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Successive iteration and positive solutions for a second-order multi-point boundary value problem on a half-line ☆

Xingqiu Zhang

School of Mathematics, Liaocheng University, Liaocheng, Shandong 252059, PR China

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ABSTRACT

This paper deals with the existence of positive solutions for some second-order multi-point boundary value problem on the half-line. Our approach is based on the fixed point theorem and the monotone iterative technique. Without the assumption of the existence of lower and upper solutions, we obtain not only the existence of positive solutions for the problems, but also establish iterative schemes for approximating the solutions.

1. Introduction

In this paper, we are concerned with the positive solutions to the following second-order multi-point boundary value problem

$$\begin{cases} x''(t) + q(t)f(t, x(t)) = 0, \quad t \in J_+, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(\infty) = x_\infty \geq 0, \end{cases} \tag{1}$$

where $J = [0, +\infty), J_+ = (0, +\infty), \alpha_i \in J$ and $\xi_i \in J_+$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty, 0 < \sum_{i=1}^{m-2} \alpha_i < 1$. Throughout this paper, we always assume that the following conditions are satisfied.

(H_1) $f \in C(J \times J, J), f(t, 0) \not\equiv 0$ on any subinterval of J and, when u is bounded, $f(t, (1+t)u)$ is bounded on J;
(H_2) $q(t)$ is a nonnegative measurable function defined in J_+ and $q(t)$ does not identically vanish on any subinterval of J_+ and

$$0 < \int_0^{+\infty} q(t)\mathrm{d}t < +\infty, \qquad 0 < \int_0^{+\infty} tq(t)\mathrm{d}t < +\infty.$$

Boundary value problems on a half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium; see [1–5], for example. In the past few years, the existence and multiplicity of positive solutions to nonlinear differential equations on the half-line have been studied by

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E-mail address: Zhxq197508@163.com.

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using several different techniques; we refer the reader to [1–7] and references therein. For example, using a fixed point theorem of cone expansion and compression of norm type, Liu [5] investigated the existence of solutions of the following second-order two-point boundary value problems on the half-line:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \quad t \in J_+, \\ x(0) = 0, \quad x'(\infty) = y_\infty \geq 0, \end{cases}$$

where $f \in C[J_+ \times J_+, J]$. Most of these papers only considered the existence of positive solutions of various boundary value problems. A natural question which arises is "How can we find the solutions when they are known to exist?" More recently, Ma, Du and Ge [8], and Sun and Ge [9,10] proved the existence of positive solutions for some second-order p-Laplacian boundary value problems which are defined on finite intervals by virtue of the iterative technique.

To the best of our knowledge, up until now, no results in the literature are available for the computation of positive solutions for boundary value problems on the half-line. On the other hand, the multi-point boundary value problems arising from applied mathematical and physical problems have been studied extensively in the literature and there are many excellent results about the existence of positive solutions for multi-point boundary value problems (see, for instance, [11–16] and references therein). Motivated by the above-mentioned papers, the purpose of this paper is to fill this gap. As we know, it is very important to check the compactness of the corresponding operator when we use the monotone iterative technique, and the Ascoli–Arzela theorem plays a very important role. However, the Ascoli–Arzela theorem is not suitable for operators on the half-line. So, we need to list some new conditions to meet the requirement of compactness.

2. Preliminaries and several lemmas

In this section, we give some preliminaries and definitions.

Definition 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that:

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$;
(2) $u, -u \in P$ implies $u = 0$.

In this paper, we will use the following space E which is denoted by

$$E = \left\{ x \in C[0, +\infty) : \sup_{t \in J} \frac{|x(t)|}{t+1} < \infty \right\},$$

to study BVP (1). Then E is a Banach space equipped with the norm $\|x\| = \sup_{t \in J} \frac{|x(t)|}{t+1}$. Let $E_+ = \{x \in E | x(t) \geq 0\}$. Define the cone $P \subset E$ by

$$P = \{x \in E_+ | x \text{ is concave and nondecreasing on } [0, +\infty) \text{ and } \lim_{t \to +\infty} x'(t) = x(\infty)\}.$$

Lemma 1. *Let conditions* (H_1) *and* (H_2) *be satisfied; then* $x \in E_+ \cap C^2[J_+, J]$ *is a solution of BVP (1) if and only if* $x \in C[J, E]$ *is a solution of the following integral equation*

$$x(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\left(\sum_{i=1}^{m-2} \alpha_i \xi_i \right) x_\infty + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \right] + \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s + t x_\infty. \tag{2}$$

Proof. Suppose that $x \in E_+ \cap C^2[J_+, J]$ is a solution of BVP (1). For $t \in J$, integrating (1) from t to $+\infty$, we have

$$x'(t) = x_\infty + \int_t^{+\infty} f(s, x(s)) \mathrm{d}s. \tag{3}$$

Integrating (3) from 0 to t, we get

$$x(t) = x(0) + t x_\infty + \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s. \tag{4}$$

Thus, we obtain

$$x(\xi_i) = x(0) + \xi_i x_\infty + \int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s,$$

which together with the boundary value condition implies that

$$x(0) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\left(\sum_{i=1}^{m-2} \alpha_i \xi_i \right) x_\infty + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \right]. \tag{5}$$

Substituting (5) into (4), we have

$$x(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\left(\sum_{i=1}^{m-2} \alpha_i \xi_i \right) x_\infty + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \right] + \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s + t x_\infty.$$

Next, we show that the integrals $\int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s$ and $\int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s$ are convergent.

Since $x \in E_+$, then there exists r_0 such that $\|x\| < r_0$. Set $B_{r_0} = \sup\{f(t, (1+t)u) | (t, u) \in J \times [0, r_0]\}$, and we have by interchanging the order of integration

$$\int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \le \int_0^{+\infty} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \le \int_0^{+\infty} s q(s) \mathrm{d}s \cdot B_{r_0}. \tag{6}$$

By (H_2), we know that $\int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s$ is convergent. Since $\int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \le \xi_i \int_0^{+\infty} q(\tau) \mathrm{d}\tau \cdot B_{r_0}$, by (H_2), we know that $\int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s$ is also convergent. Thus, we have proved that the right term in (2) is well defined.

Conversely, if x is a solution of the integral equation, then direct differentiation gives the proof. □

Now, we define an operator $A : P \to C[0, +\infty)$ by

$$(Ax)(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\left(\sum_{i=1}^{m-2} \alpha_i \xi_i \right) x_\infty + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \right] + \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s + t x_\infty. \tag{7}$$

To obtain the complete continuity of A, the following lemma is still needed.

Lemma 2 (*See [5]*). *Let W be a bounded subset of P. Then W is relatively compact in E if $\{W(t)/(1+t)\}$ are equicontinuous on any finite subinterval of $[0, +\infty)$ and for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$\left| \frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2} \right| < \varepsilon,$$

uniformly with respect to $x \in W$ as $t_1, t_2 \ge N$, where $W(t) = \{x(t) | x \in W\}$, $t \in [0, +\infty)$.

Lemma 3. *Let (H_1) and (H_2) be satisfied. Then $A : P \to P$ is completely continuous.*

Proof. It is clear that $(Ax)(t) \ge 0$ for any $x \in P$, $t \in J$. By (7), we have

$$(Ax)'(t) = \int_t^{+\infty} q(s) f(s, x(s)) \mathrm{d}s + x_\infty \ge 0, \tag{8}$$

and

$$(Ax)''(t) = -q(t) f(t, x(t)) \le 0. \tag{9}$$

(8) and (9) imply that $(AP) \subset P$. Now, we prove that A is continuous and compact respectively. Let $x_n \to x$ as $n \to \infty$ in P; then there exists r_0 such that $\sup_{n \in N \setminus \{0\}} \|x_n\| < r_0$. Let $B_{r_0} = \sup\{f(t, (1+t)u) | (t, u) \in J \times [0, r_0]\}$. By (H_2), we have

$$\int_0^t \int_s^{+\infty} q(\tau) |f(\tau, x_n(\tau)) - f(\tau, x(\tau))| \mathrm{d}\tau \mathrm{d}s \le \int_0^{+\infty} \int_s^{+\infty} q(\tau) |f(\tau, x_n(\tau)) - f(\tau, x(\tau))| \mathrm{d}\tau \mathrm{d}s \le 2 B_{r_0} \cdot \int_0^{+\infty} s q(s) \mathrm{d}s < +\infty. \tag{10}$$

By (7), (10) and the Lebesgue dominated convergence theorem, we get

$$\|Ax_n - Ax\| = \sup_{t\in J}\left\{\frac{1}{1+t}\left|\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}q(\tau)(f(\tau,x_n(\tau))-f(\tau,x(\tau)))\mathrm{d}\tau\mathrm{d}s\right]\right.\right.$$

$$\left.\left.+\int_0^t\int_s^{+\infty}q(\tau)(f(\tau,x_n(\tau))-f(\tau,x(\tau)))\mathrm{d}\tau\mathrm{d}s\right|\right\}$$

$$\le \sup_{t\in J}\left\{\frac{1}{1+t}\cdot\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}q(\tau)|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\mathrm{d}\tau\mathrm{d}s\right]\right.$$

$$\left.+\int_0^t\int_s^{+\infty}q(\tau)|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\mathrm{d}\tau\mathrm{d}s\right\}\to 0,\quad n\to\infty. \tag{11}$$

Therefore, A is continuous.

Let Ω be any bounded subset of P. Then, there exists $r > 0$ such that $\|x\| \le r$ for any $x \in \Omega$. Therefore, we have

$$\|Ax\| = \sup_{t\in J}\frac{1}{1+t}\left|\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\right]\right.$$

$$\left.+\int_0^t\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s+tx_\infty\right|$$

$$\le \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)\int_0^{+\infty}q(\tau)\mathrm{d}\tau\cdot B_r+\int_0^{+\infty}q(\tau)\mathrm{d}\tau\cdot B_r+x_\infty$$

$$=\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right)x_\infty+\left[\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)+1\right]\int_0^{+\infty}q(\tau)\mathrm{d}\tau B_r.$$

So, $T\Omega$ is bounded. Notice that the integral $\int_0^{+\infty}\int_s^{+\infty}q(\tau)\mathrm{d}\tau\mathrm{d}s$ is convergent. So, for any $T\in J_+$ and $t_1, t_2\in[0,T]$, by the absolute continuity of the integral, we have

$$\left|\frac{(Ax)(t_1)}{1+t_1}-\frac{(Ax)(t_2)}{1+t_2}\right| \le \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\right]\cdot\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|$$

$$+\left|\frac{1}{1+t_1}\int_0^{t_1}\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s-\frac{1}{1+t_2}\int_0^{t_2}\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\right|$$

$$+\left|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\right|x_\infty$$

$$\le \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}q(\tau)\mathrm{d}\tau\mathrm{d}sB_r\right]\cdot\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|$$

$$+\frac{1}{1+t_1}\left|\int_{t_1}^{t_2}\int_s^{+\infty}q(\tau)\mathrm{d}\tau\mathrm{d}sB_r\right|+\int_0^{t_2}\int_s^{+\infty}q(\tau)\mathrm{d}\tau\mathrm{d}sB_r\cdot\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|$$
$$+\left|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\right|x_\infty\to 0,\quad \text{uniformly as } t_1\to t_2.$$

Thus, we have proved that $T\Omega$ is equicontinuous on any finite subinterval of $[0,+\infty)$.

Next, we prove that for any $\varepsilon>0$, there exists a sufficiently large $N>0$ such that

$$\left|\frac{(Ax)(t_1)}{1+t_1}-\frac{(Ax)(t_2)}{1+t_2}\right|<\varepsilon\quad \text{for all } t_1,t_2\geq N,\ \forall\, x\in\Omega. \tag{12}$$

For any $x\in\Omega$, we have

$$\lim_{t\to\infty}\left|\frac{(Ax)(t)}{1+t}\right|=\lim_{t\to\infty}\frac{1}{1+t}\int_0^t\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s+x_\infty. \tag{13}$$

Similarly to (6), we get

$$\int_0^t\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\leq B_r\int_0^{+\infty}\tau q(\tau)\mathrm{d}\tau<+\infty,$$

which shows that

$$\lim_{t\to\infty}\frac{1}{1+t}\int_0^t\int_s^{+\infty}q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s=0. \tag{14}$$

It follows from (13) and (14) that $\left|\frac{(Ax)(t)}{1+t}\right|$ tend to x_∞ uniformly as $t\to\infty$. So, for any $\varepsilon>0$, $x\in Q$ there exists $N>0$ such that

$$\left|\frac{(Ax)(t)}{1+t}-x_\infty\right|<\frac{\varepsilon}{2},\quad \forall\, t\geq N.$$

Consequently, for any $t_1,t_2\geq N$, we have

$$\left|\frac{(Ax)(t_1)}{1+t_1}-x_\infty\right|<\frac{\varepsilon}{2},\qquad \left|\frac{(Ax)(t_2)}{1+t_2}-x_\infty\right|<\frac{\varepsilon}{2}. \tag{15}$$

Therefore, (12) can be easily seen from (15). In conclusion, by Lemma 2 we know that $A:P\to P$ is completely continuous. □

3. Main results

For notational convenience, we denote

$$m=\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right)x_\infty,\qquad n=\frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty}q(\tau)\mathrm{d}\tau+\max\left\{\int_0^{+\infty}q(\tau)\mathrm{d}\tau,\int_0^{+\infty}\tau q(\tau)\mathrm{d}\tau\right\}.$$

We will prove the following existence results.

Theorem 1. *Assume that* (H_1) *and* (H_2) *hold, and there exists* $a>2m$ *such that:*

(S_1) $f(t,x_1)\leq f(t,x_2)$ *for any* $0\leq t<+\infty,\ 0\leq x_1\leq x_2\leq a$;
(S_2) $f(t,(1+t)u)\leq\frac{a}{2n},\ \ (t,u)\in[0,+\infty)\times[0,a]$.

Then the boundary value problem (1) *has two positive nondecreasing on* $[0,+\infty)$ *and concave solutions* w^* *and* v^*, *such that* $0<\|w^*\|\leq a$, *and* $\lim_{n\to\infty}w_n=\lim_{n\to\infty}A^nw_0=w^*$, *where*

$$w_0(t)=\frac{a}{2}+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}x_\infty+tx_\infty,\quad t\in J,$$

and $0<\|v^*\|\leq a$, $\lim_{n\to\infty}v_n=\lim_{n\to\infty}A^nv_0=v^*$, *where* $v_0(t)=0,\ t\in J$.

Proof. By Lemma 3, we know that $A : P \to P$ is completely continuous. For any $x_1, x_2 \in P$ with $x_1 \le x_2$, from the definition of A and (S_1), we can easily get that $Ax_1 \le Ax_2$. We denote

$$\overline{P}_a = \{x \in P \mid \|x\| \le a\}.$$

Then, in what follows, we first prove that $A : \overline{P}_a \to \overline{P}_a$. If $x \in \overline{P}_a$, then $\|x\| \le a$. By (2), (S_1) and (S_2), we get

$$\begin{aligned}
\|Ax\| &= \sup_{t\in J} \frac{1}{1+t}\left| \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty + \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty} q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\right]\right.\\
&\quad \left. + \int_0^t\int_s^{+\infty} q(\tau)f(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty\right|\\
&\le \left[\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)+1\right]x_\infty + \left[\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)+1\right]\int_0^{+\infty} q(\tau)\mathrm{d}\tau\cdot\frac{a}{2n}\\
&= m + n\cdot\frac{a}{2n} \le a.
\end{aligned}$$

Hence, we have proved that $A : \overline{P}_a \to \overline{P}_a$.

Let $w_0(t) = \frac{a}{2} + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}x_\infty + tx_\infty, 0 \le t < +\infty$; then $w_0(t) \in \overline{P}_a$. Let $w_1 = Aw_0$, $w_2 = A^2w_0$; then by Lemma 3, we have that $w_1 \in \overline{P}_a$ and $w_2 \in \overline{P}_a$. We denote $w_{n+1} = Aw_n = A^n w_0, n = 0, 1, 2, \ldots$. Since $A : \overline{P}_a \to \overline{P}_a$, we have $w_n \in A(\overline{P}_a) \subset \overline{P}_a, n = 1, 2, 3, \ldots$. It follows from the complete continuity of A that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set.

By (2) and (S_2), we get

$$\begin{aligned}
w_1(t) &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty + \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty} q(\tau)f(\tau,w_0(\tau))\mathrm{d}\tau\mathrm{d}s\right]\\
&\quad + \int_0^t\int_s^{+\infty} q(\tau)f(\tau,w_0(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty\\
&\le \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\xi_i\right)x_\infty + \left[\frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty} q(\tau)\mathrm{d}\tau + \int_0^{+\infty}\tau q(\tau)\mathrm{d}\tau\right]\frac{a}{2n} + tx_\infty\\
&= \frac{a}{2} + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}x_\infty + tx_\infty = w_0(t), \quad 0 \le t < +\infty.
\end{aligned} \tag{16}$$

So, by (16) and (S_1) we have

$$w_2(t) = Aw_1(t) \le Aw_0(t) = w_1(t), \quad 0 \le t < +\infty.$$

By induction, we get

$$w_{n+1} \le w_n, \quad 0 \le t < +\infty,\ n = 0, 1, 2, \ldots.$$

Thus, there exists $w^* \in \overline{P}_a$ such that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of A and $w_{n+1} = Aw_n$, we get that $Aw^* = w^*$.

Let $v_0(t) = 0, 0 \le t < +\infty$; then $v_0(t) \in \overline{P}_a$. Let $v_1 = Av_0$, $v_2 = A^2v_0$; then by Lemma 3, we have that $v_1 \in \overline{P}_a$ and $v_2 \in \overline{P}_a$. We denote $v_{n+1} = Av_n = A^n v_0, n = 0, 1, 2, \ldots$. Since $A : \overline{P}_a \to \overline{P}_a$, we have $v_n \in A(\overline{P}_a) \subset \overline{P}_a, n = 1, 2, 3, \ldots$. It follows from the complete continuity of A that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $v_1 = Av_0 \in \overline{P}_a$, we have

$$v_1(t) = (Av_0)(t) = (A0)(t) \geq 0, \quad 0 \leq t < +\infty.$$

So, we have

$$v_2(t) = Av_1(t) \geq (A0)(t) = v_1(t), \quad 0 \leq t < +\infty.$$

By induction, we get

$$v_{n+1} \geq v_n, \quad 0 \leq t < +\infty,\ n = 0, 1, 2, \ldots.$$

Thus, there exists $v^* \in \overline{P}_a$ such that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of A and $v_{n+1} = Av_n$, we get that $Av^* = v^*$.

If $f(t, 0) \not\equiv 0,\ 0 \leq t < \infty$, then the zero function is not the solution of BVP (1). Thus, v^* is a positive solution of BVP (1).

It is well known that each fixed point of A in P is a solution of BVP (1). Hence, we assert that w^* and v^* are two positive, nondecreasing on $[0, +\infty)$ and concave solutions of the BVP (1). □

Remark 1. The iterative schemes in Theorem 1 are $w_0(t) = \frac{a}{2} + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}x_\infty + tx_\infty$, $w_{n+1} = Aw_n = A^n w_0$, $n = 0, 1, 2, \ldots$ and $v_0(t) = 0$, $v_{n+1} = Av_n = A^n v_0$, $n = 0, 1, 2, \ldots$. They start off with a known simple linear function and the zero function respectively. This is convenient for application.

4. An example

Example 1. Consider the boundary value problem of the differential equation

$$\begin{cases} x''(t) + \dfrac{1}{\sqrt{t}(1+t)^2} f(t, x(t)) = 0, & t \in J_+, \\ x(0) = \dfrac{1}{6}x(1) + \dfrac{1}{6}x(3), & x'(\infty) = 0, \end{cases} \tag{17}$$

where

$$f(t, x) = \begin{cases} 10^{-3}|\cos(t+2)| + \left(\dfrac{x}{1+t}\right)^4, & x \leq 2, \\ 10^{-3}|\cos(t+2)| + \left(\dfrac{2}{1+t}\right)^4, & x \geq 2. \end{cases}$$

Set $q(t) = \frac{1}{\sqrt{t}(1+t)^2}$. It is clear that (H_1) and (H_2) hold. Let $\alpha_1 = \alpha_2 = \frac{1}{6}$, $\xi_1 = 1$, $\xi_2 = 3$, $x_\infty = 0$. By direct computation, we can obtain

$$\int_0^{+\infty} q(t)\mathrm{d}t = \int_0^{+\infty} \frac{1}{\sqrt{t}(1+t)^2}\mathrm{d}t < \int_0^1 \frac{1}{\sqrt{t}}\mathrm{d}t + \int_1^{+\infty} \frac{1}{\sqrt{t}\cdot t^2}\mathrm{d}t = \frac{8}{3}, \tag{18}$$

and

$$\int_0^{+\infty} tq(t)\mathrm{d}t = \int_0^{+\infty} \frac{t}{\sqrt{t}(1+t)^2}\mathrm{d}t < \int_0^1 \sqrt{t}\mathrm{d}t + \int_1^{+\infty} \frac{\sqrt{t}}{t^2}\mathrm{d}t = \frac{8}{3}. \tag{19}$$

By (18) and (19) we have $m = 0,\ n < \frac{16}{3}$. Take $a = 300$. In the following we check (S_2).

Since the nonlinear term f satisfies

$$f(t, (1+t)x) \leq \frac{1}{10^3} + 2^4 < \frac{300}{2\cdot\frac{16}{3}} < \frac{300}{2n}, \quad t \in [0, +\infty), x \in [0, 300],$$

the conditions in Theorem 1 are all satisfied. Therefore, the conclusion of Theorem 1 holds.

References

[1] S. Chen, Y. Zhang, Singular boundary value problems on a half-line, J. Math. Anal. Appl. 195 (1995) 449–468.
[2] X. Liu, Solutions of impulsive boundary value problems on the half-line, J. Math. Anal. Appl. 222 (1998) 411–430.
[3] D. O'Regan, Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1994.
[4] M. Zima, On positive solution of boundary value problems on the half-line, J. Math. Anal. Appl. 259 (2001) 127–136.
[5] Y. Liu, Existence and unboundedness of positive solutions for singular boundary value problems on half-line, Appl. Math. Comput. 144 (2003) 543–556.
[6] H. Lian, H. Pang, W. Ge, Triple positive solutions for boundary value problems on infinite intervals, Nonlinear Anal. 67 (2007) 2199–2207.
[7] B. Yan, Boundary value problems on the half-line with impulses and infinite delay, J. Math. Anal. Appl. 259 (2001) 94–114.

[8] D. Ma, Z. Du, W. Ge, Existence and iteration of monotone positive solutions for multipoint boundary value problems with p-Laplacian operator, Comput. Math. Appl. 50 (2005) 729–739.
[9] B. Sun, W. Ge, Existence and iteration of positive solutions for some p-Laplacian boundary value problems, Nonlinear Anal. 67 (2007) 1820–1830.
[10] B. Sun, W. Ge, Successive iteration and positive pseudo-symmetric solutions for a three-point second-order p-Laplacian boundary value problems, Appl. Math. Comput. 188 (2007) 1772–1779.
[11] V.A. Il'in, E.I. Moiseer, Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects, Differ. Equ. 23 (1987) 803–810.
[12] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, J. Math. Anal. Appl. 168 (1992) 540–551.
[13] Z.X. Zhang, J.Y. Wang, The upper and lower solution method for a class of singular nonlinear second order three-point boundary value problems, J. Comput. Appl. Math. 147 (2002) 41–52.
[14] X. Xu, Positive solutions for singular m-point boundary value problems with positive parameter, J. Math. Anal. Appl. 291 (2004) 352–367.
[15] J.X. Sun, X. Xu, D. O'Regan, Nodal solutions for m-point boundary value problems using bifurcation methods, Nonlinear Anal. 68 (2008) 3034–3046.
[16] X.G. Zhang, L.S. Liu, Positive solutions of fourth-order multi-point boundary value problems with bending term, Appl. Math. Comput. 194 (2007) 321–332.