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Kernels of seminorms in constructive analysis

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Abstract

The kernel of a seminorm on a normed space is examined constructively—that is, using intuitionistic logic. In particular, conditions are given that ensure that (i) the kernel is located and (ii) the kernel is nontrivial.

In the standard approach to a computational development of mathematics one uses classical logic, which allows "decisions", such as whether or not a given real number equals 0, that cannot be made by a physical computer; in order to avoid such computationally illicit uses of the logic, one has to work with carefully specified rules of computation such as those of recursive function theory. This approach has two disadvantages. First, at times it requires a lot of precise book-keeping which can distract one's attention from the main mathematical issues under consideration and which can often make the mathematics rather hard to read. Secondly, the use of specified rules of computation cuts down the possible interpretations of the results; for example, a result proved within the framework of recursive function theory cannot, in general, be interpreted outside that framework.

In this paper we follow the alternative approach to computational mathematics, that of Bishop's constructive mathematics, in which classical logic is replaced by intuitionistic logic. The latter logic automatically takes care of the problem of noncomputational "decisions": for example, the proposition

$$\forall x \in \mathbf{R} \quad (x = 0 \vee x \neq 0)$$

is not derivable in the axiomatic theory of the real line $\mathbf{R}$ with intuitionistic logic [3]. The change-of-logic approach has two advantages over the one described in the first paragraph. First, it enables us to work, with any mathematical objects we choose (not just some special type of the so-called "constructive" objects), in the familiar, readable

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style of the analyst, algebraist, or geometer. Secondly, and perhaps more importantly, by not requiring us to specify any rules of computation, it leads to a multiplicity of interpretations: a proof of a proposition P in constructive mathematics is automatically one of P in classical mathematics and in Brouwer's intuitionistic mathematics, and is easily fitted into the recursive model or the framework of Weihrauch's type two effectivity theory of computation. Indeed, we believe that constructive mathematics can be modelled in any reasonable framework for computable mathematics.

For more information about constructive mathematics, the reader is referred to [1, 2, 7, 10]. A concise introduction to the main issues is found in an earlier article [3] in this journal.

What we are going to look at in this paper is a couple of problems related to the kernel K' of a seminorm on a linear space X already provided with a norm. In doing this, we are not claiming that our results have any major applications. Our discussion does, however, illustrate a significant feature of constructive mathematics: the appearance of problems that are classically vacuous but constructively nontrivial. With our first problem, that of computing the distances from points of x to K', we are able to provide necessary and sufficient conditions for these computations to be possible. Our solution of the second problem, that of constructing points of K', depends on a result (Proposition 8) that is classically almost ridiculous and yet, constructively, requires a subtle and nowadays widely used application of the completeness of the ambient space; it also has some simple applications, notably in connection with supplements of linear subspaces of X.

Let $(X, \|\cdot\|)$ be a normed linear space, and $\|\cdot\|'$ a seminorm on X. We are interested in the *kernel* of $\|\cdot\|'$,

$$K' = \ker \|\cdot\|' = \{x \in H: \ \|x\|' = 0\}.$$

In particular, we seek

- conditions which ensure that K' is located, in the sense that

  $$\rho(x, K') = \inf\{\|x - y\|: \ \|y\|' = 0\}$$

  exists; and
- a criterion for testing to see whether K' is nontrivial.

For convenience, we write $\rho(x, K') < \varepsilon$ to signify that $\|x - v\| < \varepsilon$ for some $v \in K'$ *even if we do not know that K' is located.* Likewise, we write $\rho(x_n, K') \to 0$ to signify that there exists a sequence (v_n) in K' such that $\lim_{n\to\infty} \|x_n - v_n\| = 0$.

We say that $\|\cdot\|'$ is

- *continuous* if the identity mapping from $(X, \|\cdot\|)$ to $(X, \|\cdot\|')$ is continuous;
- *open* if the identity mapping from $(X, \|\cdot\|)$ to $(X, \|\cdot\|')$ is an open mapping.

It is easy to see that $\|\cdot\|'$ is continuous if and only if there exists $c > 0$ such that $\|\cdot\|' \leqslant c\|\cdot\|$, and to prove the following lemma.

Lemma 1. *The following are equivalent conditions on the seminorm $\|\cdot\|'$, with kernel K', on $(X, \|\cdot\|)$.*

(i) $\|\cdot\|'$ *is open.*
(ii) *There exists* $r>0$ *such that if* $\|x\|'<r$, *then* $\rho(x,K')<1$; *in which case if* $\varepsilon>0$ *and* $\|x\|'<r\varepsilon$, *then* $\rho(x,K')<\varepsilon$.

Classically, $\|\cdot\|'$ is an open seminorm if and only if the seminorm $x\mapsto\rho(x,K')$ is continuous with respect to $\|\cdot\|'$; if also $\|\cdot\|'$ is continuous relative to $\|\cdot\|$, then $\|\cdot\|'$ is equivalent to the seminorm $x\mapsto\rho(x,K')$. The only constructive problem with these remarks is that K' may not be located. However, as the next three results enable us to show, an open seminorm $\|\cdot\|'$ on a finite-dimensional space has located kernel and is equivalent to the seminorm $x\mapsto\rho(x,K')$.

Lemma 2. *Let* $(X,\|\cdot\|)$ *be a finite-dimensional normed space, and* $\|\cdot\|'$ *a seminorm on* X. *Then the mapping* $x\mapsto\|x\|'$ *is uniformly continuous on* X.

Proof. We omit the routine computations. □

The next lemma reflects the fact that if x and y are orthogonal unit vectors in a Hilbert space, then the balls $B(x,1/\sqrt{2})$ and $B(y,1/\sqrt{2})$ are the largest disjoint balls with centres x and y. Note that vectors $x_1,\dots,x_n$ in a complex normed linear space are said to be *linearly dependent* if $\sum_{i=1}^{n}|\lambda_i x_i|>0$ for all scalars $\lambda_1,\dots,\lambda_n$ with $\sum_{i=1}^{n}|\lambda_i|>0$. This is, constructively, a stronger property than the usual classical one (to which, of course, it is classically equivalent).

Lemma 3. *Let* $a_1,\dots,a_k$ *be pairwise orthogonal unit vectors in a Hilbert space* H, *and let* $b_1,\dots,b_k$ *be vectors in* H *such that* $\|a_i-b_i\|<1$ $(1\leqslant i\leqslant k)$. *Then the vectors* $b_1,\dots,b_k$ *are linearly independent.*

Proof. Let $\lambda_1,\dots,\lambda_k$ be complex numbers such that $\sum_{i=1}^{k}|\lambda_i|>0$; we must show that $\|\sum_{i=1}^{k}\lambda_i b_i\|>0$. To this end, we may assume that $\sum_{i=1}^{k}|\lambda_i|^2=1$. Since

$$\left\|\sum_{i=1}^{k}\lambda_i a_i-\sum_{i=1}^{k}\lambda_i b_i\right\|\leqslant\sum_{i=1}^{k}|\lambda_i|\,\|a_i-b_i\|<\sum_{i=1}^{k}|\lambda_i|,$$

we have

$$\left\|\sum_{i=1}^{k}\lambda_i b_i\right\|>\left\|\sum_{i=1}^{k}\lambda_i a_i\right\|-\sum_{i=1}^{k}|\lambda_i|.$$

The result follows since

$$\left\|\sum_{i=1}^{k}\lambda_i a_i\right\|^2-\left(\sum_{i=1}^{k}|\lambda_i|\right)^2=\sum_{i=1}^{k}|\lambda_i|^2-\sum_{i,j=1}^{k}|\lambda_i|\,|\lambda_j|$$

$$\geqslant 1-\left(\sum_{i=1}^{k}|\lambda_i|^2\right)^{1/2}\left(\sum_{j=1}^{k}|\lambda_j|^2\right)^{1/2}=0. \qquad \square$$

Proposition 4. *Let* $\|\cdot\|'$ *be an open seminorm with kernel* K' *on a Hilbert space* H, *and let* X *be a finite-dimensional subspace of* H. *Then for each* $\varepsilon > 0$ *either* $K' \cap X = \{0\}$ *or else there exist*

- *a positive integer* $k \leqslant \dim X$,
- *pairwise orthogonal unit vectors* $x_1, \ldots, x_k$ *in* X,
- *linearly independent vectors* $v_1, \ldots, v_k$ *in* K', *and*
- *a subspace* Y *of* X

such that $\|x_i - v_i\| < \varepsilon$ $(1 \leqslant i \leqslant k)$, $K' \cap Y = \{0\}$, *and* Y *is orthogonal to* $\{x_1, \ldots, x_k\}$.

Proof. We may assume that $\varepsilon < 1$. Let $0 \leqslant n \leqslant \dim X$, and $x_0 = v_0 = 0$. Suppose that, in the case $n \geqslant 0$, we have found pairwise orthogonal unit vectors $x_1, \ldots, x_n$ in X, and linearly independent vectors $v_1, \ldots, v_n$ in K', such that $\|x_i - v_i\| < \varepsilon$ $(1 \leqslant i \leqslant n)$. If $\dim X = n$, we complete the proof by taking $k = n$ and $Y = \{0\}$. If $\dim X > n$, let S be the finite-dimensional subspace of X orthogonal to $\{x_0, x_1, \ldots, x_n\}$. Then $\{x \in S\colon \ \|x\| = 1\}$ is compact; since also, by Lemma 2, the mapping $x \mapsto \|x\|'$ is uniformly continuous on X,

$$m = \inf\{\|x\|'\colon\ x \in S,\ \|x\| = 1\}$$

exists ([2, p. 94, (4.3)]). Choosing $r > 0$ as in part (ii) of Lemma 1, we see that either $m > 0$ or $m < r\varepsilon$. In the first case we must have $K' \cap S = \{0\}$; so if $n = 0$, we are finished; while if $n \geqslant 1$, we complete the proof by taking $k = n$ and $Y = S$. In the case $m < r\varepsilon$ there exist $x_{n+1} \in S$ and $v_{n+1} \in K'$ such that $\|x_{n+1}\| = 1$ and $\|x_{n+1} - v_{n+1}\| < \varepsilon$. By Lemma 3, the vectors $v_1, \ldots, v_{n+1}$ are linearly independent, and so our inductive construction of the vectors x_i and v_i is complete. This procedure must end at the construction of x_k and v_k for some $k \leqslant \dim X$. □

Theorem 5. *If* H *is a Hilbert space, and* $\|\cdot\|'$ *is an open seminorm on* H *whose kernel* K' *is contained in a finite-dimensional subspace of* H, *then* K' *is located.*

Proof. Assume that K' is contained in a finite-dimensional subspace X of H. Then the foregoing proposition shows that $K' = K' \cap X$ is finite-dimensional and hence located. □

Corollary 6. *The kernel of a seminorm on a finite-dimensional Banach space* X *is located if and only if the seminorm is open.*

Proof. The "if" part follows from the preceding theorem, since all norms on X are equivalent and we can choose a norm with respect to which X is a Hilbert space. The "only if" part is, as classically, virtually trivial. □

Let a be any real number, and define a seminorm $|\cdot|'$ on $\mathbf{R}$ by

$$|x|' = |ax|.$$

Suppose that this seminorm is open. There exists $r>0$ such that if $|x|'<r$, then $\rho(x, \ker|\cdot|')<1$. Either $|1|'>0$, in which case $ax \neq 0$ and therefore $a \neq 0$, or else $|1|'<r$. In the latter case there exists $x \in \ker|\cdot|'$ such that $|1-x|<1$, from which it readily follows that $a=0$. Thus the openness of every seminorm on **R**—and, equivalently (in view of Corollary 6), the locatedness of the kernel of every seminorm on **R**—implies that

$$\forall x \in \mathbf{R} \quad (x = 0 \vee x \neq 0).$$

This, in turn, is equivalent to the *limited principle of omniscience* (LPO),

For any binary sequence (a_n) either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

which, being false in the intuitionistic and recursive models of constructive mathematics, is essentially nonconstructive.

We do not know if the kernel of an open seminorm on an infinite-dimensional Hilbert space is located. However, we can prove a related result in the context of a Hilbert space of arbitrary dimension. For this we note the following.

- A *semidefinite inner product* on a vector space X (with an inequality relation $\neq$) is a mapping $(x, y) \mapsto \langle x, y\rangle$ that satisfies all the properties of an inner product except that we may have $\langle x,x\rangle = 0$ for some $x \neq 0$.
- A linear operator T on a Hilbert space H is *decent* if for each bounded sequence (x_n) such that $Tx_n \to 0$ there exists a sequence (y_n) in $\ker(T)$ such that $x_n + y_n \rightharpoonup 0$ (where, as usual, $\rightharpoonup$ denotes weak convergence).

  It is shown in [6] that if T has an adjoint and located range, then T is decent; and, in the case where H is separable, that if T is decent and selfadjoint and has located kernel, then it has located range.
- The Riesz Representation Theorem for a linear functional on a Hilbert space holds constructively if the functional f has a norm

  $$\|f\| = \sup\{|f(x)|;\ x \in X, \|x\| \leqslant 1\}$$

  (see [2, p. 419, (2.3); 8]).
- The unit ball B of a Hilbert space H is weakly totally bounded; that is, for each $y \in H$ the set $\{\langle x, y\rangle\colon x \in B\}$ is totally bounded. This is a special case of Proposition 1 of [9].

Proposition 7. *Let H be a separable Hilbert space. Let $\langle\cdot,\cdot\rangle'$ be a semidefinite inner product on H, $\|\cdot\|'$ the corresponding seminorm, and $K' = \ker\|\cdot\|'$. Suppose that for each $y \in H$ the mapping $x \mapsto \langle x, y\rangle'$ is uniformly continuous with respect to the weak topology on the unit ball B of H; and that if (x_n) is a bounded sequence in H such that $x_n \rightharpoonup' 0$ (where $\rightharpoonup'$ denotes weak convergence with respect to $\langle\cdot,\cdot\rangle'$), then there exists a sequence (y_n) in K' such that $x_n + y_n \rightharpoonup 0$. Then the following are equivalent conditions.*

(i) *K' is located.*

(ii) *For each orthonormal basis* $(e_n)_{n=1}^{\infty}$ *of* H, *the set*

$$\left\{\sum_{n=1}^{\infty}\langle x,e_n\rangle' e_n\colon\ x\in H\right\}$$

is located.

Proof. Since the unit ball B of H is weakly totally bounded, it follows from the uniform continuity hypothesis that for each $y\in H$ the subset

$$\{|\langle x,y\rangle'|\colon\ x\in B\}$$

of $\mathbf{R}$ is totally bounded and hence has a supremum ([2, p. 94, (4.3)]). Thus the linear functional $x\mapsto\langle x,y\rangle'$ on H has a norm, and we can apply the Riesz Representation Theorem in the usual way to show that there exists a linear mapping $T:H\to H$ such that $\langle x,y\rangle'=\langle x,Ty\rangle$ for all $x,y\in H$. Note that T is a positive selfadjoint operator: for we have

$$\langle x,Ty\rangle=\langle x,y\rangle'=(\langle y,x\rangle')^{*}=\langle y,Tx\rangle^{*}=\langle Tx,y\rangle$$

and $\langle Tx,x\rangle=\langle x,x\rangle'\geqslant 0$. Now let (x_n) be a bounded sequence in H such that $Tx_n\to 0$. For each $y\in H$ we have

$$|\langle x_n,y\rangle'|=|\langle Tx_n,y\rangle|\ \leqslant\ \|Tx_n\|\|y\|\to 0,$$

so, by the hypothesis about weak convergence, there exists a sequence (y_n) in K' such that $x_n+y_n\rightharpoonup 0$. Hence T is decent. To obtain the equivalence of (i) and (ii), we now note that

$$Tx=\sum_{n=1}^{\infty}\langle Tx,e_n\rangle e_n=\sum_{n=1}^{\infty}\langle x,e_n\rangle' e_n\quad (x\in H)$$

and refer to the second of the remarks preceding this proposition. □

A continuous open seminorm with kernel $\{0\}$ on a normed space $(X,\|\cdot\|)$ is equivalent to the given norm on X. So it is natural to seek conditions that ensure that a given seminorm on X has a nontrivial kernel—that is, one containing an element x with $\|x\|>0$. A classically bizarre result provides us with such conditions.

Proposition 8. *Let* $(X,\|\cdot\|)$ *be a Banach space*, *let* S *be a linear subset of* X, *and let* $\|\cdot\|'$ *be an open seminorm on* X, *with kernel* K', *such that* $S+K'$ *is closed. For each nonzero* $s\in S$ *there exists* $\delta>0$ *such that if* $0<\|s\|'<\delta$, *then* K' *contains a nonzero element of* S.

Proof. By Lemma 1, there exists $r>0$ such that for each $x\in X$ and each $\varepsilon>0$, if $\|x\|<r\varepsilon$, then $\rho(x,K')<\varepsilon$. Given $s\neq 0$ in S, define an increasing binary sequence

$(\lambda_n)_{n=1}^{\infty}$ such that

$$\lambda_n = 0 \;\Rightarrow\; \|s\|' < r/n^2,$$
$$\lambda_n = 1 \;\Rightarrow\; \|s\|' > r/(n+1)^2.$$

If $\lambda_1 = 1$, we can take $\delta = \|s\|'$; so we may assume that $\lambda_1 = 0$. Suppose we have constructed certain elements $x_0 = 0, x_1, \ldots, x_{n-1}$ of X. If $\lambda_n = 0$, choose $v_n \in K'$ such that $\|s - v_n\| < 1/n^2$ and set $x_n = x_{n-1} + (s - v_i)$; if $\lambda_n = 1$, set $x_n = x_{n-1}$. This completes the inductive construction of a sequence (x_n) in $S + K'$.

Since

$$\|x_k - x_j\| < \sum_{i=j+1}^{k} 1/n^2 \quad (k > j),$$

(x_n) is a Cauchy sequence, and therefore converges to a limit x_∞, in the complete subset $S + K'$ of X. Thus there exist $s_\infty \in S$ and $v \in K'$ such that $x_\infty = s_\infty + v$. Choosing a positive integer N such that $N\|s\| > \|s_\infty\|$, and setting $\delta = r/(N+1)^2$, suppose that $0 < \|s\|' < \delta$. Then we must have $\lambda_N = 0$. On the other hand, $\|s\|' > r/n^2$, and therefore $\lambda_n = 1$, for all sufficiently large n; so there exists $v \geqslant N$ such that $\lambda_{v+1} = 1 - \lambda_v$ and therefore $x_\infty = \sum_{i=1}^{v} (s - v_i)$. Hence

$$vs - s_\infty = v + \sum_{i=1}^{v} v_i,$$

which belongs to the linear set $S \cap K'$. Finally,

$$\|vs - s_\infty\| \geqslant v\|s\| - \|s_\infty\| \geqslant N\|s\| - \|s_\infty\| > 0,$$

so $vs - s_\infty \neq 0$. □

Corollary 9. *Let $\|\cdot\|'$ be a continuous open seminorm, with kernel K', on a Banach space X, and let S be a finite-dimensional subspace of X such that $K' \cap S = \{0\}$. Then $\|s\|' > 0$ for each $s \in S$ with $\|s\| > 0$.*

Proof. Since $\|\cdot\|'$ is continuous, K' is closed. Since S is finite-dimensional and $K' \cap S = \{0\}$, $S + K'$ is closed in X. Given $s \in S$ with $\|s\| > 0$, and applying Proposition 8, we obtain $\delta > 0$ such that if $0 < \|s\|' < \delta$, then $K' \cap S$ contains a nonzero vector. It follows that either $\|s\|' = 0$ or else $\|s\|' \geqslant \delta$. The former case is ruled out, since it implies that $K' \cap S$ contains the nonzero vector s. □

Corollary 10. *Let V be a closed located subspace of a normed space X, and S a linear subset of X such that $V \cap S = \{0\}$ and $S + V$ is closed. Then $\rho(s, V) > 0$ for each nonzero $s \in S$.*

Proof. Taking $\|x\|' = \rho(x, V)$ in Proposition 8, use an argument like that in the proof of Corollary 9. □

Let V and S be linear subsets of a linear space X. We say that S is a *supplement* of V (in X) if $V \cap S = \{0\}$ and $V + S = X$. In that case each element of X can be uniquely expressed in the form $v + s$ with $v \in V$ and $s \in S$.

Corollary 11. *Let V be a closed located subspace of a Banach space $(X, \|\cdot\|)$, and S a supplement of V in X. Then every linearly independent subset of S is linearly independent in X/V.*

Proof. Let $\{e_1, \ldots, e_n\}$ be linearly independent vectors in S, and $\lambda_1, \ldots, \lambda_n$ scalars such that $\sum_{i=1}^n |\lambda_i| > 0$; then $\|\sum_{i=1}^n \lambda_i e_i\| > 0$. By Corollary 10, $\rho(\sum_{i=1}^n \lambda_i e_i, V) > 0$; that is, $\sum_{i=1}^n \lambda_i e_i$ is a nonzero element of X/V. □

Corollary 9, even without the hypothesis that $V + S$ is complete, is a simple consequence of *Markov's Principle*: for any binary sequence (a_n),

$$\neg \forall n \ (a_n = 0) \ \Rightarrow \ \exists n \, (a_n = 1).$$

In fact, Markov's Principle is equivalent to the proposition,

> If V is a closed located subspace, and S a linear subset, of a normed space such that $V \cap S = \{0\}$, then $\rho(x, V) > 0$ for each nonzero $x \in S$.

To see this, let θ be a real number such that $\neg(\theta = 0)$, let $X = \mathbf{R}^2$, let $V = \mathbf{R} \times \{0\}$, and let $S = \mathbf{R}e$, where $\mathbf{e} = (\cos\theta, \sin\theta)$. Being 1-dimensional, V is closed in X. Also, $V \cap S = \{0\}$ and $\|\mathbf{e}\| = 1$. But if $\rho(\mathbf{e}, V) > 0$, then $|\theta| > 0$. So the aforementioned proposition entails

$$\forall \theta \in \mathbf{R} \quad (\neg(\theta = 0) \Rightarrow \theta \neq 0),$$

which is equivalent to Markov's Principle.

Since Markov's Principle embodies an unbounded search, can neither be proved nor disproved within Heyting arithmetic (that is, Peano arithmetic with intuitionistic logic; see [7, pp. 137–138]), and is therefore not usually accepted as a principle of constructive mathematics, Corollary 10 is not as trivial, constructively, as it may seem.

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