times m$ subdeterminants are ± 1. Hence for a Dantzig matrix A with full row rank, $A \in$ UNM if and only if there is one such subdeterminant with absolute value 1. In general we define a rank-r integral matrix to be *subunimodular* if it contains at least one $r \times r$ subdeterminant with absolute value 1. The collection of all such matrices is denoted by SUB. This generalizes the concept of strong unimodularity introduced by Truemper and Chandrasekaran [11]. An $m \times n$ rank-r matrix A is said to be strongly unimodular if every basis of A contains an $r \times r$ submatrix with determinant ± 1. We denote the collection of all such matrices by SUM.

3. CHARACTERIZATIONS

The containment relations between the various classes are displayed in Figure 1(a). Figure 1(b) shows the relations when one restricts the classes to matrices with full row rank.

That SUB $\subset$ BSP is shown in Lemma 1 below. The other containment relations are clear from the definitions and characterizations given in the last section. A few examples suffice to show that these collections are distinct.

Let $L = (2\ \ 3)$, $M = (1\ \ 2)$,

$$N = \begin{pmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 & -1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}.$$

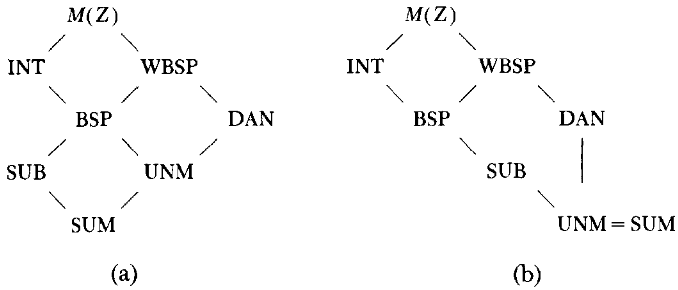

FIG. 1.

Then $M \in \text{SUB}$ and $U \in \text{UNM}$, but $M \notin \text{UNM}$ and $U \notin \text{SUB}$. Also $L \in \text{INT}$ but $L \notin \text{BSP}$. Finally it is clear that $N \notin \text{SUB}$ and $N \notin \text{UNM}$, but we can show that $N \in \text{BSP}$. Note that $\text{Z}(N) = \text{Z}^2 = \text{S}_1 \cup \text{S}_2 \cup \text{S}_3$, where Z^2 is the Z-module of all ordered pairs of integers written as columns, S_1 is the submodule of all such ordered pairs whose first coordinates are even, S_2 is the submodule of all such ordered pairs whose second coordinates are even, and S_3 is the submodule of all such ordered pairs whose coordinates have the same parity. Now the first two columns of N generate S_1, the third and fourth columns of N generate S_2, and the fourth and fifth columns of N generate S_3. Thus $N \in \text{BSP}$. The remaining distinctions are either clear or can be obtained by combining the above examples with the results of Theorem 2 below.

LEMMA 1. SUB $\subset$ BSP, *that is if A is subunimodular, then it also has the basic solution property.*

Proof. Assuming A has an $r \times r$ nonsingular unimodular submatrix, say A_1, there exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix}.$$

Then, since A has rank r, we have

$$PAQ = \begin{pmatrix} A_1 & A_1 N \\ MA_1 & MA_1 N \end{pmatrix} \tag{1}$$

for some M and N. Since A_1 is nonsingular unimodular, M and N must be integral. Now suppose the system $Ax = b$, b integral, is consistent. Note that

$Ax = b$ if and only if

$$\begin{pmatrix} A_1 & A_1N \\ MA_1 & MA_1N \end{pmatrix} Q^T x = Pb. \tag{2}$$

Partition conformably

$$Q^T x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad Pb = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Then, Equation (2) yields

$$A_1 x_1 + A_1 N x_2 = b_1 \tag{3}$$

and

$$MA_1 x_1 + MA_1 N x_2 = b_2. \tag{4}$$

Equation (4) says that $Mb_1 = b_2$, so that we need only to find solutions of (3) to obtain solutions of (2). We then simply observe that

$$\begin{pmatrix} A_1^{-1} b_1 \\ 0 \end{pmatrix}$$

satisfies (3), and so

$$Q \begin{pmatrix} A_1^{-1} b_1 \\ 0 \end{pmatrix}$$

is an integer basic solution of the original system. Thus, A has the basic solution property. ∎

The following result states that the collections displayed in Figure 1 form a meet semilattice with respect to set inclusion, that is:

THEOREM 1. (i) BSP = INT $\cap$ WBSP.
(ii) UNM = BSP $\cap$ DAN = INT $\cap$ DAN.
(iii) SUM = SUB $\cap$ UNM = SUB $\cap$ DAN.

Proof. (i): That $\mathrm{BSP} \subset \mathrm{INT} \cap \mathrm{WBSP}$ is clear from the inclusion relations. Now $A \in \mathrm{INT}$ implies that $Z_0(A) = Z(A)$, and thus $A \in \mathrm{INT} \cap \mathrm{WBSP}$ implies that $A \in \mathrm{BSP}$. Part (ii) is obtained similarly, and the proof of part (iii) is given after that of Theorem 2. ■

Each of the classes of Figure 1 is characterized in terms of integral full-rank factorizations in Theorem 2. That such factorizations exist is given in [2, Lemma 2]. Note the similarity of these characterizations for classes on the same diagonal (from upper right to lower left) of Figure 1(a). Related characterizations of UNM are given by Truemper in [10].

THEOREM 2. *Let $A = FG$ be an integral full-rank factorization of A. Then*:

(i) $A \in \mathrm{WBSP} \Leftrightarrow G \in \mathrm{WBSP}$.
(ii) $A \in \mathrm{DAN} \Leftrightarrow G \in \mathrm{DAN}$.
(iii) $A \in \mathrm{INT} \Leftrightarrow F \in \mathrm{INT}$ and $G \in \mathrm{INT}$.
(iv) $A \in \mathrm{BSP} \Leftrightarrow F \in \mathrm{INT}$ and $G \in \mathrm{BSP}$.
(v) $A \in \mathrm{UNM} \Leftrightarrow F \in \mathrm{INT}$ and $G \in \mathrm{UNM}$.
(vi) $A \in \mathrm{SUB} \Leftrightarrow F, G \in \mathrm{SUB}$.
(vii) $A \in \mathrm{SUM} \Leftrightarrow F, G \in \mathrm{SUM}$.

Furthermore, for any matrix F with full column rank, $F \in \mathrm{INT} \Leftrightarrow F \in \mathrm{BSP} \Leftrightarrow F \in \mathrm{UNM}$ and $F \in \mathrm{SUB} \Leftrightarrow F \in \mathrm{SUM}$.

Proof. For matrices with full column rank the characterizations of INT and SUB in terms of subdeterminants are identical to those of UNM and SUM respectively. Hence the "Furthermore" portion holds.

For part (i) suppose that $A \in \mathrm{WBSP}$ and that $b \in Z_0(G)$, thus there is an integral vector z such that $Gz = b$. Now $Az = FGz = Fb \in Z_0(A)$, and there is an integral basic solution x_0 such that $Ax_0 = FGx_0 = Fb$. Since F has full column rank, this implies that $Gx_0 = b$. Now the nonzero entries of x_0 correspond to linearly independent columns of A, which correspond to linearly independent columns of G. Hence x_0 is a basic solution to $Gx = b$, and $G \in \mathrm{WBSP}$. Conversely, if $G \in \mathrm{WBSP}$ and $b = Az \in Z_0(A)$ where z is integral, then $Gz \in Z_0(G)$. Hence there is an integral basic solution x_0 such that $Gx_0 = Gz$ and $Ax_0 = FGx_0 = FGz = b$. Thus $A \in \mathrm{WBSP}$.

For part (ii) note that the row space of G equals the row space of A, which is also given as the row space of any submatrix A_0 of r linearly independent rows of A. Hence there is a nonsingular matrix T such that $TG = A_0$. Now $A \in \mathrm{DAN}$ if and only if $A_0 \in \mathrm{DAN}$ [6, (4.3)] if and only if $G \in \mathrm{DAN}$ [6, (4.2)].

Part (iii) is given in [2, Theorem 2].

For part (iv) suppose that $A \in \mathrm{BSP}$. Then $A \in \mathrm{INT}$, and hence $F \in \mathrm{INT}$ by (iii). Also $A \in \mathrm{BSP}$ implies $G \in \mathrm{BSP}$ by an argument similar to (i). Conversely, suppose that $G \in \mathrm{BSP}$ and $F \in \mathrm{INT}$. Then $G \in \mathrm{INT}$, and thus $A \in \mathrm{INT}$ by (iii). Also $A \in \mathrm{WBSP}$ by (i), so that $A \in \mathrm{BSP}$ by Theorem 1(i).

For part (v) note that there is a correspondence between bases B of A and bases C of G such that $B = FC$. Also there is a correspondence between $r \times r$ submatrices B_i of B and $r \times r$ submatrices F_i of F such that $B_i = F_i C$. Thus $\gcd_i\{\det B_i\} = \gcd_i\{\det F_i \det C\} = (\det C)\gcd_i\{\det F_i\} = \det C$ whenever $F \in \mathrm{INT}$. Now $A \in \mathrm{UNM}$ implies $F \in \mathrm{INT}$ by (iii) and thus $G \in \mathrm{UNM}$. Conversely, when $F \in \mathrm{INT}$ the above identity for determinants holds and hence $G \in \mathrm{UNM}$ implies $A \in \mathrm{UNM}$.

Observe that any $r \times r$ submatrix of A is obtained by multiplying $r \times r$ submatrices of F and G. Thus part (vi) follows, since the determinant of a product is the product of determinants, and all subdeterminants of F and G are integers. Finally since there is a correspondence between bases of A and bases of G, part (vii) is obtained in a manner similar to part (vi). ■

Proof of Theorem 1(iii). That $\mathrm{SUM} \subset \mathrm{SUB} \cap \mathrm{DAN}$ is clear. Let $A \in \mathrm{SUB} \cap \mathrm{DAN}$, and let $A = FG$ be an integral full-rank factorization. Then $F, G \in \mathrm{SUB}$ and $G \in \mathrm{DAN}$ by Theorem 2(ii), (vi). Now G contains some $r \times r$ submatrix with determinant ± 1, and hence all nonzero $r \times r$ subdeterminants of G are ± 1. If C is a basis for G, then $\det C \neq 0$ and hence $\det C = \pm 1$, that is $G \in \mathrm{SUM}$. Also $F \in \mathrm{SUM}$, since it has full column rank, and $A = FG \in \mathrm{SUM}$ by Theorem 2(vii). ■

Let

$$A = P\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} Q,$$

where P and Q are nonsingular unimodular matrices, and D_r is an $r \times r$ integral nonsingular diagonal matrix (the Smith canonical form will yield such a factorization). Then a great deal of information is contained in the first r rows of Q. Let P_0 be the submatrix consisting of the first r columns of P, and let Q_0 be the submatrix consisting of the first r rows of Q. Then $A = FQ_0$, where $F = P_0 D_r$ gives an integral full-rank factorization of A. Now the Smith canonical form of Q_0 is $(I_r \;\; 0)$, so that $Q_0 \in \mathrm{INT}$ and we obtain $A \in \mathrm{WBSP} \Leftrightarrow Q_0 \in \mathrm{WBSP} \Leftrightarrow Q_0 \in \mathrm{BSP}$ and $A \in \mathrm{DAN} \Leftrightarrow Q_0 \in \mathrm{DAN} \Leftrightarrow Q_0 \in \mathrm{UNM}$. Furthermore if

$$T = \begin{pmatrix} D_r^{-1} & 0 \\ 0 & I_k \end{pmatrix} P^{-1},$$

where $k = m - r$, then

$$TA = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$

belongs to INT and has the full-rank factorization

$$TA = \begin{pmatrix} I_r \\ 0 \end{pmatrix} Q_0.$$

Thus we obtain

COROLLARY 1. *Given an integral matrix A, there exists a nonsingular (not necessarily integral) matrix T with an integral inverse such that $TA \in$ INT, $TA \in$ BSP $\Leftrightarrow$ $A \in$ WBSP, and $TA \in$ UNM $\Leftrightarrow$ $A \in$ DAN. Furthermore, if S is any nonsingular matrix with an integral inverse such that SA is integral, then $SA \in BSP$ $\Rightarrow$ $A \in$ WBSP and $SA \in$ UNM $\Rightarrow$ $A \in$ DAN.*

Proof. All that remains is the "Furthermore" portion. Suppose that $SA = FG$ is an integral full-rank factorization of SA. Then $A = (S^{-1}F)G$ is an integral full-rank factorization of A. Now $SA \in$ BSP implies that $G \in$ BSP and hence $A \in$ WBSP. Similarly, $SA \in$ UNM implies that $G \in$ UNM and hence $A \in$ DAN. ∎

Finally we give a characterization of subunimodular matrices in terms of both full-rank factorizations and generalized inverses.

THEOREM 3. *Let A be an $m \times n$ integral matrix of rank r. Then A is a subunimodular matrix if and only if*

(i) *A has an integral full-rank factorization $A = FG$ where*

$$PF = \begin{pmatrix} B \\ MB \end{pmatrix} \quad \text{and} \quad GQ = (C \quad CN)$$

for some permutation matrices P and Q, some $r \times r$ matrices B and C, and some integral matrices M and N, respectively.

(ii) *A has an integral $\{1\}$-inverse.*

Proof. $\Rightarrow$: Use Equation (1) again to write

$$PAQ = \begin{pmatrix} A_1 & A_1N \\ MA_1 & MA_1N \end{pmatrix},$$

where P and Q are permutation matrices, M and N are integral, and A_1 is nonsingular unimodular. Hence writing

$$A = P^T \begin{pmatrix} A_1 \\ MA_1 \end{pmatrix} \begin{pmatrix} I_r & N \end{pmatrix} Q^T$$

yields (i). Also (ii) holds since the gcd of all $r \times r$ subdeterminants of A is 1.

$\Leftarrow$: Assuming (ii), from Theorem 2(iii) we then have that both F and G have an integral {1}-inverse. Hence, the gcd of all $r \times r$ subdeterminants of F, and of G, is 1.

Now, the gcd of all $r \times r$ subdeterminants of

$$F = P^T \begin{pmatrix} B \\ MB \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B \\ MB \end{pmatrix}$$

are the same. But then

$$\begin{pmatrix} I & 0 \\ -M & I \end{pmatrix} \begin{pmatrix} B \\ MB \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

and since M is integral it follows from [8, Lemma 7.5] that the gcd of all $r \times r$ subdeterminants of

$$\begin{pmatrix} B \\ MB \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

are the same. Thus, the gcd of all $r \times r$ subdeterminants of $\begin{pmatrix} B \\ 0 \end{pmatrix}$ is 1, and hence B must be nonsingular unimodular. (Note that $r = \operatorname{rank} F = \operatorname{rank} B$, which implies B is nonsingular.) Similarly, C is nonsingular unimodular. The proof then follows from Theorem 2(vi). ■

REMARKS.

(1) The characterization of SUB given in Theorem 3 is similar to a characterization given in [4] for nonnegative integral matrices with a nonnegative integral {1}-inverse. It is also shown in [4] that a rank-r nonnegative integral matrix is in this latter class if and only if it contains under permutation I_r as a submatrix, and thus this latter class is contained in SUB.

(2) It is well known [7, Theorem II.1] that if v is an integral vector whose coordinates have a gcd of 1, then there is a nonsingular unimodular matrix

containing v as one of its rows. This result is generalized by:

PROPOSITION 1. *Let G be an $r \times n$ integral rank-r matrix. Then $G \in$ INT if and only if there exists an $(n-r)\times n$ integral matrix M such that*

$$Q = \begin{pmatrix} G \\ M \end{pmatrix}$$

is nonsingular unimodular.

Proof. By [7, Theorem II.2] there is a nonsingular unimodular matrix U such that $GU = (H\ \ 0)$, where H is an integral upper triangular matrix with positive diagonal entries and with each element to the right of the diagonal being a least residue modulo the diagonal element on its left. If $G \in$ INT, then by [8, Lemma 7.5] $\det H = 1$. Thus the diagonal entries of H are 1's and $H = I_r$. It follows that $G = (I_r\ \ 0)U^{-1}$ and G is the first r rows of U^{-1}. Conversely, if $Q = \begin{pmatrix} G \\ M \end{pmatrix}$ is nonsingular unimodular, then the Smith canonical form of G is $GQ^{-1} = (I_r\ \ 0)$ and thus $G \in$ INT. ∎

(3) We note that in characterizing the classes BSP, SUB, and UNM, the full-row-rank case is very important (see Theorem 2 and the remark preceding its corollary). It could be helpful to have characterizations for the full-row-rank matrices in these classes similar to that given in Proposition 1 for INT. That is, what additional conditions would one have to put on Q in Proposition 1 in order to replace INT by BSP, SUB, or UNM?

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