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Splitting off rational parts in homotopy types

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Abstract

It is known algebraically that any abelian group is a direct sum of a divisible group and a reduced group (see Theorem 21.3 of [L. Fuchs, Infinite Abelian Groups, vol. I, Academic Press, New York–London, 1970]). In this paper, conditions to split off rational parts in homotopy types from a given space are studied in terms of a variant of Hurewicz map, say $\bar{\rho}:[S^n_{\mathbb{Q}}, X] \to H_n(X;\mathbb{Z})$ and generalised Gottlieb groups. This yields decomposition theorems on rational homotopy types of Hopf spaces, T-spaces and Gottlieb spaces, which has been known in various situations, especially for spaces with finiteness conditions.

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Introduction

The Gottlieb group is introduced by Gottlieb [6,7] and the generalised Gottlieb set is introduced by Varadarajan [18]. Dula and Gottlieb obtained a general result on splitting a Hopf space off from a fibration as Theorem 1.3 of [5].

In this paper, we work in the category of spaces having homotopy types of CW complexes with base points and pointed continuous maps. A relation $f \sim g$ indicates a pointed homotopy relation of maps f and g and a relation $X \simeq Y$ indicates a homotopy equivalence relation of spaces X and Y. We also denote by $[X, Y]$ the set of pointed homotopy classes of maps from X to Y.

We adopt some more conventional notations: $X_{\mathbb{Q}}$ stands for the rationalisation of a space X, $K(\pi, n)$ for the Eilenberg–Mac Lane space of type (π, n), $G(V, X)$ for the generalised Gottlieb subset of $[V, X]$ and $H_n(X)$ for $H_n(X; \mathbb{Z})$. We introduce a variant of Hurewicz map $\bar{\rho} : [S^n_{\mathbb{Q}}, X] \to H_n(X)$ by $\bar{\rho}(\alpha) = \alpha_*([S^n] \otimes 1)$ for $\alpha \in [S^n_{\mathbb{Q}}, X]$, where α_* is the homomorphism given by $\alpha_* : H_n(S^n) \otimes \mathbb{Q} = H_n(S^n_{\mathbb{Q}}) \to H_n(X)$. Our main result is described as follows:

Theorem 2.2. *Let $R = \bigoplus_{\lambda \in \Lambda} \mathbb{Q}$ be a $\mathbb{Q}$-vector space of dimension $\# \Lambda \leqslant \infty$. Let X be 0-connected and $R \subset \bar{\rho}(G(S^n_{\mathbb{Q}}, X)) \subseteq H_n(X)$, $n \geqslant 2$. Then X decomposes as*

$$X \simeq Y \times K(R, n).$$

Theorem 2.2 gives unified proof to the splitting phenomena on rational G-space, T-space and Hopf space without assuming any finiteness conditions, which are proved under various situations by a number of authors: Scheerer [16] obtained decomposition theorems of rational Hopf spaces without assuming the finite type assumptions. Oprea [15] obtained decomposition theorems by using minimal model method in rational homotopy theory. Aguadé [2] obtained a decomposition theorems on rational T-spaces of finite type.

1. Preliminaries

We regard the one pnoint union $X \vee Y$ of spaces X and Y as a subspace $X \times * \cup * \times Y$ of the product space $X \times Y$ with the inclusion map $j : X \vee Y \to X \times Y$. For any collection of a finitely or infinitely many spaces X_λ ($\lambda \in \Lambda$), we denote the *wedge sum* (or one point union) by $\bigvee_{\lambda \in \Lambda} X_\lambda$ and the *direct sum* (or weak product) by $\bigoplus_{\lambda \in \Lambda} X_\lambda = \{(x_\lambda) \in \prod_{\lambda \in \Lambda} X_\lambda \mid x_\lambda = * \text{ except for finitely many } \lambda\}$. Then we have $\bigvee_{\lambda \in \Lambda} X_\lambda \subset \bigoplus_{\lambda \in \Lambda} X_\lambda$, where $\bigoplus_{\lambda \in \Lambda} X_\lambda$ is a dense subset of the product space $\prod_{\lambda \in \Lambda} X_\lambda$ and has the weak topology with respect to finite products of X_λ's.

Let X_∞ be the James reduced product space of a 0-connected space X of finite type, so that $X_\infty \simeq \Omega(\Sigma X)$ by James [10]. Then X_∞ is a nice CW approximation of a space $\Omega \Sigma X$ to work in the category of spaces having homotopy types of CW complexes.

We apply rationalisation or $\mathbb{Q}$-localisation to any 0-*connected nilpotent spaces* or any *nilpotent groups* (see [4,9] or [13] for the precise definition of the rationalisation of a space or a nilpotent group). The rationalisation $\ell_{\mathbb{Q}} : X \to X_{\mathbb{Q}}$, or simply $X_{\mathbb{Q}}$ does exist for such spaces X such that $\ell_{\mathbb{Q}}$ induces the following isomorphisms:

$$\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q} \quad \text{and} \quad H_n(X_{\mathbb{Q}}) \cong H_n(X) \otimes \mathbb{Q}$$

for any integer $n \geqslant 1$, where $G \otimes \mathbb{Q}$ denotes the rationalisation of a nilpotent group G (cf. [4,9] or [13]). Moreover the universality of rationalisation yields a bijection

$$\ell_{\mathbb{Q}}^*: [X_{\mathbb{Q}}, Y_{\mathbb{Q}}] \cong [X, Y_{\mathbb{Q}}]$$

for any such spaces X and Y. The rationalisation enjoys the following fact.

Fact 1.1.

(1) $S_{\mathbb{Q}}^{2m+1} \simeq K(\mathbb{Q}, 2m+1)$ *for any integer* $m \geqslant 0$.
(2) $\Omega(S_{\mathbb{Q}}^{2m+1}) \simeq (\Omega S^{2m+1})_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2m)$ *for any integer* $m \geqslant 1$.
(3) $(X_\infty)_{\mathbb{Q}} \simeq (X_{\mathbb{Q}})_\infty$ *for* X *a* 0*-connected nilpotent space of finite type.*

Proof. (1) and (2) are well-known. We give here a brief explanation for (3): The suspension functor Σ and the loop functor Ω enjoys the properties $\Sigma(X_{\mathbb{Q}}) \simeq (\Sigma X)_{\mathbb{Q}}$ for any 0-connected space X and $\Omega(X_{\mathbb{Q}}) \simeq (\Omega X)_{\mathbb{Q}}$ for any 1-connected space X. Then it follows that $(X_\infty)_{\mathbb{Q}} \simeq (\Omega(\Sigma X))_{\mathbb{Q}} \simeq \Omega(\Sigma(X_{\mathbb{Q}})) \simeq (X_{\mathbb{Q}})_\infty$. □

We state two propositions to be used in the proof of the main theorem.

Proposition 1.2. *Let* X *be a* 0*-connected space of finite type and* $f : X \to Y$ *a map. If* $f \in G(X, Y)$*, then there is an extension* $\overline{f} : X_\infty \to Y$ *of* f *such that* $\overline{f} \in G(X_\infty, Y)$.

Proof. We may assume that there is a map $\mu : Y \times X \to Y$ such that $\mu|Y \times \{*\} = 1_Y : Y \to Y$ and $\mu|\{*\} \times X = f : X \to Y$. We put $\mu_1 = \mu$ and, for any n we define

$$\mu_n = \mu \circ (\mu_{n-1} \times 1_X) : Y \times X^n = \left(Y \times X^{n-1}\right) \times X \to Y \times X \to Y$$

by induction on n. Then we observe that μ_n factors through $Y \times X^n \to Y \times X_n$, where X_n denotes the set of products of at most n elements of X in the James reduced product space X_∞ (cf. James [10]). Since X_∞ has a weak topology with respect to X_n, we have done. □

Proposition 1.3. *Let* $\alpha_\lambda : X_\lambda \to Z$ *be a map for any* $\lambda \in \Lambda$*. If* $\alpha_\lambda \in G(X_\lambda, Z)$ *for each* $\lambda \in \Lambda$*, then the map* $\alpha : \bigvee_{\lambda \in \Lambda} X_\lambda \to Z$ *defined by* $\alpha|X_\lambda = \alpha_\lambda : X_\lambda \to Z$ *can be extended to a map* $\bar{\alpha} : \bigoplus_{\lambda \in \Lambda} X_\lambda \to Z$ *with* $\bar{\alpha} \in G(\bigoplus_{\lambda \in \Lambda} X_\lambda, Z)$.

Proof. Since each X_λ has a homotopy type of a CW complex, we may assume that there is a map $\mu_\lambda : Z \times X_\lambda \to Z$ such that $\mu_\lambda|\{*\} \times X_\lambda = \alpha_\lambda : X_\lambda \to Z$ and $\mu_\lambda|Z \times \{*\} = 1_Z : Z \to Z$ for each $\lambda \in \Lambda$. For any n and $\lambda_1, \lambda_2, \ldots, \lambda_n$, we define

$$\mu_{\lambda_1,\ldots,\lambda_n} = \mu_{\lambda_n} \circ (\mu_{\lambda_1,\ldots,\lambda_{n-1}} \times 1_{X_{\lambda_n}}) : Z \times (X_{\lambda_1} \times \cdots \times X_{\lambda_{n-1}} \times X_{\lambda_n}) \to Z$$

by induction on n. For any index set Λ, we assume that Λ is totally-ordered. Since $\bigoplus_{\lambda\in\Lambda} X_\lambda$ has a weak topology with respect to $X_{\lambda_1}\times\cdots\times X_{\lambda_n}$, $\lambda_1,\ldots,\lambda_n$ ($n\geqslant 0$), the collection of maps $\mu_{\lambda_1,\ldots,\lambda_n}$ defines a pairing $\mu : Z\times(\bigoplus_{\lambda\in\Lambda} X_\lambda)\to Z$ with axes $(1_Z,\bar\alpha)$ (cf. [14]). □

2. Proof of the main result

Proposition 2.1. *Let P be an idempotent endomorphism of $H_n(X)$, $n\geqslant 2$. Suppose that $R=\operatorname{im}P\subseteq H_n(X)$ is a rational vector space and is in $\operatorname{im}\bar\rho$. Then we have maps $\alpha : S^n(R)\to X$ and $\beta : X\to K(R,n)$ such that*

$$\beta\circ\alpha\sim\iota^n_R : S^n(R)\to K(R,n),\quad \text{and}$$
$$P=\alpha_*\circ\left(\iota^n_{R*}\right)^{-1}\circ\beta_* : H_n(X)\to H_n\bigl(K(R,n)\bigr)\xleftarrow{\cong} H_n\bigl(S^n(R)\bigr)\to H_n(X),$$

where $S^n(R)$ denotes the Moore space of type (R,n) and ι^n_R corresponds to the identity element in $\operatorname{Hom}(R,R)=\operatorname{Hom}(\pi_n(S^n(R)),\pi_n(K(R,n)))\cong[S^n(R),K(R,n)]$.

Proof. Let $\{\bar\rho(\alpha_\lambda)\mid\lambda\in\Lambda\}$ be a basis of $R=\operatorname{im}P$, and hence $R\cong\bigoplus_{\lambda\in\Lambda}\mathbb{Q}$. Since $S^n(R)=\bigvee_{\lambda\in\Lambda}S^n_{\mathbb{Q}}$, we define $\alpha : S^n(R)\to X$ by its restrictions to all factors:

$$\alpha|_{S^n_{\mathbb{Q}}}=\alpha_\lambda : S^n_{\mathbb{Q}}\to X.$$

Since α_* is an isomorphism onto $R\subseteq H_n(X)$, we have its inverse $\phi : R\to H_n(S^n(R))$ so that $\phi\circ\alpha_*=\mathrm{id}_{H_n(S^n(R))}$ and $\alpha_*\circ\phi=\mathrm{id}_R$. Now we define a homomorphism $s : H_n(X)\to\operatorname{im}P\cong H_n(S^n(R))$ by $s=\phi\circ P$: Since $\operatorname{im}\alpha_*$ is in the image of an idempotent endomorphism P, we have $s\circ\alpha_*=\phi\circ P\circ\alpha_*=\phi\circ\alpha_*=\mathrm{id}$. Also we have $\alpha_*\circ s=\alpha_*\circ\phi\circ P=P$. Thus s satisfies the following formulae:

$$s\circ\alpha_*=\mathrm{id} : H_n\bigl(S^n(R)\bigr)\to H_n\bigl(S^n(R)\bigr),$$
$$\alpha_*\circ s=P : H_n(X)\to H_n(X).$$

Let us recall that α induces the following commutative diagram:

$$\begin{array}{ccc}
[X,K(R,n)] & \xrightarrow[\cong]{\Psi'} & \operatorname{Hom}(H_n(X),H_n(K(R,n))) \\
\Big\downarrow{\scriptstyle\alpha^*} & & \Big\downarrow{\scriptstyle(\alpha_*)^*} \\
[S^n(R),K(R,n)] & \xrightarrow[\cong]{\Psi} & \operatorname{Hom}(H_n(S^n(R)),H_n(K(R,n))),
\end{array}\tag{2.1}$$

where Ψ and Ψ' are homomorphisms defined by taking the nth homology groups, and are isomorphisms by the universal coefficient theorem. Since Ψ' is an isomorphism, we define β to be the unique element $\Psi'^{-1}(\iota^n_{R*}\circ s)$ so that $\beta_*=\iota^n_{R*}\circ s$.

Firstly by $P=\alpha_*\circ s$, we have $P=\alpha_*\circ s=\alpha_*\circ(\iota^n_{\mathbb{Q}*})^{-1}\circ\beta_*$.

Next we show $\beta\circ\alpha\sim\iota^n_{\mathbb{Q}}$. By the commutativity of the diagram (2.1), we have

$$\Psi\bigl(\alpha^*(\beta)\bigr)=(\alpha_*)^*\circ\Psi'(\beta)=(\alpha_*)^*\bigl(\iota^n_{\mathbb{Q}*}\circ s\bigr)=\iota^n_{\mathbb{Q}*}\circ s\circ\alpha_*=\iota^n_{\mathbb{Q}*}=\Psi\bigl(\iota^n_{\mathbb{Q}}\bigr).$$

Since Ψ is an isomorphism, we also have $\beta\circ\alpha=\alpha^*(\beta)\sim\iota^n_{\mathbb{Q}}$. □

Let us recall that $G(S^n_{\mathbb{Q}}, X) \subset [S^n_{\mathbb{Q}}, X] \xrightarrow{\bar\rho} H_n(X)$. In the following theorem, we do *not* assume that X is rationalised nor that X is $(n-1)$-connected.

Theorem 2.2. *Let $R = \bigoplus_{\lambda\in\Lambda}\mathbb{Q}$ be a $\mathbb{Q}$-vector space of dimension $\#\Lambda \leqslant \infty$. Let X be 0-connected and $R \subset \bar\rho(G(S^n_{\mathbb{Q}}, X)) \subseteq H_n(X)$, $n \geqslant 2$. Then X decomposes as*

$$X \simeq Y \times K(R, n).$$

Proof. Since a divisible submodule R is a direct summand of $H_n(X)$, there is an idempotent endomorphism $P : H_n(X) \to H_n(X)$ with $\operatorname{Im} P = R$. We fix a basis of R as $\{\bar\rho(\alpha_\lambda) \mid \alpha_\lambda \in G(S^n_{\mathbb{Q}}, X),\ \lambda \in \Lambda\}$.

By Proposition 2.1, there are maps $\alpha : S^n(R) \to X$, $\beta : X \to K(R, n)$ such that

$$\beta\circ\alpha \sim \iota^n_R : S^n(R) \to K(R, n),$$
$$P = \alpha_* \circ \left(\iota^n_{R*}\right)^{-1} \circ \beta_* : H_n(X) \to H_n\bigl(K(R, n)\bigr) \xleftarrow{\cong} H_n\bigl(S^n(R)\bigr) \to H_n(X).$$

Then we extend the map α onto $K(R, n) \supseteq S^n(R)$ as $\bar\alpha : K(R, n) \to X$ by dividing our arguments in two cases:

Case 1: n is an odd positive integer > 1, namely, $n = 2m + 1$ for some $m \geqslant 1$. Then we have $K(\mathbb{Q}, 2m + 1) \simeq S^{2m+1}_{\mathbb{Q}}$, and hence by Proposition 1.3 we obtain the desired map.

Case 2: n is an even positive integer, namely, $n = 2m$ for some $m \geqslant 1$. Since $\alpha_\sigma \in G(S^{2m}_{\mathbb{Q}}, X)$, the map $\alpha_\sigma : S^{2m}_{\mathbb{Q}} \to X$ can be extended to the James reduced product space by Proposition 1.2, say,

$$\bar\alpha_\sigma : \bigl(S^{2m}_{\mathbb{Q}}\bigr)_\infty \to X, \quad \bar\alpha_\sigma \in G\bigl(\bigl(S^{2m}_{\mathbb{Q}}\bigr)_\infty, X\bigr),$$

where we know

$$\bigl(S^{2m}_{\mathbb{Q}}\bigr)_\infty \simeq \bigl(S^{2m}_\infty\bigr)_{\mathbb{Q}} \simeq \bigl(\Omega\Sigma S^{2m}\bigr)_{\mathbb{Q}} \simeq \bigl(\Omega S^{2m+1}\bigr)_{\mathbb{Q}}$$
$$\simeq \Omega\bigl(S^{2m+1}_{\mathbb{Q}}\bigr) \simeq \Omega K(\mathbb{Q}, 2m{+}1) \simeq K(\mathbb{Q}, 2m).$$

Thus we have $\bar\alpha_\sigma \in G(K(\mathbb{Q}, 2m), X)$. Hence by Proposition 1.3, there is a map

$$\bar\alpha : K(R, 2m) = \bigoplus_{\lambda\in\Lambda} K(\mathbb{Q}, 2m) \to X$$

extending $\alpha : S^n(R) \to X$. Then we obtain $\beta\circ\bar\alpha \sim \mathrm{id}_{K(R,n)}$, since the identity map $\mathrm{id} : K(R, n) \to K(R, n)$ is the unique extension of $\iota^n_R : S^n(R) \to K(R, n)$, up to homotopy.

Thus in either case, we obtain a map $\bar\alpha \in G(K(R, n), X)$ such that

$$\beta\circ\bar\alpha \sim \mathrm{id} : K(R, n) \to K(R, n).$$

Let Y be the homotopy fibre of $\beta : X \to K(R, n)$. Then by Theorem 1.3 of Dula and Gottlieb [5], we obtain

$$X \simeq Y \times \bigoplus_{\lambda\in\Lambda} K(\mathbb{Q}, n) \simeq Y \times K(R, n).$$

This completes the proof of the theorem. □

3. Applications

A 0-connected space X is called a *T-space* if the fibration $\Omega X \to X^{S^1} \to X$ is trivial in the sense of fibre homotopy type (Aguadé [2]). If X is a 0-connected Hopf space, then X is a T-space. Aguadé showed that 1-connected space X of finite type is a rational T-space if and only if X has the same rational homotopy type as a generalised Eilenberg–Mac Lane space, i.e., a product of (infinitely many) Eilenberg–Mac Lane spaces (Theorem 3.3 of [2]). Woo and Yoon showed that a space X is a T-space if and only if $G(\Sigma A, X) = [\Sigma A, X]$ for any space A by Theorem 2.2 of [19]. Then we have the following result by Theorem 2.2.

Theorem 3.1. *Let $R = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}$ be a finite or an infinite dimensional $\mathbb{Q}$-vector space. Let X be a 0-connected T-space and $R \subset \pi_n(X)$, $n \geqslant 2$. If $\bar{\rho}|R : R \to H_n(X)$ is an injection and $[S^n_{\mathbb{Q}}, X] = G(S^n_{\mathbb{Q}}, X)$, then X decomposes as*

$$X \simeq Y \times \bigoplus_{\lambda\in\Lambda} K(\mathbb{Q}, n) \simeq Y \times K(R, n), \quad \textit{for some } T\textit{-space } Y.$$

Proof. Firstly, we observe the image of $\ell^*_{\mathbb{Q}}$ contains R: Let a be a generator of the $\mathbb{Q}$-vector space $R \subseteq \pi_n(X)$. Then we can use the telescope construction (cf. Adams [1], Sullivan [17]) to obtain a map $\alpha : S^n_{\mathbb{Q}} \to X$ such that $\alpha \circ \ell_{\mathbb{Q}} \sim a : S^n \to X$. Thus we can choose a $\mathbb{Q}$-vector space $\bar{R} \subseteq [S^n_{\mathbb{Q}}, X]$ such that

$$\bar{R} \overset{\ell^*_{\mathbb{Q}}}{\cong} R,$$

and hence we have $\bar{\rho}(\bar{R}) = \rho(R) \cong R$. Then by Theorem 2.11 of [19] and Theorem 2.2, we obtain the result. □

Theorem 3.1 implies the following result as a direct consequence.

Corollary 3.2. *Let $n \geqslant 2$. Let $R = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}$ be a finite or an infinite dimensional $\mathbb{Q}$-vector space and assume that $R \subset \pi_n(X)$. If X is an $(n-1)$-connected T-space, then X splits as*

$$X \simeq Y \times \bigoplus_{\lambda\in\Lambda} K(\mathbb{Q}, n) \simeq Y \times K(R, n), \quad \textit{for some } T\textit{-space } Y.$$

A space X is called a *G-space* if $G_n(X) = \pi_n(X)$ for all n (cf. [7]). As a special case of Theorem 2.2, we have the following result for rational G-space. We remark that $\pi_n(X_{\mathbb{Q}}) = G_n(X_{\mathbb{Q}})$ implies $[S^n_{\mathbb{Q}}, X_{\mathbb{Q}}] = G(S^n_{\mathbb{Q}}, X_{\mathbb{Q}})$ for any n.

Theorem 3.3. *Let $n \geqslant 2$. Assume that a rational space $X_{\mathbb{Q}}$ is an $(n-1)$-connected G-space. If $\pi_n(X_{\mathbb{Q}}) \cong \bigoplus_{\lambda\in\Lambda} \mathbb{Q}$, a finite or an infinite dimensional $\mathbb{Q}$-vector space, then $X_{\mathbb{Q}}$ decomposes as*

$$X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}} \times \bigoplus_{\lambda\in\Lambda} K(\mathbb{Q}, n) \simeq Y_{\mathbb{Q}} \times K\big(\pi_n(X_{\mathbb{Q}}), n\big),$$

for some rational space $Y_{\mathbb{Q}}$ an n-connected G-space.

Theorem 3.3 implies the following theorem (cf. [16]). For finite complexes or finite Postnikov pieces, it is known by Haslam [8] and Mataga [12].

Theorem 3.4. *If X is a 1-connected space, then the following are equivalent*:

(1) *$X_{\mathbb{Q}}$ is a G-space.*
(2) *$X_{\mathbb{Q}}$ is a T-space.*
(3) *$X_{\mathbb{Q}}$ is a Hopf space.*
(4) *$X_{\mathbb{Q}}$ has the homotopy type of a generalised Eilenberg–Mac Lane space.*

Corollary 3.5. *Any k-invariant of a 1-connected G-space is rationally trivial.*

We remark that Corollary 3.5 does not imply that a k-invariant of a 1-connected G-space is of finite order. Now, $H_*(K(\bigoplus_{\lambda\in\Lambda}\mathbb{Q}, 2m+1); \mathbb{Q})$ is isomorphic to an exterior algebra and $H_*(K(\bigoplus_{\lambda\in\Lambda}\mathbb{Q}, 2m); \mathbb{Q})$ is isomorphic to a polynomial algebra as Hopf algebras. Thus we obtain a generalisation of Theorem 3.2 of Borel [3]:

Corollary 3.6. *Let X be a 1-connected rational G-space, i.e., a G-space in the rational homotopy category. Then X is a Hopf space and the Hopf algebra $H^*(X; \mathbb{Q})$ is isomorphic* (*as an algebra*) *to the tensor product of the dual algebra of a polynomial algebra on even degree generators and the dual algebra of an exterior algebra on odd degree generators.*

We remark that $\pi_q(X) \otimes \mathbb{Q}$ may be infinite dimensional for each $q \geqslant 1$, and hence $H_q(X; \mathbb{Q})$ and its dual $H^q(X; \mathbb{Q}) \cong \mathrm{Hom}(H_q(X; \mathbb{Q}); \mathbb{Q})$ may be distinct as $\mathbb{Q}$-modules for each $q \geqslant 1$. For example, the dual of an exterior algebra on $\{\alpha_\lambda\}$ is not an exterior algebra on $\{\bar{\alpha}_\lambda\}$, in general, where $\bar{\alpha}_\lambda$ is the dual to α_λ (cf. [11]).

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