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UNITARIZING PROBABILITY MEASURES FOR REPRESENTATIONS OF VIRASORO ALGEBRA

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ABSTRACT. – Determination of a formula of integration by part insuring the unitarity. © 2001 Éditions scientifiques et médicales Elsevier SAS

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Introduction

André Weil has shown [9] that the realm of invariant integration on a group G is the class of locally compact groups; it seems therefore that the realization on a space of square integrable functions of representations of G has to be limited to the case where G is locally compact. Nevertheless the classical theory of Segal and Bargmann shows that the infinite-dimensional Heisenberg group has an L^2 representation on a space holomorphic functions defined on an Hilbert space and square integrable for a Gaussian measure. An analogous Gaussian realization for representations of Loop groups has been established in [3].

The purpose of this work is to get out of the Gaussian circle of ideas; we show that holomorphic square integrable realization is equivalent to the construction of a *unitarizing probability measure satisfying an a priori given formula of integration by part*, fully determined by the automorphy factor of the representation. The purpose of this paper is to work out quite explicitly this correspondance in the framework of some highest weight representation of Virasoro algebra; this paper is a preliminary work in the sense that the construction of the measure satisfying the prescribed formula of integration by part will not be treated here.

The Virasoro algebra depends upon the choice of the cocycle defining the central extension. The most general cocyle depends upon two parameters. We shall limit ourselves to a one-parameter family which is directly linked with the *universal Teichmuller space*, that is the quotient of the group of diffeomorphisms of the circle by the group of Möbius transformations of the unit disk.

A given representation has several realizations which appear isomorphic through intertwinning operators. It is not clear that intertwinning between two realizations implies transference for the corresponding unitarizing measures.

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For this reason we shall discuss below unitarizing measures for two distinct realizations of the discrete series, the first based on Neretin polynomials in the context of the Kirillov homogeneous space $\mathcal{M}$ of univalent functions and the second based on an infinite-dimensional version of Berezin quantization scheme. We hope that the algebraic meaning of unitarizing measures will appear more clearly by confrontation of its developpements in these two different realizations.

The Appendix is for us a key step in proving the effectiveness of Neretin formulas; we find here explicitly the generating functions associated to representation commutation rules; on these computations rely all the new results of our paper.

Part I: Resolution of a $\bar{\partial}$-problem on the space of univalent functions

1. Kirillov action of Virasoro on the manifold of univalent functions

1.1. The Lie algebra diff(S^1)

The group of C^∞, orientation preserving diffeomorphisms of the circle S^1 will be denoted by Diff(S^1), its Lie algebra by diff(S^1). We shall identify diff(S^1) with the real valued C^∞ functions ϕ on S^1, the infinitesimal action being $\theta \mapsto \theta + \varepsilon\phi(\theta)$; more geometrically we associate to ϕ the vector field $\phi(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}$. Under this identification the Lie bracket in diff(S^1) is the the bracket of the corresponding vector fields given by:

$$[\phi_1, \phi_2] = \phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1. \tag{1.1.1}$$

The infinitesimal rotation $\theta \mapsto \theta + \varepsilon$ corresponds to the constant function equal to 1; we shall denote by e_0 this function.

We denote:

$$U^{\phi}_{t\leftarrow 0}(\theta_0) := u(t) \text{ the flow defined by } \dot{u}(t) = \phi\big(u(t)\big), \quad u(0) = \theta_0.$$

The Jacobian (1×1)-matrix associated to this flow is obtained by solving the linearized differential equation, solution which can expressed as:

$$J^{\phi}_{t\leftarrow 0} = \exp\left(\int_0^t \dot{\phi}\big(u(s)\,\mathrm{d}s\right).$$

We deduce that

$$\frac{\partial^2}{\partial t_1 \partial t_2}_{t_1=t_2=0}\big(U^{\phi_1}_{t_1\leftarrow 0} \circ U^{\phi_2}_{t_2\leftarrow 0} - U^{\phi_2}_{t_2\leftarrow 0} \circ U^{\phi_1}_{t_1\leftarrow 0}\big) = -[\phi_1, \phi_2]. \tag{1.1.2}$$

Therefore diff(S^1) with the bracket defined in (1.1.1) has to be considered as the space of left invariant vector fields on Diff(S^1).

The Lie bracket has the following expression in the trigonometric basis:

$$\begin{aligned}
2[\cos j\theta, \cos k\theta] &= (j-k)\sin(j+k)\theta + (j+k)\sin(j-k)\theta,\\
2[\sin j\theta, \sin k\theta] &= (k-j)\sin(j+k)\theta + (j+k)\sin(j-k)\theta,\\
2[\sin j\theta, \cos k\theta] &= (k-j)\cos(j+k)\theta - (j+k)\cos(j-k)\theta.
\end{aligned}$$

1.2. The fundamental cocyle

We define on $\mathrm{diff}(S^1)$ the bilinear antisymmetric form:

$$\omega(f,g) := -\int_{S^1} \big(f' + f^{(3)}\big) g \frac{\mathrm{d}\theta}{4\pi}. \tag{1.2.1}$$

LEMMA. –

$$\omega\big([f_1, f_2], f_3\big) + \omega\big([f_2, f_3], f_1\big) + \omega\big([f_3, f_1], f_2\big) = 0. \tag{1.2.2}$$

Proof. – We remark that $[f_1, f_2]' = f_1 \ddot{f_2} - f_2 \ddot{f_1}$. As the wanted identity is linear relatively to the bracket, it is sufficient to check it for the two following cases:

in the first case the first term of the identity is $f_1 \ddot{f_2} f_3 - f_2 \ddot{f_1} f_3$ and this term cancelled with the others terms obtained by circular permutation;

in the second case we make an integration by part to replace the third derivative in the expression of ω by a second derivative; then the first term of the wanted identity can be written as $f_1 \ddot{f_2} \ddot{f_3} - f_2 \ddot{f_1} \ddot{f_3}$ and we get again cancellation by circular permutation. □

Fixing a positive constant c, called the *central charge*, the *Virasoro algebra* $\mathcal{V}_c$ is defined as vector space $\mathcal{V}_c := R \oplus \mathrm{diff}(S^1)$; we denote by κ the central element and define the bracket by:

$$[\xi\kappa + f, \eta\kappa + g]_{\mathcal{V}_c} := \frac{c}{12}\omega(f,g)\kappa + [f,g] \tag{1.2.3}$$

and according (1.2.2) the Jacobi identity is satisfied and we get a structure of Lie algebra.

LEMMA. – *The fundamental cocycle is invariant under the adjoint action by* e_0:

$$\big(\mathrm{Ad}(\exp t e_0)\big)^* \omega = \omega. \tag{1.2.4}$$

Proof. – We have

$$[e_0, f] = f' \quad \text{therefore} \quad \big(\mathrm{Ad}(\exp t e_0) f\big)(\theta) = f(\theta + t)$$

and it is clear that ω is invariant. □

We denote by S^1 the vector space of constant vector field which constitutes the Lie algebra of the group S^1 of rotations; we denote by $\mathrm{diff}_0(S^1)$ the quotient of $\mathrm{diff}(S^1)/S^1$.

We have $\langle e_0 \wedge \zeta, \omega\rangle = 0\ \forall\zeta$, therefore it is possible to quotient ω and we get a well defined 2-differential form on $\mathrm{diff}_0(S^1)$ which will be by abuse of notations still be denoted by ω.

Sometimes we shall identify $\mathrm{diff}_0(S^1)$ with the functions having mean value zero; using Fourier series this identification leads to write $\phi \in \mathrm{diff}_0(S^1)$ as:

$$\phi(\theta) = \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta,$$

where a_k, b_k are rapidly decreasing sequences of real numbers. We define:

$$J(\phi) = \sum_{k=1}^{\infty} -a_k \sin k\theta + b_k \cos k\theta. \tag{1.2.5}$$

Then $J^2 = -\text{Identity}$ and we get a complex structure on $\mathrm{diff}_0(S^1)$. On $\mathrm{diff}_0(S^1)\otimes C$, the operator J diagonalizes; we call vector of type $(1,0)$ (resp. of type $(0,1)$) the eigenvectors associated to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). A vector of type $(1,0)$ is of the form:

$$\phi = f - \sqrt{-1}J(f) = \sum_{k>0}(a_k - \mathrm{i}b_k)\cos k\theta + (b_k + \mathrm{i}a_k)\sin k\theta = \sum_{k>0}(a_k - \mathrm{i}b_k)\exp(\mathrm{i}k\theta),$$

which means that ϕ has a prolongation inside the unit disk as an holomorphic function.

We take for basis of $\mathrm{diff}(S^1)\otimes C$ the complex exponential $e_k(\theta) := \exp(\mathrm{i}k\theta),\ k\in Z$; in this basis the Lie bracket is:

$$[e_m, e_n] = \sqrt{-1}(n-m)e_{m+n} + \sqrt{-1}\delta^n_{-m}\left(\frac{c}{12}\big(m^3 - m\big)\right)\kappa. \tag{1.2.6}$$

The expression of ω can be written on for $f_1, f_2 \in \mathrm{diff}_0(S^1)\otimes C$ as:

$$\begin{aligned}\omega(f_1, f_2) &= \frac{\sqrt{-1}}{2}\sum_{k\in Z}\big(k^3 - k\big)c_k(f_1)c_{-k}(f_2)\\ &= \frac{\sqrt{-1}}{2}\sum_{k>0}\big(k^3 - k\big)\langle f_1\wedge f_2, d\zeta_k\wedge d\bar{\zeta}_k\rangle,\end{aligned} \tag{1.2.7}$$

where $f = \sum_{k\in Z} c_k(f)e_k$ and where $d\zeta_k$ denotes the R-linear map of $\mathrm{diff}_0(S^1)\mapsto C$ defined by $f\mapsto c_k(f),\ k>0$; in the same way $\langle f, d\bar{\zeta}_k\rangle := c_{-k}(f)$.

We have

$$\langle d\zeta_k, Jf\rangle = \sqrt{-1}\langle d\zeta_k, f\rangle; \qquad \langle d\bar{\zeta}_k, Jf\rangle = -\sqrt{-1}\langle d\bar{\zeta}_k, f\rangle, \quad k>0;$$

therefore

$$\omega(f, Jf) = \sum_{k>0}\big(k^3 - k\big)\big|c_k(f)\big|^2. \tag{1.2.8}$$

The positivity of (1.2.8) leads to impose the positiveness of the central charge c.

1.3. The manifold of univalent functions

We denote by $\mathcal{F}$ the vector space of functions f which are holomorphic in the unit disk $D := \{z; |z|<1\}$ and C^∞ on its closure $\bar{D}$ and such that $f(0) = 0$; we denote by $\mathcal{F}_0$ the subspace of functions of f satisfying $f'(0) = 0$. We denote:

$$\mathcal{M} := \big\{f\in\mathcal{F};\ f'(0) = 1 \text{ and } f \text{ injective on } \bar{D},\ f'(z)\neq 0\ \forall z\in\bar{D}\big\}.$$

Then $\mathcal{M}$ is an open set of the affine space $f_0 + \mathcal{F}_0$ where $f_0(z) = z,\ \forall z\in D$. As $\mathcal{F}_0$ is a complex vector space, $\mathcal{M}$ inherits of an infinite-dimensional structure of complex manifold. The embeding $\mathcal{M}\mapsto C^N$ defined by writting

$$f(z) = z\left(1 + \sum_{n=1}^{+\infty} c_n z^n\right) \tag{1.3.1}$$

introduces the affine coordinates $f\mapsto\{c_*\}$. Granted De Branges Theorem, $\mathcal{M}$ is identified to an open subset of $\{|c_n| < n + \varepsilon\}$.

LEMMA 1.3.2. – *$\mathcal{M}$ is a contractible manifold.*

Proof. – We define for $t \in]0,1]$ and $f \in \mathcal{M}$ the function $f_t(z) := t^{-1} f(tz)$; then $f_t \in \mathcal{M}$ and $\lim_{t\to 0} f_t = f_0$. $\square$

1.4. Kirillov identification of $\mathrm{Diff}(S^1)/S^1$ with $\mathcal{M}$

THEOREM 1.4.1. – *There exists a canonic identification of $\mathcal{M}$ with* $\mathrm{Diff}(S^1)/S^1$. *As a consequence $\mathcal{M}$ is an homogeneous space under the left action of* $\mathrm{Diff}(S^1)$. *Furthermore there exists a global section $\sigma : \mathcal{M} \mapsto$* $\mathrm{Diff}(S^1)$.

Proof. – Given $f \in \mathcal{M}$, we denote $\Gamma = f(\partial D)$. Then Γ is a smooth Jordan curve which splits the complex plane into two connected open sets: Γ^+ which contains 0 and Γ^- which contains the point at infinity of the Riemann sphere (the map $z \mapsto z^{-1}$ sends Γ^- onto a bounded domain countaining 0).

By the Riemann mapping theorem there exists an holomorphic map $\phi_f : D^c \mapsto \bar{\Gamma}^-$ such that $\phi_f(\infty) = \infty$, ϕ being holomorphic nearby ∞.

A diffeomorphism $g_f \in \mathrm{Diff}(S^1)$ is defined by:

$$g_f(\theta) := \left(f^{-1} \circ \phi_f\right)\left(\mathrm{e}^{\mathrm{i}\theta}\right).$$

We remark that ϕ_f is uniquely defined up to a rotation of D^c, which means up to an element of S^1; therefore we get a canonic map:

$$\mathcal{K} : \mathcal{M} \mapsto \mathrm{Diff}\left(S^1\right)/S^1.$$

In fact it has been proved by Kirillov [6] that $\mathcal{K}$ is bijective. The stabilizer of the left action of S^1 is the identity function $f_0(z) = z$.

We obtain the section σ by choosing ϕ_f^0 such that $u(z) := \phi_f^0(z^{-1})$ satisfy that $u'(0)$ is real and positive. $\square$

THEOREM 1.4.2. – *The fundamental cocycle* (1.2.1) *defines a* 2*-differential closed form Θ on $\mathcal{M}$. This form is invariant under the action of* $\mathrm{Diff}(S^1)$.

Proof. – Given $f \in \mathcal{M}$ denote $S_f := \{g \in \mathrm{Diff}(S^1)$ such that $gf_0 = f\}$; define $\Theta_f = (g^{-1})^*\omega$ for some $g \in S_f$; if γ also belongs to S_f we have $\gamma = gu$ with $u \in S^1$; then $(u^{-1}g^{-1})^*\omega = (g^{-1})^*(u^{-1})^*\omega$ expression equal, according (1.2.4) to $(g^{-1})^*\omega$; therefore our definition is independent of the choice of $g \in S_f$ and Θ is a well defined 2-differential form on $\mathcal{M}$, which is closed according (1.2.2).

The invariance under the action of $\mathrm{Diff}(S^1)$ results of the invariance of Maurer–Cartan differential form or more elementary from the identities

$$\Theta_{\gamma g(f_0)} = \left(g^{-1}\gamma^{-1}\right)^*\omega = \left(\left(\gamma^{-1}\right)^*\Theta\right)_{\gamma g(f_0)}. \qquad \square$$

1.5. Kirillov infinitesimal action of $\mathrm{diff}(S^1)$ on $\mathcal{M}$

THEOREM. – *Given $v \in$* $\mathrm{diff}(S^1)$, *and given $f \in \mathcal{M}$ we define $K_v(f) \in \mathcal{F}_0$ by the formula*:

$$K_v(f)(z) := \frac{f^2(z)}{2\pi} \int_{\partial D} \left[\frac{tf'(t)}{f(t)}\right]^2 \frac{v(t)}{f(t) - f(z)} \frac{\mathrm{d}t}{t}, \tag{1.5.1}$$

then K_v is the infinitesimal expression of the Kirillov action at the point f.

Proof. – We follow [5]; given $v \in \operatorname{diff}(S^1)$ we consider the infinitesimal action

$$\exp(\varepsilon v)\big(f^{-1} \circ \phi_f\big).$$

Denote by ∂_v the first-order differential operator associated to v; then we have the intertwinning formula $\partial_v f^{-1} = f^{-1}\partial_w$ where $w_\tau = \mathrm{i} f'(t) t v_t$, with $t \in \partial D$ and $\tau = f(t) \in \Gamma$; we can split $w := w^+ + w^-$ where w^+ has an holomorphic prolongement w^{++} to Γ^+ satisfying $0 = w^{++}(0) = (w^{++})'(0)$ and w^- is meromorphic on Γ^- with a unique pole at infinity which is simple; the Cauchy formula gives:

$$w^{++}(\zeta) = \frac{\zeta^2}{2\pi \mathrm{i}} \int_\Gamma \frac{w(\tau)}{\tau^2(\tau - \zeta)}\, \mathrm{d}\tau.$$

Doing again an intertwinning by defining $f_\varepsilon := \exp(\varepsilon w^{++}) \circ f = f \circ \exp(\varepsilon w^*)$, w^* defined by the relation $f'(z) w^*(z) = w^{++}(\zeta)$, we get then $\exp(\varepsilon v) f \simeq f_\varepsilon^{-1} \circ \exp(\varepsilon w^-)\phi_f$ which implies $\mathcal{K} f_\varepsilon \simeq \exp(\varepsilon v)\mathcal{K}(f)$. □

THEOREM 1.5.2. – *Fixing $g_0 \in \operatorname{Diff}(S^1)$ the map $f \mapsto g_0 f$ is an holomorphic map $\mathcal{M} \mapsto \mathcal{M}$.*

Proof. – We prove this fact when $g_0 = \exp(v_0)$; introducing $h_\tau := \exp(\tau v_0) f$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} h_\tau = K_{v_0}(h_\tau), \quad h_0 = f.$$

As the solution of an holomorphic differential equation depends holomorphically of its initial condition, we have only to prove that the map

$$\mathcal{M} \mapsto \mathcal{F}_0 \text{ defined by } f \mapsto K_{v_0}(f)$$

depends holomorphically upon f, fact which results from the formula given in (1.5.1).

PROPOSITION 1.5.3. –

$$(K_{e_0} f)(z) = \sqrt{-1}\big(z f'(z) - f(z)\big).$$

Proof. – A residue calculus of (1.5.1) for $v = 1$. □

Remark 1.5.4. – The operator K_1 is a real vector field on $\mathcal{M}$; this means that it defines a map of $\mathcal{M} \mapsto \mathcal{F}_0$; it is indeed legitimate to use on the target $\mathcal{F}_0$ the multiplication by $\sqrt{-1}$.

From another hand the *real* vector field K_1 is of a different nature from the *complex* vector field L_0^c introduced in (1.6.1)! The beginning of next section emphasizes this distinction between complex and real vector fields.

1.6. Kählerian structure on $\mathcal{M}$

We denote by $T(\mathcal{M})$ the vector space of *real* first-order differential operators on $\mathcal{M}$ (a real differential operator transforms real functionals into real functionals). We call also the elements of $T(\mathcal{M})$ the *real vector fields* on $\mathcal{M}$.

Then the complex structure J on $\mathcal{M}$ induces the splitting:

$$T(\mathcal{M}) \otimes C = T^{(1,0)}(\mathcal{M}) \oplus T^{(0,1)}(\mathcal{M}).$$

We have a map $\tilde{J}: T(\mathcal{M}) \otimes C \mapsto T(\mathcal{M})$ defined by $(X + \sqrt{-1}Y) \mapsto X + J(Y)$; then $T^{(0,1)}(\mathcal{M})$ can be characterized as the kernel of $\tilde{J}$; furthermore for an holomorphic functionnal Ψ we have $Z.\Psi = (\tilde{J}(Z)).\Psi$.

THEOREM. – *Define*

$$L_k^c := -\sqrt{-1} K_{\cos k\theta} + K_{\sin k\theta}, \quad \forall k \neq 0; \qquad L_0^c := -\sqrt{-1} K_{e_0}; \tag{1.6.1}$$

$$L_k := \tilde{J}\big(L_k^c\big), \tag{1.6.2}$$

then

$$L_k(f) = f'(z) z^{k+1}, \quad k > 0; \qquad L_0 = f'(z) z - f(z); \tag{1.6.3}$$

$$L_k^c(f_0) \in T_{f_0}^{(1,0)}(\mathcal{M}) \quad \textit{for } k > 0; \tag{1.6.4}$$

$$\big[L_j^c, L_k^c\big] = (j-k) L_{j+k}^c, \quad \forall j, k \in Z, \tag{1.6.5}$$

$$[L_m, L_n] = (m-n) L_{m+n}, \quad \forall m, n \in Z. \tag{1.6.6}$$

Define L_k^h *as the* $(1,0)$ *component of the real vector field* L_k*, then* $\forall k \in Z$*,*

$$L_k^h \Phi = L_k^c \Phi \quad \forall \Phi \ \textit{holomorphic}; \qquad \big[L_m^h, L_n^h\big] = (m-n) L_{m+n}^h \quad \forall m, n \in Z. \tag{1.6.7}$$

Proof. – The identity (1.6.3) results from the computation of the integral (1.5.1) by residue calculus.

Denote v^l the right invariant vector field associated to $v \in \mathrm{diff}(S^1)$; in particular to $e_k = \exp(\mathrm{i}k\theta)$ is associated e_k^l, and by (1.1.2) we have

$$\big[e_m^l, e_n^l\big] = \sqrt{-1}(m-n) e_{m+n}^l,$$

therefore

$$\big[L_j^c, L_k^c\big] = -\sqrt{-1}\,(j-k) e_{j+k}^l = (j-k) L_{j+k}^c.$$

We have the fact that the complex structure on $\mathcal{M}$ is invariant under the left action of $\mathrm{Diff}(S^1)$; therefore denoting $\mathcal{L}_\phi$ the Lie derivative associated to the left action of $\phi \in \mathrm{diff}(S^1)$ we have

$$\mathcal{L}_\phi\big(\tilde{J}\big(L_k^c\big)\big) = \tilde{J}\big(\mathcal{L}_\phi\big(L_k^c\big)\big).$$

Therefore

$$\mathcal{L}_\phi L_k = \tilde{J}\big(\big[\phi, L_k^c\big]\big)$$

which by C-linearity implies (1.6.6).

Denote by $\mathcal{J}^{-1}$ the linear map identifying $T_{f_0}(\mathcal{M})$ with $\mathrm{diff}_0(S^1)$ on which we put the complex structure defined in (1.2.5). To prove that $\mathcal{J}$ is holomorphic is equivalent to verify that the map $v \mapsto K_v(f_0)$ is C-linear or that $\forall k > 0$ $L_{\bar{k}}(f_0) = 0$; indeed

$$\frac{2\pi}{z^2} L_{\bar{k}}(f_0) = \int_{\partial D} \frac{t^{-k}}{z-t} \frac{\mathrm{d}t}{t} = -\sum_{s \geqslant 0} z^s \int_{\partial D} \frac{\mathrm{d}t}{t^{s+k+1}} = 0.$$

Denote $L_n^a = L_n - L_n^h$ the antiholomorphic component of L_n which is equal to $\bar{L}_n^h$. As we shall see in (1.8) the operators L_n^h in affine coordinates have holomorphic coefficients which imply that $[L_n^h, L_m^a] = 0$; therefore

$$\big([L_m, L_n]\big)^h = \big[L_m^h, L_n^h\big] \quad \text{which establish (1.6.7).} \qquad \square$$

Remark. – For $f \neq f_0$, (1.6.4) is not true.

Remark. – In the Appendix, as in [5], L_k^h will be shorthanded as L_k.

THEOREM. – *The* 2-*differential form*

$$\Theta \text{ defined in (1.4.2) is of type } (1,1), \text{ positive definite} \tag{1.6.8}$$

and

$$\Theta_{f_0}\big(L_n^c(f_0), \bar{L}_m^c(f_0)\big) = \sqrt{-1}\delta_n^m \gamma_n, \quad \text{where } \gamma_n = \big(n^3 - n\big). \tag{1.6.9}$$

Proof. – As Θ is defined by transport through the holomorphic action of Diff(S^1), it is sufficient to check (1.6.8) at f_0, which will result from (1.6.9).

$$\omega\big(-\mathrm{i}\exp(\mathrm{i}n\theta), \mathrm{i}\exp(-\mathrm{i}m\theta)\big) = \omega\big(\exp(\mathrm{i}n\theta), \exp(-\mathrm{i}m\theta)\big) = -i\big(n - n^3\big)\delta_n^m. \qquad \square$$

Remark. –

$$\big\langle \Theta_{g(f_0)},\ vg(f_0) \wedge \bar{w}g(f_0)\big\rangle = \big\langle \omega,\ g^{-1}vg \wedge g^{-1}\bar{w}g\big\rangle \neq \big\langle \omega,\ v \wedge \bar{w}\big\rangle.$$

1.7. The vector fields L_n, $n > 0$, in affine coordinates

The formula (1.3.1) gives a global chart $\mathcal{M} \mapsto C^N$ in terms of the Taylor coefficients $f \mapsto \{c_*(f)\}$. We want to express the vector fields L_n^h in this chart.

Realization of the flow associated to L_n^h, $n > 0$

Firstly we remark that the vector field L_n^c, L_n^h are not real vector fields; this fact has been already discussed at the beginning of Section 1.6; we can also refer to Kirillov [5], page 738, ten lines before the end where this fact is underlined. Flows of complex vector fields need to be defined by analytic prolongation; when there exist they are very singular. It will be indeed our case.

Consider on a neighborhood of 0 the following holomorphic function:

$$M_t^{(k)}(z) = \frac{z}{(1 - tkz^k)^{1/k}}, \tag{1.7.1}$$

where t is a smal real parameter. Then for z small enough we have

$$\big(M_t^{(k)} \circ M_{t'}^{(k)}\big)(z) = M_{t+t'}^{(k)}(z). \tag{1.7.2}$$

Then given $f \in \mathcal{M}$ the composition $f \circ M_t^{(k)} \notin \mathcal{M}$ but it is an holomorphic function defined for t small enough on an arbitrarily large disk contained in the unit disk.

THEOREM. – *We have* $\forall z_0$, $|z_0| < 1$ *the following identity*:

$$\lim_{t \to 0} \frac{f(M_t^{(k)}(z_0)) - f(z_0)}{t} = f'(z_0) z_0^{k+1}.$$

We call cylindrical functional a map $\Phi : \mathcal{M} \to R$ such that $\Phi(f) = \phi(\ldots, f(z_i), \ldots)$ where $\phi : D^r \mapsto R$, ϕ being smooth, and $i \in [1, r]$, $|z_i| < 1$. Then

$$(L_k^h \Phi)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi\big(f \circ M_t^{(k)}\big), \quad k > 0. \tag{1.7.3}$$

Proof. – As the vector field L_k^h is of type $(1, 0)$ it is sufficient to check the identity assuming furthermore that ϕ is holomorphic, then

$$M_t^{(k)}(z) = z + tz^{k+1} + \mathrm{o}(t). \qquad \square$$

LEMMA. –

$$f\big(M_t^{(k)}(z)\big) = \big(z + tz^k\big)\left(1 + \sum_{n=1}^{\infty} c_n z^n \big(1 + ntz^k\big)\right) + \mathrm{o}(t). \tag{1.7.4}$$

THEOREM. – *Denote by ∂_k the holomorphic partial derivative relatively to the affine coordinate c_k (if $c_k = \xi_k + \sqrt{-1}\eta_k$ then $2\partial_k = \frac{\partial}{\partial \xi_k} - \sqrt{-1}\frac{\partial}{\partial \eta_k}$), then*

$$L_k = \partial_k + \bar{\partial}_k + \sum_{n=1}^{+\infty} (n+1)(c_n \partial_{k+n} + \bar{c}_n \bar{\partial}_{k+n}), \quad k > 0. \tag{1.7.5}$$

Proof. – Consider a cylindrical function Φ, which is assumed furthermore to be holomorphic, which means that ϕ is an holomorphic map $D^r \mapsto C$. Then all $\bar{\partial}$ vanishes on such Φ. Any holomorphic cylindrical funtional can be approximated by polynomial in the c_*. We take $\Psi := c_{k_0}^q$. By (1.7.4) we have:

$$\Psi\big(\big(M_t^{(k)}\big)^* f\big) = \begin{cases} c_{k_0}(f) & \text{if } k > k_0; \\ \big(t + c_{k_0}(f)\big)^q + \mathrm{o}(t), & \text{if } k = k_0; \\ \big(c_{k_0}(f) + (k_0 - k + 1)tc_{k_0 - k}(f)\big)^q + \mathrm{o}(t) & \text{if } k < k_0. \end{cases}$$

Differentiating relatively to t and making $t = 0$ we obtain respectively 0 or $qc_{k_0}^{q-1}$ or $q(k_0 - k + 1)[c_{k_0}(f)]^{q-1} c_{k_0}(f)$, relations which prove the theorem for holomorphic functionals. The left-hand side of (1.7.5) can be written as the sum of a $(1, 0)$ vector field Z plus a $(0, 1)$ vector field Y; as L_k is a real vector field we have $Y = \bar{Z}$ relation which proves the theorem. $\square$

Using the definition of L_*^h made in (1.6.7) we obtain:

$$L_k^h := \partial_k + \sum_{n=1}^{+\infty} (n+1) c_n \partial_{k+n}, \quad k > 0, \qquad L_0^h = \sum_{n \geqslant 1} n c_n \partial_n. \tag{1.7.6}$$

1.8. Analyticity of the holomorphic action

THEOREM. – *For all $k \in Z$ there exist holomorphic polynomials $\varphi_{k,s}(c)$ such that denoting*

$$L_k^h = \sum_{s=1}^{+\infty} \varphi_{k,s} \partial_s$$

then for every holomorphic functional Φ on $\mathcal{M}$ we have $L_k^c\Phi = L_k^h\Phi$.

Proof. – This result is a consequence of (1.7.6) for $k \geqslant 0$; the case $k < 0$ is proved in the Appendix. □

2. Unitarizing measure for the Neretin representation

2.1. Representation associated to Neretin polynomials

In the affine coordinates $\{c_k\}$ on $\mathcal{M}$, Neretin introduced [8] (see also [5]) the sequence of polynomials P_n defined by the following double indices recurrence relations:

$$L_k^h P_n = (n+k)P_{n-k} + \gamma_k \delta_k^n, \quad \text{where } \gamma_k = \frac{c}{12}\left(k^3 - k\right),\ P_0 = P_1 = 0,\ P_n(0) = 0, \tag{2.1.1}$$

where the central charge c has been fixed.

LEMMA. –

$$\textit{The polynomial } P_n \textit{ is of weight } n. \tag{2.1.1$_\mathrm{a}$}$$

Proof. – The weight $w(P) := \sum_k k \times$ (degree of P relatively c_k). Then the recurrence implies that $L_1^h P_n = (n+1)P_{n-1}$; the lemma will result of the identities:

$$w\left(L_1^h c_1^\alpha\right) = \alpha - 1 = w\left(c_1^\alpha\right) - 1; \qquad w\left(L_1^h c_k^\alpha\right) = k(\alpha-1) + (k-1) = w\left(c_k^\alpha\right) - 1. \qquad \square$$

THEOREM. – *Denote $\mathcal{H}(\mathcal{M})$ the vector space of holomorphic functionals defined on $\mathcal{M}$; we associate to $\Phi \in \mathcal{H}(\mathcal{M})$*

$$\begin{aligned} &\rho(\kappa)\Phi = \sqrt{-1}\frac{c}{12}\Phi, \qquad \rho(e_0) = \sqrt{-1}L_0\Phi, \\ &\rho\left(\exp(\mathrm{i}k\theta)\right)\Phi = \sqrt{-1}L_k^c\Phi, \qquad \rho\left(\exp(-\mathrm{i}k\theta)\right) = \sqrt{-1}\left(L_{-k}^c + P_k\right)\Phi \quad \forall k > 0. \end{aligned} \tag{2.1.2}$$

Then ρ is an anti-representation of $\mathcal{V}_c \otimes C$ on $\mathcal{H}(\mathcal{M})$ which means that

$$\left[\rho(v_1), \rho(v_2)\right] = -\rho\left([v_1, v_2]_{\mathcal{V}_c}\right), \tag{2.1.3}$$

the Lie bracket in the right-hand side have been defined in (1.2.3).

Proof. – Granted the holomorphy of Φ we can replace by L_n^c by L_n^h; as L_n^h are operators with holomorphic coefficients we deduce that L_v^c send an holomorphic functional into an holomorphic functional; as P_n is holomorphic we obtain that the operator ρ operates on $\mathcal{H}(\mathcal{M})$.

The vector fields L_v^c realize an anti-representation of $\mathrm{diff}(S^1)$; this proves (2.1.3) when v_1, v_2 are two exponential with positive frequencies.

Consider the case of v_1 of positive frequency and v_2 of negative frequency:

$$\left[\rho\left(\exp \mathrm{i}k\theta\right), \rho(\exp - \mathrm{i}s\theta)\right] = -\left(L_k^h P_s + (k+s)L_{k-s}^c\right) := B_{k,s}.$$

For $k > s$ we deduce from (2.1.1)$_\mathrm{a}$ and (1.7.6) that $L_k^h P_s = 0$.

For $k = s$ we have

$$B_{k,k} = \rho(\gamma_k\kappa + 2ke_0).$$

For $k < s$ we have

$$B_{k,s} = (k+s)\big(P_{s-k} + L^c_{k-s}\big) = \rho\big((k+s)e_{k-s}\big).$$

Finally it remains to compute, for $k, s > 0$ the expression:

$$\begin{aligned}&\big[\rho\big(\exp(-\mathrm{i}k\theta)\big), \rho\big(\exp(-\mathrm{i}s\theta)\big)\big] - \big[L^h_{e_k}, L^h_{e_s}\big]\\ &\quad = A_{k,s} := L^h_{-k}P_s - L^h_{-s}P_k = (s-k)P_{k+s} \quad \forall s, k \geqslant 0\end{aligned} \tag{2.1.4}$$

basic identity which is proved in the Section A.7 of the Appendix. □

2.2. Differential form on Diff(S^1) associated to Neretin polynomials

We associate to $v \in \mathrm{diff}(S^1)$ the right invariant tangent vector field to $\mathrm{Diff}(S^1)$ defined by $(v^l)_g = \exp(\varepsilon v)g$; in particular to $e_{2k} = \cos k\theta,\ e_{2k+1} = \sin k\theta,\ e_0 = 1 \in \mathrm{diff}(S^1)$ we associate the right invariant vector fields $e^l_* := \exp(\varepsilon e_*)g$. We consider the following first-order differential operator with complex coefficient defined on $\mathrm{Diff}(S^1)$: $\tilde{L}^c_n := -\sqrt{-1}e^l_{2s} + e^l_{2s+1},\ s \neq 0$, and $\tilde{L}_0 = -\sqrt{-1}e^l_0$. Denote ψ_{-s} is the dual basis of $\tilde{L}^c_n$, that is

$$\big\langle \tilde{L}^c_n, \psi_s\big\rangle = \delta^0_{n+s}. \tag{2.2.1}$$

It results from the duality that the differential of a function Ψ defined on $\mathrm{Diff}(S^1)$ has for expression

$$\mathrm{d}\Psi = \sum_{k\in Z}\big(\tilde{L}^c_{-k}\Psi\big)\psi_k. \tag{2.2.2}$$

Furthermore the $\{\psi_s\}$ satisfy the structural equation:

$$\big\langle \mathrm{d}\psi_k, \tilde{L}^c_q \wedge \tilde{L}^c_r\big\rangle = (r-q)\big\langle \psi_k, \tilde{L}^c_{q+r}\big\rangle = (r-q)\delta^{-k}_{q+r}$$

or finally

$$\mathrm{d}\psi_k = -\frac{1}{2}\sum_{s\in Z}(k+2s)\psi_{-s}\wedge\psi_{k+s}. \tag{2.2.3}$$

A complex valued 1-differential form Ω on $\mathrm{Diff}(S^1)$ will be built from Neretin polynomials by the formula:

$$\Omega = \sum_{k>0}(P_k \circ \pi)\psi_k, \tag{2.2.4}$$

where π is the projection map $\mathrm{Diff}(S^1) \mapsto \mathrm{Diff}(S^1)/S^1$.

THEOREM. –

$$\big\langle v^l, \Omega\big\rangle_g = \frac{1}{2\pi}\int_{\partial D} S_f(t)v(\log t)t^2\frac{\mathrm{d}t}{t}, \tag{2.2.5}$$

where $g = f^{-1}\circ\phi_f$ and where S_f is the Schwarzian derivative:

$$S_f := \frac{f^{(3)}}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.2.6}$$

Proof. – It is possible to find in the Appendix the following Neretin generatrix function:

$$\sum_{n>0} t^n P_n = t^2 S_f(t).$$

As $\tilde{L}_n^c = -\sqrt{-1}(\exp(\mathrm{i}n\theta))^l$ we get $\psi_s(\exp(\mathrm{i}n\theta)) = \sqrt{-1}\delta_{n+s}^0$ and finally the formula reduce to prove

$$P_n = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} t^2 S_f(t) t^{-n} \frac{\mathrm{d}t}{t}. \qquad \square$$

THEOREM. – *Denote by $\tilde{\Theta}$ the image of the symplectic form defined on G by the transformation $g \mapsto g^{-1}$, then*

$$\tilde{\Theta} = -\frac{\sqrt{-1}}{2} \sum_{s>0} \gamma_s \psi_{-s} \wedge \psi_s, \tag{2.2.7}$$

where γ_s is defined in (2.1.1); *we have*

$$\mathrm{d}\Omega = \sqrt{-1}\,\tilde{\Theta}. \tag{2.2.8}$$

Proof. – The form Θ is left invariant. Therefore $\tilde{\Theta}$ is right invariant. The forms $\{\psi_q\}$ constitute a basis of right invariant differential forms; therefore there exists constants $c_{k,s}$ such that:

$$\tilde{\Theta} = \frac{1}{2} \sum_{s,k \in Z} c_{k,s} \psi_{-k} \wedge \psi_{-s}.$$

In order to compute the constants $c_{k,s}$ we look at this identity at the point f_0. Then by (1.6.9) we have, for $s > 0$:

$$\langle \tilde{\Theta}, L_s^c \wedge \bar{L}_s^c \rangle_{f_0} = \sqrt{-1}\gamma_s = c_{s,-s} \langle \bar{L}_s^c, \psi_s \rangle_{f_0} = -c_{s,-s}.$$

Define $P_s^* = P_s \circ \pi$ for $s > 0$ and $P_s^* = 0$ for $s \leqslant 0$; with these notations $\Omega = \sum_s P_s^* \psi_s$. Using (2.2.2)

$$\mathrm{d}\Omega = \sum_{k,s \in Z} (\tilde{L}_{-k}^c P_s^*) \psi_k \wedge \psi_s + R,$$

where R, granted (2.2.3), has the following expression:

$$R = -\frac{1}{2} \sum_{r,t \in Z} P_r^*(r + 2t) \psi_{-t} \wedge \psi_{r+t} = -\frac{1}{2} \sum_{k,s \in Z} P_{s+k}^*(s - k) \psi_k \wedge \psi_s.$$

Therefore

$$\mathrm{d}\Omega = \frac{1}{2} \sum_{k,s \in Z} B_{k,s} \psi_k \wedge \psi_s,$$

where

$$B_{k,s} := \tilde{L}_{-k}^c P_s^* - \tilde{L}_{-s}^c P_k^* - (s - k) P_{s+k}^*. \tag{2.2.9}$$

For $k > 0$ and $s > 0$ we obtain, granted (2.1.4), $B_{k,s} = 0$.

For $k<0$ and $s<0$ we obtain that all the P^* vanish and $B_{k,s}=0$.

Consider the last case $k<0$ and $s>0$; if $-k>s$ the polynomial P_s being of weight s do not depend the variables c_j for $j>s$, therefore $L_{-k}P_s=0$; for $-k\leqslant s$ the defining relations (2.1.1) proves that $B_{-k,s}=\gamma_{-k}\delta^s_{-k}$. □

2.3. Unitarizing measure

DEFINITION. – *A probability measure μ on $\mathcal{M}$ is an* unitarizing measure for the representation ρ of $\mathcal{V}_c\otimes C$ in $\mathcal{H}L^2_\mu$ *if and only if for all v real (i.e. $v\in\mathcal{V}_c$) the operator $\rho(v)$ is anti-Hermitian*:

$$\rho(v)+\big(\rho(v)\big)^*=0. \tag{2.3.1}$$

The relation (2.3.1) is equivalent to:

$$\big(\rho\big(\exp(\mathrm{i}k\theta)\big)\big)^*=-\rho\big(\exp(-\mathrm{i}k\theta)\big). \tag{2.3.2}$$

In fact denote $A_s=\rho(\exp(\mathrm{i}s\theta)$, then $2\rho(\cos\theta)=A_s+A_{-s}$; its adjoint is $-A_{-s}-A_s$; finally $2(\rho(\sin s\theta)^*=\mathrm{i}(-A_{-s}+A_s)=-\rho(\sin s\theta)$.

THEOREM. – *A probability measure μ is unitarizing if it satisfies the following relation*:

$$\begin{aligned}&\operatorname{div}_\mu\big(L^c_k\big)=\bar P_k,\quad k\geqslant 0,\quad \textit{or equivalently}\\ &\operatorname{div}_\mu(K_{\cos k\theta})(f)=\Im P_k,\qquad \operatorname{div}_\mu(K_{\sin k\theta})=\Re P_k.\end{aligned} \tag{2.3.3}$$

Proof. – The unitarity condition (2.3.2) is given in terms of the L^c_k through the formulas (2.1.2); then the appearance in those formulas of a factor $\sqrt{-1}$ changes the sign of (2.3.2) which finally can be written as follows: $\forall k>0$ we have

$$\int_{\mathcal{M}}\big[\big(L^c_k\Phi\big)\bar\Psi-\Phi\big(\big(L^c_{-k}+P_k\big)\Psi\big)^*\big]\,\mathrm{d}\mu=0, \tag{2.3.4}$$

where the * above the parenthesis indicate that we take the imaginary conjugate.

Firstly we replace $L^c_k\mapsto L^h_k$, $L^c_{-k}\mapsto L^h_{-k}$ and we use the identities $L^h_k(\Phi\bar\Psi)=(L^h_k\Phi)\bar\Psi$ and $\Phi\bar L^h_{-k}\bar\Psi=\bar L^h_{-k}(\Phi\bar\Psi)$; then (2.3.4) takes the shape:

$$\int_{\mathcal{M}}\big[\big(L^h_k-\bar L^h_{-k}-\bar P_k\big)(\Phi\bar\Psi)\big]\,\mathrm{d}\mu=0\quad\text{or}\quad \operatorname{div}_\mu(Z_k)=\bar P_k, \tag{2.3.5}$$

where $Z_k=L^h_k-\bar L^h_{-k}$. The operator $L^c_k=L^h_k+L^a_k$ where L^a_k is a vector field of type $(0,1)$. Using the fact that $L^c_{-k}=-\bar L^c_k$ we get by conjugation that $\bar L^h_{-k}=-L^a_k$, therefore $Z_k=L^c_k$ and finally $\operatorname{div}_\mu(L^c_k)=\bar P_k$.

We explicit the differential operators in terms of real differential operators: the decomposition

$$L^c_k=-\sqrt{-1}K_{\cos k\theta}+K_{\sin k\theta}$$

implies, as the divergence of a probability measure is a real operator, that the second part of (2.3.3) holds true. □

THEOREM 2.3.6. – *The differential form Ω is invariant under the left action of S^1.*

Proof. – We shall prove the infinitesimal invariance under the action of L_0. Applying (2.1.1) with $k = 0$, we get

$$L_0 P_s = s P_s. \tag{i}$$

The linear forms $\{\psi_{-s}\}$ are defined as the dualizing base of the $\{L_k^c\}$; we have by (1.6.5)

$$\left[L_0, L_s^c\right] = -s L_s^c; \quad \text{by duality} \quad L_0 \psi_{-s} = -s \psi_s; \tag{ii}$$

it results from (i) and (ii) that $L_0(P_s \psi_s) = 0$. □

THEOREM 2.3.7. – *Every unitarizing measure μ is invariant under the action on the left* $\exp(\theta e_0)$.

Proof. – The unitarity condition $\rho(e_0) + (\rho(e_0))^* = 0$ together with (2.1.2) imply the invariance. □

2.4. Resolution of a $\bar\partial$ problem on $\mathcal{M}$

THEOREM. – *Denote $\mathcal{I}$ the endomorphism of* $\mathrm{Diff}(S^1)$ *defined by $\mathcal{I}(g) := g^{-1}$. Then there exists a unique differential form Ω_1 defined on $\mathcal{M}$ such that*:

$$\mathcal{I}^* \Omega = \pi^* \Omega_1. \tag{2.4.1}$$

Furthermore Ω_1 for the complex structure on $\mathcal{M}$ is of type $(0, 1)$ *and satisfies*

$$\partial \Omega_1 = \sqrt{-1}\Theta, \qquad \bar\partial \Omega_1 = 0. \tag{2.4.2}$$

Proof. – The endomorphism $\mathcal{I}$ changes differentiation on left into differentation on the right. Using (2.3.6) we obtain that $\mathcal{I}^*\Omega$ is invariant under the *right action of* S^1. Therefore it defines a differential form Ω_1 on $\mathrm{Diff}(S^1)$ invariant under the right action of e_0 therefore coming by π^* of a form Ω_1 on $\mathcal{M}$.

As $\mathcal{I}^*$ commutes with the coboundary operator, we deduce that $\mathrm{d}\Omega_1 = \Theta$ which by bidegree splitting proves (2.4.2). □

Part II: Symplectic embedding and Kähler potential

3. Embedding of the diffeomorphism group into the Siegel disk

3.1. Symplectic action of the diffeomorphism group

We consider the space V of real valued C^1-functions defined on the circle with mean value equal to 0. On V we define a bilinear alternate form:

$$\omega(u, v) = \frac{1}{\pi} \int_0^{2\pi} u v' \, \mathrm{d}\theta.$$

THEOREM. – *If g is an orientation preserving diffeomorphism of S^1 then*

$$\omega(g^* u, g^* v) = \omega(u, v). \tag{3.1.1}$$

Proof. –

$$(g^*v)' = \big(g^*(v')\big)g',$$

$$\int_0^{2\pi} \big(g^*(uv')\big)g'\,\mathrm{d}\theta = \int_0^{2\pi} uv'\,\mathrm{d}\theta. \qquad \square$$

We define an action of $\mathrm{Diff}(S^1)$ on V by:

$$U_{g^{-1}}(u) = g^*u - \frac{1}{2}\int_0^{2\pi} (g^*u)\,\mathrm{d}\theta. \tag{3.1.2}$$

Then

$$\omega\big(U_g(u), U_g(v)\big) = \omega(u, v). \tag{3.1.3}$$

We have in this way defined an embedding of $\mathrm{Diff}(S^1)$ into the automorphism of V which preserves the symplectic form ω. We introduce on V a complex structure defined by the Hilbert transform:

$$\mathcal{J} : \sin(k\theta) \mapsto \cos(k\theta), \quad \cos(k\theta) \mapsto -\sin(k\theta).$$

We define on V an Hilbertian metric:

$$\|u\|^2 = -\omega(u, \mathcal{J}u).$$

We have

$$\left|\sum_{k>0} a_k \cos(k\theta) + b_k \sin(k\theta)\right|^2 = \sum_{k>0} k\big(a_k^2 + b_k^2\big).$$

Then $\mathcal{J}$ is an orthogonal transformation of V.

We consider the complex Hilbert space $H = V \otimes C$; then H can be identified with complex valued function defined on the circle having mean value 0; on H the operation of conjugation $f \mapsto \bar{f}$ is well defined.

The orthogonal transformation $\mathcal{J}$ can be diagonalized in H; as $\mathcal{J}^2 = -1$ only appears the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. We denote H^+ the eigenspace associated to the eigenvalue $\sqrt{-1}$; then we can identify H^+ to the vectors of type $(1,0)$ that is the vectors of the form $v - \sqrt{-1}\mathcal{J}(v),\ v \in V$. We can also identify H^+ with the functions having an holomorphic extension inside the unit disk. Then define $H^- = \bar{H^+}$; then H^- can be identified with the functions on the circle which possess an holomorphic extension ouside the unit disk, regular at the point at ∞ of the complex plane. The bilinear form ω extends to a bilinear form $\tilde{\omega}$ defined on H and we have:

$$\tilde{\omega}(w, w') = 0 \quad \text{if } w, w' \in H^+ \text{ or } w, w' \in H^-.$$

We can express $\tilde{\omega}$ in term of the Hilbertian structure

$$\tilde{\omega}(h^+,\ h^-) = \sqrt{-1}\ (h^+|\bar{h}^-), \quad \forall h^+ \in H^+,\ h^- \in H^-,$$

identity which is proved by checking on $h^+ = \mathrm{e}^{\mathrm{i}n\theta},\ h^- = \mathrm{e}^{\mathrm{i}m\theta},\ n > 0,\ m < 0.$

We define a symmetric C-bilinear form on $H \times H$ by:

$$\langle h_1, h_2\rangle = (h_1|\bar{h}_2), \quad \text{then} \quad (h_1|h_2) = \langle h_1, \bar{h}_2\rangle. \tag{3.1.4}$$

Then

$$\tilde{\omega}(h_1, h_2) = \sqrt{-1}\big(\langle h_1^+, h_2^-\rangle - \langle h_1^-, h_2^+\rangle\big). \tag{3.1.5}$$

The restriction to $H^+ \times H^-$ of the bilinear form $(*)$ defines a duality coupling:

$$\langle h^+, h^-\rangle := (h^+|\bar{h}^-). \tag{3.1.6}$$

Given $A \in \mathrm{End}(H)$ we denote A^{T} the transposed defined by:

$$\langle Ah_1, h_2\rangle = \big\langle h_1, A^{\mathrm{T}}h_2\big\rangle.$$

Given $a \in \mathrm{End}(H^+)$, then the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 0\end{smallmatrix}\right)$ makes possible to identify $\mathrm{End}(H^+) \subset \mathrm{End}(H)$; then $a^{\mathrm{T}} \in \mathrm{End}(H)$ is well defined; furthermore we have through the duality coupling (3.1.6)

$$\langle ah^+, h^-\rangle = \big\langle h^+, a^{\mathrm{T}}h^-\big\rangle;$$

which means that $a^{\mathrm{T}} \in \mathrm{End}(H^+)$. The adjoint $a^\dagger \in \mathrm{End}(H^+)$ is defined by:

$$(aw_1|w_2) = (w_1|a^\dagger w_2), \quad \forall w_1, w_2 \in H^+.$$

The conjugation operator sends $H^+ \mapsto H^-$ therefore $\bar{a} \in \mathrm{End}(H^-)$ and we have the fact that the adjoint is obtained by conjugation followed by transpostion

$$a^\dagger = (\bar{a})^{\mathrm{T}} = \bar{a}^{\mathrm{T}}.$$

The automorphism U_g of V extends to an endomorphism $\tilde{U}_g$ of H. Denoting π^+, π^- the projection of H on H^+, H^- we introduce:

$$a(g) := \pi^+\tilde{U}_g\pi^+; \qquad b(g) := \pi^+\tilde{U}_g\pi^-.$$

As the endomorphism $\tilde{U}_g$ commutes with the conjugation it is represented by the matrix

$$\tilde{U}_g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix}. \tag{3.1.7}$$

3.2. The Siegel disk in infinite dimension

The conservation of the symplectic form (3.1.5) is equivalent to:

$$(\bar{a})^{\mathrm{T}}(a+b) - b^{\mathrm{T}}(\bar{a}+\bar{b}) = \pi^+, \qquad (\bar{b})^{\mathrm{T}}(a+b) - a^{\mathrm{T}}(\bar{a}+\bar{b}) = \pi^-$$

we remark that the first relation is the conjugate of the second. Therefore we have only to take care of the second relation which by splitting on the components H^+, H^- gives:

$$a^{\mathrm{T}}\bar{a} - b^\dagger b = \pi^-, \tag{3.2.1}$$

$$a^{\mathrm{T}}\bar{b} - b^{\dagger}a = 0. \tag{3.2.2}$$

We call *Symplectic Group of infinite order*, let $\mathrm{Sp}(\infty)$, the collection of bounded, invertible, operators $a \in \mathrm{End}(H^+)$, $b \in \mathcal{L}(H^-; H^+)$ satisfying the relations (3.2.1), (3.2.2); then $\mathrm{Sp}(\infty)$ is a group for the composition of matrices.

We consider the infinite-dimensional Siegel disk $\mathcal{D}_\infty$ consisting of operators $Z \in \mathcal{L}(H^-, H^+)$ such that:

$$Z^{\mathrm{T}} = Z, \quad 1 - Z^{\dagger}Z > 0, \quad \operatorname{trace}(Z^{\dagger}Z) < \infty; \text{ its based point is the matrix } Z_0 = 0. \tag{3.2.3}$$

We shall identify $Z \in \mathcal{D}_\infty$ to $\hat{Z} \in \mathrm{End}(H)$ through the matrix $\hat{Z} := \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$.

THEOREM. – *The group* $\mathrm{Sp}(\infty)$ *operates on* $\mathcal{D}_\infty$ *by*

$$Z \mapsto Y = (aZ + b)(\bar{b}Z + \bar{a})^{-1}. \tag{3.2.4}$$

Remark. – In the above formula a, b are identified with the corresponding elements of $\mathrm{End}(H)$.

Proof. – We have firstly to show that $Y \in \mathcal{L}(H^-; H^+)$: we have $\bar{a} \in \mathrm{End}(H^-)$, $\bar{b}Z \in \mathrm{End}(H^-)$; therefore $(\bar{b}Z + \bar{a})^{-1} \in \mathrm{End}(H^-)$.

We have secondly to show that $Y^{\mathrm{T}} = Y$:

$$Y^{\mathrm{T}} = (Zb^{\dagger} + a^{\dagger})^{-1}\left(Za^{\mathrm{T}} + b^{\mathrm{T}}\right),$$

therefore the identity $Y^{\mathrm{T}} = Y$ is equivalent to:

$$\begin{aligned} 0 &= \left(Za^{\mathrm{T}} + b^{\mathrm{T}}\right)(\bar{b}Z + \bar{a}) - (Zb^{\dagger} + a^{\dagger})(aZ + b) \\ &= Z\left(a^{\mathrm{T}}\bar{b} - b^{\dagger}a\right)Z + \left(b^{\mathrm{T}}\bar{b} - a^{\dagger}a\right)Z + Z\left(a^{\mathrm{T}}\bar{a} - b^{\dagger}b\right) + b^{\mathrm{T}}\bar{a} - a^{\dagger}b, \end{aligned}$$

the first coefficient vanishes accordingly (3.2.2); by conjugating (3.2.2) we obtain the vanishing of the fourth coefficient; using (3.2.1) and its conjugation we obtain $= \pi^- Z + Z\pi^+$; these two terms are zero according the fact that Z is in fact the matrix $\left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$.

We have to check that $\Delta := Y^{\dagger}Y - \pi^- < 0$. We denote $D := (\bar{b}Z + \bar{a})$ then:

$$\begin{aligned} D^{\dagger}\Delta D &= (Z^{\dagger}a^{\dagger} + b^{\dagger})(aZ + b) - \left(Z^{\dagger}b^{\mathrm{T}} + a^{\mathrm{T}}\right)\pi^-(\bar{b}Z + \bar{a}) \\ &= Z^{\dagger}\left(a^{\dagger}a - b^{\mathrm{T}}\bar{b}\right)Z + Z^{\dagger}\left(a^{\dagger}b - b^{\mathrm{T}}\bar{a}\right) + \left(b^{\dagger}a - a^{\mathrm{T}}\bar{b}\right)Z \\ &\quad + \left(b^{\dagger}b - a^{\mathrm{T}}\bar{a}\right) = Z^{\dagger}Z - \pi^-; \\ \pi^- - Y^{\dagger}Y &= \left(D^{-1}\right)^{\dagger}(\pi^- - Z^{\dagger}Z)D^{-1}, \end{aligned} \tag{3.2.5}$$

as the conjugation of a positive operator stay positive we get $Y \in \mathcal{D}_\infty$. □

The orbit through $\mathrm{Sp}(\infty)$ of the based point Z_0 defined in (3.2.3) is the space of matrices of the form

$$Z = b\left(\bar{a}^{-1}\right). \tag{3.2.6}$$

We remark that $(a, 0) \in \mathrm{Sp}(\infty)$ iff $a \in U(H^+)$ the unitary group of H. Therefore the orbit of Z_0 can be identified to $\mathrm{Sp}(\infty)/U(H^+)$.

We remark that $a \in u(H^+)$ implies $\bar{a}^{-1} = \bar{a^\dagger} = a^{\mathrm{T}}$; therefore the action of $U(H^+)$ can be describe by:

$$Z \mapsto aZa^{\mathrm{T}} \quad \text{and} \quad \bar{Z}Z \mapsto c\bar{Z}Zc^\dagger,$$

with $c = \bar{a}$; therefore $\det(1 - Z^\dagger Z)$ is invariant under the action of $U(H^+)$.

The Kähler potential on $\mathcal{D}_\infty$ is defined as:

$$K(Z) = -\log\det(1 - Z^\dagger Z) = -\operatorname{trace}\log(1 - Z^\dagger Z),$$

the last equality is intrinsic and does not depend upon a basis; it will be proved using a basis diagonalizing $Z^\dagger Z$. As by (3.2.3) the operator $Z^\dagger Z$ has a trace, the determinant is well defined.

THEOREM 3.2.7. – *Associating to the complex structure of $\mathcal{D}_\infty$ the corresponding $\partial\bar{\partial}$ operator, we have that $\partial\bar{\partial}K$ is invariant under the left action of* $\mathrm{Sp}(\infty)$.

Proof. – Using (3.2.5) we get, assuming that b is a trace class operator and that $a = \text{Identity} +$ trace class operator, that

$$K(Y) - K(Z) = 2\Re \operatorname{trace}\log(\bar{b}Z + \bar{a})$$

as the right-hand side is the real part of a function holomorphic in Z we get that its $\partial\bar{\partial}$ vanishes. The general case is deduced by density. □

THEOREM. – *Using the identification* (3.2.6) *we have*

$$K(Z) = \operatorname{trace}\log(1 + b^\dagger b). \tag{3.2.8}$$

Proof. – We have

$$Z = b\bar{a}^{-1}, \qquad Z^\dagger = \left(\left(a^{\mathrm{T}}\right)^{-1}\right)b^\dagger,$$

therefore

$$\det(1 - Z^\dagger Z) = \det\left(1 - \left(a^{\mathrm{T}}\right)^{-1} b^\dagger b\bar{a}^{-1}\right);$$

then we get

$$= \det\left(\left(a^{\mathrm{T}}\right)^{-1}\left(a^{\mathrm{T}}\bar{a} - b^\dagger b\right)\bar{a}^{-1}\right) = \left(\det\left(a^{\mathrm{T}}\bar{a}\right)\right)^{-1},$$

and using (3.2.1) we get the result. □

4. Kähler potential and Berezinian representation

4.1. Kähler potential

To the map U_g defined in (3.1.7) we associate a map $\Psi : \mathrm{Diff}(S^1) \mapsto \mathcal{D}_\infty$ defined by:

$$\Psi(g) = U_g(Z_0) \quad \text{where } Z_0 \text{ is the based point defined in (3.2.3).} \tag{4.1.0}$$

By abuse of notations we shall denote $K \circ \Psi$ still by K. Given $g \in \mathrm{Diff}(S^1)$, we define the kernel:

$$B_g(z,\zeta) = \sum_{j,k>0} c_{j,k}(g) z^j \zeta^k,$$

(4.1.1)

$$\text{where } c_{j,k}(g) = \frac{1}{2\pi}\int_0^{2\pi} \exp\bigl(-\sqrt{-1}\bigl(j\theta + k g^{-1}(\theta)\bigr)\bigr)\,\mathrm{d}\theta.$$

THEOREM. – *The operator b_g has the following expression*:

$$\bigl(b_g(\phi)\bigr)(z) = \frac{1}{2\pi}\int_{S^1} B_g(z,\zeta)\phi(\zeta)\frac{\mathrm{d}\zeta}{\zeta}, \quad \phi \in H^-. \tag{4.1.2}$$

In the same spirit

$$\bigl(b_g^\dagger b_g(\phi)\bigr)(\bar z) = \frac{1}{2\pi}\int_0^{2\pi} C_g(\bar z,\zeta)\phi(\zeta)\frac{\mathrm{d}\zeta}{\zeta}, \quad C_g(\bar z,\zeta) := \sum_{j,l>0} \bar z^j \zeta^l \sqrt{jl}\sum_k \frac{1}{k}\bar c_{k,j} c_{k,l}. \tag{4.1.3}$$

Then

$$\mathrm{trace}(b_g^\dagger b_g) = \sum_{j,k>0} \frac{j}{k}|c_{j,k}|^2. \tag{4.1.4}$$

We have the invariance properties:

$$K(g) = K\bigl(g^{-1}\bigr), \qquad K(g\gamma) = K(g), \quad \gamma \in S^1, \tag{4.1.5}$$

in particular $K(g)$ defines a function on $\mathrm{Diff}(S^1)/S^1$.

Proof. – We consider the orthonormal basis $\alpha_k \exp(\mathrm{i}k\theta)$, where $\alpha_k = k^{-1/2}$. Consider the coefficients of the operator U_g in this basis: $\hat c_{j,k} = \alpha_j\alpha_k(\exp(\mathrm{i}j\theta)|U_g(\exp -\mathrm{i}k\theta)))$; then

$$\hat c_{j,k} = \frac{\alpha_k}{\alpha_j} c_{j,k}.$$

The operator $b^\dagger b$ has coefficients in this orthonormal basis:

$$\frac{1}{\alpha_j\alpha_l}\sum_k \alpha_k \bar c_{j,k}\alpha_k c_{l,k}.$$

By an integration by part

$$c_{k,l}(g) = \frac{\mathrm{i}}{l}\int \exp\bigl(-\mathrm{i}\bigl(l\theta + k g^{-1}(\theta)\bigr)\bigr)\bigl(-\mathrm{i}k\bigl(g^{-1}\bigr)'(\theta)\bigr)\,\mathrm{d}\theta = \frac{k}{l}c_{l,k}\bigl(g^{-1}\bigr), \tag{4.1.6}$$

the last equality is obtained by making the change of variable $g^{-1}(\theta) = \phi$.

Therefore

$$\sum \frac{l}{k}\bigl|c_{k,l}(g)\bigr|^2 = \sum \frac{k}{l}\bigl|c_{l,k}\bigl(g^{-1}\bigr)\bigr|^2,$$

relation which is the first order term of the expansion nearby $b = 0$ of $\operatorname{trace}(\log(I + b_g^\dagger b_g))$. For $\|b_g\| < 1$ we develop the logarithm in entire series which has for second term:

$$\frac{-1}{2}\operatorname{trace}\bigl((b_g^\dagger b_g)^2\bigr) = \frac{-1}{2} \sum_{j,k,l,m>0} \frac{jk}{lm} \bar{c}_{l,j}(g)\bar{c}_{m,j}(g)c_{l,k}(g)c_{m,k}(g),$$

using (4.1.6) we prove the invariance of this last quantity when $g \mapsto g^{-1}$. Proceeding along the same lines we prove the same equality for all the coefficients of the power series and we get $(4.1.5)_{\mathrm{a}}$ for $\|b_g\| < 1$ and by analytic prolongation for all g.

Finally we want to prove the S^1 invariance.

$$c_{k,l}(g\gamma) = \int_0^{2\pi} \exp\bigl(-\mathrm{i}\bigl(k\theta - l\gamma + lg^{-1}(\theta)\bigr)\bigr)\,\mathrm{d}\theta = \exp(\mathrm{i}l\gamma)c_{k,l}(g),$$

which imply $K(g\gamma) = K(g)$. □

The functional K is relatively settled. It is of interest to know *a priori* some case where it is finite as this is done in the next theorem.

THEOREM 4.1.7. – *Given M and $\varepsilon > 0$, consider the class $\mathcal{D}_{M,\varepsilon}$ of all diffeomorphisms satisfying $\|g\|_{C^3} \leqslant M$ and $g'(\theta) \geqslant \varepsilon > 0$, $\forall \theta \in S^1$. Then there exists $M' < \infty$ which can be computed from M, ε such that $K(g) \leqslant M'$ for all $g \in \mathcal{D}_{M,\varepsilon}$.*

Proof. –

$$c_{k,l}\bigl(g^{-1}\bigr) = \int_0^{2\pi} \exp\bigl(-\mathrm{i}(k+l)\psi_{k,l}(\theta)\bigr)\,\mathrm{d}\theta,$$

where $\psi_{k,l}$ is defined as the convex combination

$$\psi_{k,l}(\theta) = \frac{l}{k+l}\theta + \frac{k}{k+l}g(\theta).$$

If we assume $\varepsilon < 1$ and $M > 1$ then $\psi_{k,l} \in \mathcal{D}_{M,\varepsilon}$. We denote χ the diffeomorphism inverse of $\psi_{k,l}$. Then making the change of variable $\theta = \chi(\theta')$,

$$c_{k,l}(g) = \int \exp\bigl(-\mathrm{i}(k+l)\theta'\bigr)\chi'(\theta')\,\mathrm{d}\theta'.$$

Making a double integration by part

$$|c_{k,l}|(k+l)^2 \leqslant \int \bigl|\chi^{(3)}(\theta')\bigr|\,\mathrm{d}\theta' \leqslant M_2,$$

where M_2 is a constant depending only upon M and ε which comes from classical computation of derivatives of the implicit function χ. Then

$$K(g) \leqslant M_2^2 \sum_{k,l>0} \frac{l}{k}\frac{1}{(k+l)^4} := M_1 < \infty. \qquad \square$$

LEMMA. – *The differential of K is given by*:

$$\mathrm{d}K = \operatorname{trace}(1+b^\dagger b)^{-1}\big((\mathrm{d}b)^\dagger b + b^\dagger \mathrm{d}b\big).$$

Proof. – Denote $c := b^\dagger b$, then

$$\mathrm{d}(c-c_0)^n = \mathrm{d}c(c-c_0)^{n-1} + (c-c_0)\,\mathrm{d}c(c-c_0)^{n-2} + \cdots + (c-c_0)^{n-1}\,\mathrm{d}c,$$

$$\operatorname{trace}\big(\mathrm{d}(c-c_0)^n\big) = \operatorname{trace}\big(n(c-c_0)^{n-1}\,\mathrm{d}c\big).$$

Developping $c \mapsto \log(1+c)$ in Taylor series around the point c_0 we get the result. □

PROPOSITION. – *Denote ∇_s the variation along the plane generated by* $\cos s\theta, \sin s\theta$, *then*:

$$\big(\nabla_s^l c_{k,l}\big)(g) = \frac{-\mathrm{i}l}{2\pi}\int_0^{2\pi} \exp\big(-\mathrm{i}kg(\phi) - \mathrm{i}l\phi\big)\big(e_s\big((0,1)\big)\cos\big(sg(\phi)\big) - e_s\big((1,0)\big)\sin\big(sg(\phi)\big)\big)\,\mathrm{d}\phi. \tag{4.1.8}$$

Proof. – Let $z \in \operatorname{diff}(S^1)$ then denote by ∂_z^l the derivative on the left associated to z, then:

$$\big(\partial_z^l c_{k,l}\big)(g) = \frac{-\mathrm{i}l}{2\pi}\int_0^{2\pi} \exp\big(-\mathrm{i}\big(k\theta + lg^{-1}(\theta)\big)\big)\big(g^{-1}\big)'(\theta)z(\theta)\,\mathrm{d}\theta.$$

Making the change of variables $\phi = g^{-1}(\theta)$, we get the result. □

THEOREM 4.1.9 (Hong and Rajeev [4]). – *The function $6K$ is the Kählerian potential of the left invariant pseudo-Kählerian metric defined on* $\operatorname{Diff}(S^1)$ *by* (1.2.8).

Proof. – Given $h \in \operatorname{diff}(S^1)$ the differentiation on the left

$$\begin{aligned}\partial_h^l c_{j,k} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon|_{\varepsilon=0}}\int_0^{2\pi} \exp\big(-\mathrm{i}l\theta - \mathrm{i}k\big(-\varepsilon h(\theta) + g^{-1}(\theta)\big)\big)\,\mathrm{d}\theta\\ &= \mathrm{i}k\int_0^{2\pi}\exp\big(-\mathrm{i}l\theta - \mathrm{i}kg^{-1}(\theta)\big)h(\theta)\,\mathrm{d}\theta.\end{aligned}$$

We shall prove the identity at the identity element e; then $c_{j,k}(e) = 0$; furthermore $(\partial_h c_{j,k})(e) = 0$ if h is antiholomorphic; therefore

$$\big\langle \partial\bar\partial K, \mathrm{e}^{\mathrm{i}n\theta} \wedge \mathrm{e}^{-\mathrm{i}n\theta}\big\rangle = \sum_{k,l}\frac{l}{k}k^2 1(k+l=n) = \sum_{k+l=n} kl = \frac{1}{6}\big(n^3 - n\big).$$

In order to prove the theorem at every point of $\operatorname{Diff}(S^1)$ we shall proceed using homogeneity argument. On the Siegel infinite-dimensional disk $\mathcal{D}_\infty$ the $(1,1)$ form $\partial\bar\partial K$ is invariant under the left action of symplectic group as shown in (3.2.7); using (4.1.0) we have $\Psi(g_0 g) = U_{g_0}(\Psi(g))$. □

4.2. Berezinian representation

We recall the notion of *Berezinian representation* on a finite-dimensional complex symmetric space $M := G/H$ admitting a Kähler potential K. Then Berezin introduced on the trivial line bundle over M the metric

$$\exp(-cK) \text{ and the corresponding measure } \nu_c := \exp(-cK)\,\mathrm{dm},$$

where dm is the Riemannian volume of M; for c large enough ν_c is of finite mass. Berezin constructed a unitary representation σ on $\mathcal{H}L^2_{\nu_c}(M)$ by

$$\sigma(v)\Phi = L_v\Phi + \langle L_v, \Omega_2\rangle\Phi, \quad v \in \mathcal{G}, \text{ the Lie algebra of } G, \tag{4.2.1}$$

where Ω_2 is a 1-differential form defined on M, of type $(0,1)$, which will uniquely determined below in (4.2.3). The unitarity condition following the line of the proof of (2.4.1) can be written as:

$$\mathrm{div}_{\nu_c}(L_v) = 2\Re\bigl(\langle L_v, \Omega_2\rangle\bigr). \tag{4.2.2}$$

Then it is possible to show that relation (4.2.2) implies that

$$\Omega_2 = \bar\partial K. \tag{4.2.3}$$

THEOREM 4.2.4. – *The Neretin representation is Berezinian*; *more precisely the differential form Ω_1 constructed in* (2.5.1) *satifies the identity*

$$6\bar\partial K_1 = \Omega_1, \tag{4.2.5}$$

where $K = K_1 \circ \pi$, $\pi : \mathrm{Diff}(S^1) \mapsto \mathrm{Diff}(S^1)/S^1$.

Proof. – Consider the $(0,1)$ form $\Gamma = -6\bar\partial K_1 + \Omega_1$; then $\partial\Gamma = \bar\partial\Gamma = 0$ which implies $\mathrm{d}\Gamma = 0$; therefore as $\mathcal{M}$ is contractible, there exists a function χ such that $\Gamma = \mathrm{d}\chi$. As Γ is of type $(0,1)$ we must have $\chi = \bar h$ with h holomorphic on $\mathcal{M}$.

Furthermore by (4.1.5), we have $\bar\partial K_1$ is invariant under the infinitesimal action of L_0.

The differential form $\Omega = \sum \psi_{-s}(P_s \circ \pi)$ has its coefficients $P_s \circ \pi$ invariant under the right action of e_0^r. The diffferential forms ψ_k are right invariant and in particular invariant under the action of e_0^r; therefore Ω_1 is invariant under L_0. Finally Γ is invariant under L_0. Using Cartan formula (where we denote by $i(*)$ the interior product),

$$0 = \mathrm{d}\bigl(i(L_0)(\Gamma)\bigr) + i(L_0)\,\mathrm{d}\Gamma \quad \text{which implies that } \langle L_0, \mathrm{d}h\rangle = \text{constant}. \tag{4.2.6}$$

We expand in Taylor series the holomorphic function h nearby the origin and we split this expansion in polynomials homogeneous in weight:

$$h = \sum_{s \geqslant 0} Q_s, \quad \text{weight}(Q_s) = s;$$

then

$$\langle L_0, \mathrm{d}h\rangle = \sum_{s \geqslant 0} s Q_s \tag{4.2.7}$$

combining (4.2.6) with (4.2.7) we obtain $Q_s = 0$ for $s > 0$. $\square$

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Appendix

We prove identity (2.1.4) of Part I. As it is not *a priori* known that Neretin construction gives rise to a representation, we give a direct proof of this fact (Section A.6 and A.7). With the embedding $\mathcal{M} \mapsto C^N$ which sends $f(z) = z(1 + \sum_{n=1}^{+\infty} c_n z^n)$ to $(c_n)_{n\geqslant 1}$, following Kirillov, we express the vector fields $(L_k)_{k\in Z}$ in terms of the $(c_i)_{i\geqslant 1}$. With this approach, we compute in Section A.4 the components of the L_k as homogeneous polynomials in the $(c_i)_{i\geqslant 1}$ and obtain generating functions for these polynomials. In Section A.5, we deduce more asymptotic expansions related to the operators $(L_k)_{k\in Z}$ and to the representation. Moreover, in Section A.7, we calculate the action of L_{-k}, $k \geqslant 0$, on the Neretin polynomials.

A.1. Polynomials associated to a univalent function

Consider:

$$f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) = z + \sum_{n=1}^{\infty} c_n z^{n+1}. \tag{A.1.0$_1$}$$

We have

$$zf'(z) = z + \sum_{n=1}^{\infty} (n+1) c_n z^{n+1}. \tag{A.1.0$_2$}$$

For $n \geqslant 0,\ k \in Z,\ j \in Z$, we consider the homogeneous polynomials P_n^{n+k}, Q_n^j in the variables $(c_i)_{i\geqslant 1}$ defined by:

$$z^2\left(\frac{f'(z)}{f(z)}\right)^2\left(\frac{f(z)}{z}\right)^k = 1 + \sum_{n\geqslant 1} P_n^{n+k} z^n \quad \text{and} \quad \frac{f(z)^{2+j}}{z^{2+j}} = \sum_{n\geqslant 0} Q_n^j z^n \tag{A.1.1}$$

then

$$\left(z\frac{f'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^k = 1 + \sum_{n\geqslant 1} P_n^k \bigl(f(z)\bigr)^n. \tag{A.1.2}$$

We obtain (A.1.2) after making the change of variable $\xi = f(z)$ with the function $h(\xi) = (z\frac{f'}{f})(\frac{f(z)}{z})^k$ in the integral contour $\frac{1}{2i\pi}\int_\gamma h(\xi)\frac{d\xi}{\xi^{j+1}}$.

If $k \geqslant 0$, (A.1.2) can be rewritten as:

$$z^{1-k} f'(z) = \sum_{j=0}^{\infty} P_{1+j+k}^k f(z)^{j+2} + \sum_{j=0}^{k} P_{k-j}^k f(z)^{1-j}. \tag{A.1.3$_1$}$$

We denote

$$\phi_k = -\sum_{j=0}^{k} P_{k-j}^k f(z)^{1-j} = -\left(\frac{1}{f^{k-j}} + \cdots + P_k^k f\right). \tag{A.1.4}$$

If we write (A.1.2) with $-k$, (A.1.2) yields

$$z^{1+k} f'(z) = \sum_{j \geqslant k-1}^{\infty} P_{1+j-k}^{-k} f(z)^{j+2}. \tag{A.1.3$_2$}$$

If we expand $f(z)^j$ in sum of powers of z in (A.1.2) or (A.1.3), then match the coefficients of z^n, we get

$$\sum_{j=0}^{k+n} P_j^k Q_{n+k-j}^{j-1-k} = (n+k+1)c_{n+k}. \tag{A.1.5}$$

The polynomials Q_n^j are given by the expansion:

$$\begin{aligned}\frac{1}{f(z)^p} = \frac{1}{z^p}\Bigg[&1 - pc_1 z + \left(\frac{p(p+1)}{2}c_1^2 - pc_2\right)z^2 \\ &+ \left(p(p+1)c_1c_2 - pc_3 - \frac{p(p+1)(p+2)}{3!}c_1^3\right)z^3 + V_4 z^4 + V_5 z^5 + \cdots\Bigg],\end{aligned} \tag{A.1.6}$$

where

$$V_4 = p(p+1)c_1c_3 - pc_4 + \frac{p(p+1)}{2}c_2^2 - \frac{p(p+1)(p+2)}{2}c_1^2c_2 + \frac{p(p+1)(p+2)(p+3)}{4!}c_1^4,$$

$$\begin{aligned}V_5 = {}& p(p+1)c_2c_3 + p(p+1)c_1c_4 - pc_5 - \frac{p(p+1)(p+2)}{2}c_1c_2^2 - \frac{p(p+1)(p+2)}{2}c_1^2c_3 \\ &+ \frac{p(p+1)(p+2)(p+3)}{3!}c_1^3c_2 - \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}c_1^5.\end{aligned}$$

We compute the P_n^k with the expansion of $z^2(\frac{f'(z)}{f(z)})^2(\frac{f(z)}{z})^j$ in powers of z, we obtain:

$$\begin{aligned}P_0^p &= 1, \\ P_1^p &= (p+1)c_1, \\ P_2^p &= (p+2)c_2 + \frac{p^2-p-4}{2}c_1^2, \\ P_3^p &= (p+3)c_3 + \left(p^2-p-8\right)c_1c_2 + \frac{(p-3)(p-5)(p+2)}{6}c_1^3, \\ P_4^p &= (p+4)c_4 + \left(p^2-p-14\right)c_1c_3 + \frac{p^2-p-12}{2}c_2^2 \\ &\quad + \frac{(p-6)(p^2-p-10)}{2}c_1^2c_2 + \frac{(p-6)(p-7)(p^2-p-8)}{24}c_1^4, \\ P_5^p &= (p+5)c_5 + \left(p^2-p-22\right)c_1c_4 + \frac{(p-7)(p^2-p-14)}{2}c_1c_2^2 \\ &\quad + \frac{(p-7)(p^2-p-16)}{2}c_1^2c_3 + \frac{(p-7)(p-8)(p+3)(p-4)}{6}c_1^3c_2\end{aligned} \tag{A.1.7}$$

$$+\frac{(p-7)(p-8)(p-9)(p^2-p-10)}{120}c_1^5+\bigl(p^2-p-18\bigr)c_2c_3.$$

For the $(\phi_k)_{k\geqslant 0}$, it gives:

$$\begin{aligned}
&\phi_0=-f,\\
&\phi_1=-1-2c_1f,\\
&\phi_2=-\frac{1}{f}-3c_1-\bigl(4c_2-c_1^2\bigr)f,\\
&\phi_3=-\frac{1}{f^2}-4c_1\frac{1}{f}-\bigl(c_1^2+5c_2\bigr)-P_3^3f,\\
&\phi_4=-\frac{1}{f^3}-5c_1\frac{1}{f^2}-\bigl(4c_1^2+6c_2\bigr)\frac{1}{f}-P_3^4-P_4^4f,
\end{aligned}\tag{A.1.8}$$

$$\begin{aligned}
&P_0^0=1, && &&\\
&P_1^1=2c_1, &&P_0^1=1, &&\\
&P_2^2=4c_2-c_1^2, &&P_1^2=3c_1, &&P_0^2=1,\\
&P_3^3=6c_3-2c_1c_2, &&P_2^3=c_1^2+5c_2, &&P_1^3=4c_1,\\
&P_4^4=8c_4-2c_1c_3-2c_1^2c_2+c_1^4, &&P_3^4=-c_1^3+4c_1c_2+7c_3, &&P_2^4=4c_1^2+6c_2,\\
&P_5^5=10c_5-2c_1c_4-6c_1c_2^2 && &&\\
&\qquad -4c_1^2c_3+8c_1^3c_2-2c_1^5+c_2c_3, && &&
\end{aligned}\tag{A.1.9}$$

A.2. The algebra of operators L_k

For a function $\phi(z,c_1,c_2,\ldots,c_n,\ldots)$, we denote by $\partial_n\phi=\frac{\partial}{\partial c_n}\phi$ the partial derivative of ϕ with respect to the variable c_n and by $'=\partial_z$ the derivative with respect to z. For $n\geqslant 1$,

$$\partial_n\bigl[f(z)\bigr]=z^{n+1},\qquad \partial_1\bigl[f'(z)\bigr]=2z.\tag{A.2.1}$$

In the same way $\partial_n[zf'(z)]=(n+1)z^{n+1}$, then $\partial_n[z^{k+1}f'(z)]=(n+1)z^{n+k+1}$ and for any $n\geqslant 1$,

$$\partial_n\partial_z=\partial_z\partial_n.\tag{A.2.2}$$

For any $n\geqslant 2$, $f(z)$ is satisfies:

$$\partial_n\partial_z\bigl[f(z)\bigr]=(n+1)\partial_{n-1}\bigl[f(z)\bigr].\tag{A.2.3}$$

Remark. – Because of (A.2.3) and the second relation in (2.1), we put $\partial_0=\frac{1}{2}\partial_1\partial_z$ and $\partial_{-1}=\partial_0\partial_z$. We have $\partial_0[f(z)]=z$ and $\partial_{-1}[f(z)]=1$. We see that $\partial_0\partial_z=\partial_z\partial_0$ and $\partial_{-1}\partial_z=\partial_z\partial_{-1}$. Moreover $\partial_{-1}\partial_z[f(z)]=0$.

For $k\geqslant 1$, we define the first-order differential operators:

$$L_k=\partial_k+\sum_{n=1}^{\infty}(n+1)c_n\partial_{n+k}.\tag{A.2.4}$$

From $(A.1.0)_2$ and (A.2.4), we see that $f(z)$ is solution of the partial differential equation:

$$L_k\big(f(z)\big) = z^{k+1} f'(z) \quad \text{for } k \geqslant 1. \tag{A.2.5}$$

Since $\partial_n \partial_z = \partial_z \partial_n$ for any $n \geqslant 1$ and since the coefficients of L_k do not depend upon z, for any $k \geqslant 1$, we have

$$L_k \partial_z = \partial_z L_k. \tag{A.2.6}$$

Thus, for any integer p, and $k \geqslant 1$,

$$\begin{aligned}
&L_k\left(\frac{z^p}{f^p}\right) = z^{k+1} z^p \left(\frac{1}{f^p}\right)' = z^{k+1}\left(\frac{z^p}{f^p}\right)' - p z^k \frac{z^p}{f^p},\\
&L_k\left(z^p\left(\frac{f'}{f}\right)^p\right) = z^{k+1}\left(z^p\left(\frac{f'}{f}\right)^p\right)' + kpz^k z^p \left(\frac{f'}{f}\right)^p,\\
&L_k\left[z^2\left(\frac{f'}{f}\right)^2\left(\frac{f}{z}\right)^p\right] = z^{k+1}\left(z^2\left(\frac{f'}{f}\right)^2\left(\frac{f}{z}\right)^p\right)' + (2k+p) z^2 \left(\frac{f'}{f}\right)^2\left(\frac{f}{z}\right)^p.
\end{aligned} \tag{A.2.7}$$

For any $k \geqslant 1$, $p \geqslant 1$, we have

$$[L_k, L_p] = (k-p) L_{k+p}. \tag{A.2.8}$$

A.3. Calculation of asymptotic expansions with recurrence formulas

$$\frac{1}{f^p} = \frac{1}{z^p}\left(1 + \sum_{n \geqslant 1} V_n^p z^n\right), \tag{A.3.1}$$

where V_n^p are homogeneous polynomials in the $(c_i)_{i \geqslant 1}$ and n, p are indices. Given the asymptotic expansion of $1/f^p$, we compute easily the asymptotic expansion of $1/f^{p+1}$ since

$$\frac{\partial}{\partial c_i}\left(\frac{1}{f^p}\right) = -p \frac{z^{i+1}}{f^{p+1}}.$$

In this way, we obtain $\frac{\partial}{\partial c_i} V_n^p = 0$ if $n < i$ and $V_n^{p+1} = -\frac{1}{p} \frac{\partial}{\partial c_i} V_{n+i}^p$ if $n \geqslant 0$.

The operators $(L_k)_{k \geqslant 1}$ allow to calculate the asymptotic expansion of $1/f^p$ independently of that of $1/f^{p-1}$. We replace $1/f^p$ by (A.3.1) in the first equation (A.2.7). We obtain:

$$L_k(V_n) = \begin{cases} 0 & \text{if } n < k, \\ (n-k-p) V_{n-k} & \text{if } n \geqslant k. \end{cases} \tag{A.3.2}$$

We have $V_1 = -pc_1$, we determine $V_2 = \alpha c_1^2 + \beta c_2$ with the conditions $L_2(V_2) = -p$ and $L_1(V_2) = (1-p)V_1$. We find $V_2 = \frac{p(p+1)}{2} c_1^2 - pc_2$. In the same way, we calculate $V_3 = p(p+1)c_1c_2 - pc_3 - \frac{p(p+1)(p+2)}{6} c_1^3 \ldots.$

To compute the polynomials P_n^P, $n \geqslant 0$, $p \in Z$, we replace in the third equation (A.2.7) the function $z^2(\frac{f'}{f})^2(\frac{f}{z})^p$ by its asymptotic expansion in powers of z and we match the coefficients, we obtain for $k \geqslant 1$:

$$L_k\big(P_n^j\big) = \begin{cases} (k+j) P_{n-k}^{j-k} & \text{if } n \geqslant k, \\ 0 & \text{if } n < k. \end{cases} \tag{A.3.3}$$

With the homogeneity conditions on the polynomials, we deduce from $P_0^p = 1$ the other polynomials $P_1^p, P_2^p, \ldots$.

A.4. The operators L_{-p} for $p \geqslant 0$

We aim to define sequence of operators L_{-p} for $p \geqslant 0$ such that L_{-p} is of the form:

$$L_{-p} = \sum_{n=1}^{\infty} A_n^p \partial_n, \tag{A.4.1}$$

where A_n^p are functions of the $(c_i)_{i\geqslant 1}$ and depend upon p.

Thus, we shall have:

$$L_{-p}\big(f(z)\big) = \sum_{n=1}^{\infty} A_n^p \partial_n\big[f(z)\big] = \sum_{n=1}^{\infty} A_n^p z^{n+1} \tag{A.4.2}$$

and for $p \geqslant 0$,

$$A_n^p = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} \frac{L_{-p}[f(z)]}{z^{n+2}}\,\mathrm{d}z = L_{-p}(c_n). \tag{A.4.3}$$

Expanding (A.4.2) in powers of $f(z)$, we see that $L_{-p}(f(z))$ is of the form:

$$L_{-p}\big(f(z)\big) = \sum_{n\geqslant 0} B_n \big(f(z)\big)^{n+2}, \tag{A.4.4}$$

where B_n is a homogeneous polynomial in the $(c_i)_{i\geqslant 1}$.

PROPOSITION. – *For* $p \geqslant 0$, *assume that* (*see* (A.1.3)$_1$)

$$L_{-p}\big(f(z)\big) = \sum_{j=0}^{\infty} P_{1+j+p}^p f(z)^{j+2} \tag*{(A.4.5)$_1$}$$

then A_n^p *is uniquely determined in* (A.4.1); *we have*

$$\begin{aligned} L_{-p} = {} & P_{p+1}^p \frac{\partial}{\partial c_1} + \big(2c_1 P_{p+1}^p + P_{p+2}^p\big)\frac{\partial}{\partial c_2} \\ & + \big(\big(c_1^2 + 2c_2\big)P_{p+1}^p + 3c_1 P_{p+2}^p + P_{p+3}^p\big)\frac{\partial}{\partial c_3} \\ & + \big((2c_1c_2 + 2c_3)P_{p+1}^p + \big(3c_1^2 + 3c_2\big)P_{p+2}^p + 4c_1 P_{p+3}^p + P_{p+4}^p\big)\frac{\partial}{\partial c_4} + \cdots. \end{aligned} \tag*{(A.4.5)$_2$}$$

For any $j \geqslant 1$, *we have*

$$[L_j, L_{-p}] = (j+p)L_{j-p}. \tag{A.4.6}$$

Proof. – We determine A_n^p in order that:

$$L_{-p}\big(f(z)\big) = \sum_{j=0}^{\infty} P_{1+j+p}^p f(z)^{j+2} = z^{1-p} f'(z) - \sum_{j=0}^{p} P_{p-j}^p f(z)^{1-j}$$

$$
\begin{aligned}
(\mathrm{A}.4.5)_3 \qquad &= z^{1-p} f'(z) - \frac{1}{f^{p-1}} - \left(\frac{P_1^p}{f^{p-2}} + \frac{P_2^p}{f^{p-3}} + \cdots + P_p^p f \right) \\
&= z^{1-p} f'(z) + \phi_p.
\end{aligned}
$$

For that purpose, we replace $f(z)^j$ in $(\mathrm{A}.4.5)_3$ by its asymptotic expansion in powers of z and we find:

$$
\begin{aligned}
A_n^p = {}& (n+2p)c_{n+p} + \text{homogeneous polynomial of degree } n+p \\
& \text{in the variables } c_1, c_2, \ldots, c_{n+p-1}.
\end{aligned}
\tag{A.4.7}
$$

If we consider the polynomials Q_n^j defined by (A.1.1), then

$$
\begin{aligned}
A_n^p &= (n+p+1)c_{n+p} - Q_{n+p}^{-1-p} - \sum_{j=1}^{p} P_j^p Q_{n+p-j}^{j-1-p} \\
&= Q_{n-1}^0 P_{p+1}^p + Q_{n-2}^1 P_{p+2}^p + \cdots + Q_0^{n-1} P_{p+n}^p,
\end{aligned}
\tag{A.4.8}
$$

where the second equality in (A.4.8) comes from (A.1.5). Let $p \geqslant 0$, with the choice (A.4.7) of A_n^p, we prove (A.4.6) as follows: Consider two operators $J_1 = \sum_{n\geqslant 1} B_n^1 \partial_n$ and $J_2 = \sum_{n\geqslant 1} B_n^2 \partial_n$, the condition $J_1[f(z)] = J_2[f(z)]$ gives $\sum_{n\geqslant 1} B_n^1 z^{n+1} = \sum_{n\geqslant 1} B_n^2 z^{n+1}$, thus by the unicity of the asymptotic expansion, we deduce $B_n^1 = B_n^2$ for any $n \geqslant 1$ and $J_1 = J_2$. Consequently, to prove (A.4.6), it is enough to verify:

$$
[L_j, L_{-p}]\big(f(z)\big) = (j+p) L_{j-p}\big(f(z)\big).
\tag{A.4.9}
$$

We compute the left side of (A.4.9),

$$
L_j\big[L_{-k}\big(f(z)\big)\big] = L_j[z^{1-k} f'(z) + \phi_k] = z^{1-k}\big(z^{j+1} f'(z)\big)' + L_j[\phi_k].
$$

We have

$$
\begin{aligned}
L_j[\phi_k] &= \frac{\partial \phi_k}{\partial f} L_j\big[f(z)\big] - \sum_{s=0}^{k} L_j\big[P_s^k\big] \frac{1}{f^{k-s-1}} \\
&= z^{j+1}\phi_k' - \sum_{s=j}^{k} L_j\big[P_s^k\big] \frac{1}{f^{k-s-1}}.
\end{aligned}
\tag{A.4.10}
$$

On the other hand

$$
L_{-k}\big[L_j\big(f(z)\big)\big] = z^{1+j}\big(L_{-k}\big[f(z)\big]\big)' = z^{1+j}\big(z^{1-k} f'(z)\big)' + z^{1+j}\phi_k'.
\tag{A.4.11}
$$

Thus, using (A.3.3), we obtain:

$$
\begin{aligned}
L_j\big[L_{-k}\big(f(z)\big)\big] - L_{-k}\big[L_j\big(f(z)\big)\big] &= (k+j) z^{1+j-k} f'(z) - \sum_{s=j}^{k} L_j\big[P_s^k\big] \frac{1}{f^{k-s-1}} \\
&= (k+j) z^{1+j-k} f'(z) + (k+j)\phi_{k-j} = (k+j) L_{j-k}\big[f(z)\big].
\end{aligned}
$$

This proves (A.4.9).

We see that the knowledge of L_0 and the condition, for any $k \geqslant 0$,

$$[L_k, L_{-p}] = (k+p)L_{k-p}$$

determines completly the coefficients A_k^p. We have:

$$L_k L_{-p} - L_{-p} L_k = L_k\big(A_1^p\big)\frac{\partial}{\partial c_1} + L_k\big(A_2^p\big)\frac{\partial}{\partial c_2} + \cdots - 2A_1^p \frac{\partial}{\partial c_{1+k}} - 3A_2^p \frac{\partial}{\partial c_{2+k}} + \cdots$$
$$= \begin{cases} (k+p)A_1^{p-k}\frac{\partial}{\partial c_1} + (k+p)A_2^{p-k}\frac{\partial}{\partial c_2} + \cdots & \text{if } p \geqslant k, \\ (k+p)\big(\frac{\partial}{\partial c_{k-p}} + 2c_1\frac{\partial}{\partial c_{k-p+1}} + \cdots + (n+1)c_n\frac{\partial}{\partial c_{k-p+n}}\cdots\big) & \text{if } k > p. \end{cases}$$

By identifying the coefficients of $\frac{\partial}{\partial c_n}$, we obtain:

$$L_k\big(A_n^p\big) = \begin{cases} (n-k+1)A_{n-k}^p + (k+p)A_n^{p-k} & \text{if } p \geqslant k, \\ (n-k+1)A_{n-k}^p + (k+p)(n-k+p+1)c_{n-k+p} & \text{if } p < k \end{cases} \tag{A.4.12}$$

with the convention $A_{-j} = 0$ if $j \geqslant 0$, $c_0 = 1$, $c_{-j} = 0$ if $j > 0$. In this way, we find (compare with $(\text{A.4.5})_2$):

$$\begin{aligned}
L_0 &= c_1\frac{\partial}{\partial c_1} + 2c_2\frac{\partial}{\partial c_2} + 3c_3\frac{\partial}{\partial c_3} + \cdots + nc_n\frac{\partial}{\partial c_n} + \cdots, \\
L_{-1} &= \big(3c_2 - 2c_1^2\big)\frac{\partial}{\partial c_1} + (4c_3 - 2c_1c_2)\frac{\partial}{\partial c_2} + \cdots + \big((n+2)c_{n+1} - 2c_1c_n\big)\frac{\partial}{\partial c_n} + \cdots, \\
L_{-2} &= \big(5c_3 + 2c_1^3 - 6c_1c_2\big)\frac{\partial}{\partial c_1} + \big(6c_4 - 5c_2^2 - 2c_1c_3 + 4c_1^2c_2 - c_1^4\big)\frac{\partial}{\partial c_2} \\
&\quad + \big(7c_5 - 6c_2c_3 + 3c_1c_2^2 - 2c_1c_4 + 4c_1^2c_3 - 4c_1^3c_2 + c_1^5\big)\frac{\partial}{\partial c_3} + \cdots, \\
L_{-3} &= \big(7c_4 - 3c_2^2 - 8c_1c_3 + 6c_1^2c_2 - c_1^4\big)\frac{\partial}{\partial c_1} \\
&\quad + \big(2c_1^5 - 8c_1^3c_2 + 4c_1^2c_3 - 2c_1c_4 + 10c_1c_2^2 - 12c_2c_3 + 8c_5\big)\frac{\partial}{\partial c_2} + \cdots, \\
L_{-4} &= \big(9c_5 - 6c_2c_3 - 10c_1c_4 + 3c_1c_2^2 + 6c_1^2c_3 - c_1^5\big)\frac{\partial}{\partial c_1} + \cdots.
\end{aligned}$$
(A.4.13)

PROPOSITION. – *Let*

$$\begin{aligned} L_k &= \partial_k + \sum_{n=1}^{\infty}(n+1)c_n\partial_{n+k} \quad \textit{for } k \geqslant 1, \\ L_{-k} &= \sum_{n\geqslant 1} A_n^k \partial_n \quad \textit{for } k \geqslant 0, \end{aligned} \tag{A.4.14}$$

where A_n^k *the homogeneous polynomial of degree* $n+k$ *in the* $(c_i)_{i\geqslant 1}$ *given by* (A.4.7)*, then for any* $m, n \in Z$:

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{A.4.15}$$

Proof. – We did the proof for $m \in Z$, $n \geqslant 1$. We extend to all values of $n \in Z$ using the Jacobi identity on the vector fields L_k, and a recursion argument.

For the polynomials $A_n^k,\ k \geqslant 1,\ n \geqslant 0$, we have generating functions:

PROPOSITION. – *For $j \geqslant 1$ and $n \geqslant 0$, let $L_{-n} = \sum_{j \geqslant 1} A_j^n \frac{\partial}{\partial c_n}$ where A_j^n are the polynomials defined by* (A.4.7). *We consider the polynomials Q_n^j defined by* (A.1.1), *then*

$$z^{n+2}\frac{(f')^2}{f^2}\left(\frac{Q_{n-1}^0}{f}+\frac{Q_{n-2}^1}{f^2}+\frac{Q_{n-3}^2}{f^3}+\cdots+\frac{nc_1}{f^{n-1}}+\frac{1}{f^n}\right) = 1+2c_1z+3c_2z^2+4c_3z^3+\cdots+nc_{n-1}z^{n-1}+\sum_{p\geqslant 0}A_n^p z^{p+n}. \tag{A.4.16}$$

In particular, we have

$$z^3\frac{(f')^2}{f^3} = 1+\sum_{n\geqslant 1}A_1^{n-1}z^n,$$

$$z^4\frac{(f')^2}{f^2}\left(\frac{2c_1}{f}+\frac{1}{f^2}\right) = 1+2c_1z+\sum_{p\geqslant 0}A_2^p z^{p+2},$$

$$z^5\frac{(f')^2}{f^2}\left(\frac{c_1^2+2c_2}{f}+\frac{3c_1}{f^2}+\frac{1}{f^3}\right) = 1+2c_1z+3c_2z^2+\sum_{p\geqslant 0}A_3^p z^{p+3},$$

$$z^6\frac{(f')^2}{f^2}\left(\frac{2c_3+2c_1c_2}{f}+\frac{3c_2+3c_1^2}{f^2}+\frac{4c_1}{f^3}+\frac{1}{f^4}\right) = 1+2c_1z+3c_2z^2+4c_3z^3+\sum_{p\geqslant 0}A_4^p z^{p+4}, \tag{A.4.17}$$

$$z^7\frac{(f')^2}{f^2}\left(\frac{c_2^2+2c_4+2c_1c_3}{f}+\frac{3c_3+6c_1c_2+c_1^3}{f^2}+\frac{4c_2+6c_1^2}{f^3}+\frac{5c_1}{f^4}+\frac{1}{f^5}\right) = 1+2c_1z+3c_2z^2+4c_3z^3+5c_4z^4+\sum_{p\geqslant 0}A_5^p z^{p+5}.$$

Proof. – We put, for $n \geqslant 1$,

$$\psi_n = z^{n+2}\frac{(f')^2}{f^2}\left(\frac{Q_{n-1}^0}{f}+\frac{Q_{n-2}^1}{f^2}+\frac{Q_{n-3}^2}{f^3}+\cdots+\frac{nc_1}{f^{n-1}}+\frac{1}{f^n}\right)$$

and as convention $\psi_{-n} = 0$, if $n \geqslant 0$. For $s \geqslant 1$, we verify that:

$$L_s(\psi_n) = z^{1+s}\psi_n' + (2s-n)z^s\psi_n + (n-s+1)z^s\psi_{n-s}. \tag{A.4.18}$$

On the other hand, the asymptotic expansion of ψ_n is of the form:

$$\psi_n = 1+\sum_{p\geqslant 1}\beta_n^p z^p, \tag{A.4.19}$$

where β_n^p is a homogeneous polynomial of degree p. The system of partial differential equations (A.4.18) determines completely ψ_n when ψ_n has the form (A.4.19) and ψ_1 is known.

Thus, to obtain (A.4.16), it is enough to prove that

$$\widetilde{\psi_n} = 1 + 2c_1 z + 3c_2 z^2 + 4c_3 z^3 + \cdots + nc_{n-1} z^{n-1} + \sum_{p \geqslant 0} A_n^p z^{p+n}$$

satisfies also (A.4.18). This comes from (A.4.12).

Remark. – We can also deduce (A.4.16) directly form the identity (A.1.5)–(A.4.8). The left side of (A.4.16) can be expressed with a integral contour.

PROPOSITION. – *We put*:

$$I_n = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} \frac{f(t)^2}{t^2} \frac{1}{(f(t) - f(z))} \frac{\mathrm{d}t}{t^n} \tag{A.4.20}$$

then

$$\begin{aligned} \psi_n &= z^{n+2} \frac{(f')^2}{f^2} \left(\frac{Q^0_{n-1}}{f} + \frac{Q^1_{n-2}}{f^2} + \frac{Q^2_{n-3}}{f^3} + \cdots + \frac{nc_1}{f^{n-1}} + \frac{1}{f^n} \right) \\ &= f'(z) - z^{n+2} \frac{(f')^2}{f^2} I_n. \end{aligned} \tag{A.4.21}$$

Proof. – A residue calculus of the integral I_k.

A.5. Asymptotic expansions related to $[L_{-k}, L_{-p}] = (p-k)L_{-(p+k)}$

The condition

$$[L_{-k}, L_{-p}] = (p-k)L_{-(p+k)} \tag{A.5.1}$$

for $k \geqslant 0$, $p \geqslant 0$, is equivalent to

$$\begin{aligned} &\phi_k \frac{\partial \phi_p}{\partial f} - \phi_p \frac{\partial \phi_k}{\partial f} \\ &\quad = \sum_{j=0}^{p} L_{-k}\big(P_j^p\big) \frac{1}{f^{p-j-1}} - \sum_{j=0}^{k} L_{-p}\big(P_j^k\big) \frac{1}{f^{k-j-1}} - (p-k) \sum_{r=0}^{p+k} P_r^{p+k} \frac{1}{f^{p+k-r-1}}. \end{aligned} \tag{A.5.2}$$

We put as a convention $P_j^p = 0$ if $j < 0$ or if $j > p$. Then we can write the sums in (A.5.2) as series, and matching equal powers of $1/f$, we obtain the condition for $0 \leqslant r \leqslant p+k$,

$$-\sum_{j,s,j+s=r} (p-k+j-s) P_j^k P_s^p = L_{-k}\big(P_{r-k}^p\big) - L_{-p}\big(P_{r-p}^k\big) - (p-k) P_r^{p+k} \tag{A.5.3}$$

with $0 \leqslant j \leqslant k$ and $0 \leqslant s \leqslant p$.

For $p + k = r$, (A.5.3) reduces to

$$L_{-k}\big(P_p^p\big) - L_{-p}\big(P_k^k\big) = (p-k) P_{p+k}^{p+k}. \tag{A.5.4}$$

To discuss (A.5.3)–(A.5.4), we shall need the following lemma:

LEMMA. – *Let $k \in Z$ and let $h_k = z^2(\frac{f'}{f})^2(\frac{f}{z})^k = 1 + \sum_{n\geqslant 1} P_n^{n+k} z^n$, we have for $j \geqslant 0$,*

$$L_{-j}(h_k) = z^{1-j}h_k' + (k-2j)z^{-j}h_k + \left[(k-2)\frac{\phi_j}{f} + 2\frac{\partial\phi_j}{\partial f}\right]h_k. \tag{A.5.5}$$

Moreover

$$\psi_j = \left[(k-2)\frac{\phi_j}{f} + 2\frac{\partial\phi_j}{\partial f}\right]h_k = \sum_{n\geqslant -j} B_n^j z^n$$

satisfies for $s \geqslant 1$:

$$L_s(\psi_j) = z^{1+s}\psi_j' + (k+2s)z^s\psi_j + (s+j)\psi_{j-s} \tag{A.5.6}$$

with the convention $\psi_n = 0$ if $n < 0$. The polynomials B_n^j satisfy

$$L_s\big(B_n^j\big) = (n+s+k)B_{n-s}^j + (s+j)B_n^{j-s}. \tag{A.5.7}$$

For (A.5.5) and (A.5.6), the proof is a verification. We deduce (A.5.7) from (A.5.6). (A.5.4) is a consequence of the following proposition:

PROPOSITION. – *Let $h = z^2(\frac{f'}{f})^2$, then for any $j \geqslant 1$,*

$$\begin{aligned} 2h\left(\frac{\partial\phi_j}{\partial f} - \frac{\phi_j}{f}\right) &= 2z^2\left(\frac{f'}{f}\right)^2\left(\frac{\partial\phi_j}{\partial f} - \frac{\phi_j}{f}\right) \\ &= 2z^2\left(\frac{f'}{f}\right)^2\sum_{0\leqslant s\leqslant j}\frac{(j-s)P_s^j}{f^{j-s}} = \frac{L_j(P_j^j)}{z^{j-1}} + \frac{L_{j-1}(P_j^j)}{z^{j-1}} + \cdots + \sum_{n\geqslant 0} L_{-n}(P_j^j)z^n. \end{aligned} \tag{A.5.8}$$

Remark that we can also express (A.5.8) as:

$$L_{-n}\big(P_j^j\big) = \frac{1}{2\mathrm{i}\pi}\int_{\partial D} 2z^2\left(\frac{f'}{f}\right)^2\left(\frac{\partial\phi_j}{\partial f} - \frac{\phi_j}{f}\right)\frac{\mathrm{d}t}{t^{n+1}}. \tag{A.5.9}$$

Proof. – From (A.5.5) with $k = 0$,

$$L_{-j}(h) = z^{1-j}h' - 2jz^{-j}h - 2\left[\frac{\phi_j}{f} - \frac{\partial\phi_j}{\partial f}\right]h \tag{A.5.10}$$

thus

$$\begin{aligned} \psi_j &= 2\left[\frac{\partial\phi_j}{\partial f} - \frac{\phi_j}{f}\right]h = 2h\left(\frac{j}{f^j} + \frac{(j-1)P_1^j}{f^{j-1}} + \cdots + \frac{(j-n)P_n^j}{f^{j-n}} + \cdots + \frac{2P_{j-2}^j}{f^2}\right) \\ &= L_{-j}(h) - z^{1-j}h' + 2jz^{-j}h = \frac{H_{-j}^j}{z^j} + \frac{H_{-j+1}^j}{z^{j+1}} + \cdots + \sum_{n\geqslant 0} H_n^j z^n. \end{aligned} \tag{A.5.11}$$

Since $L_j(P_j^j) = 2j$ and $L_{j-1}(P_j^j) = 2c_1(2j-1)$, we verify immediately that in the asymptotic expansion (A.5.8)–(A.5.11), we have $H_{-j}^j = L_j(P_j^j)$ and $H_{-j+1}^j = L_{j-1}(P_j^j)$. We prove now (A.5.8): For $s \geqslant 1$,

$$L_s(\psi_j) = z^{1+s}\psi_j' + 2sz^s\psi_j + (s+j)\psi_{j-s} \tag{A.5.12}$$

with the convention that $\psi_j = 0$ if $j < 0$. With (A.5.12), we deduce that the polynomials H_n^j in the asymptotic expansion (A.5.11) satisfy for $s \geqslant 1$:

$$L_s\big(H_n^j\big) = (n+s)H_{n-s}^j + (s+j)H_n^{j-s}. \tag{A.5.13}$$

On the other hand, let

$$K_n^j = L_{-n}\big(P_j^j\big)$$

we verify that K_n^j satisfy also (A.5.13). Since the two sequences of polynomials have the same initial data, they are equal. This proves (A.5.8).

COROLLARY. – *The relation* (A.5.4) *holds.*

Proof. – For $h = z^2(\frac{f'}{f})^2$, then $L_{-j}(h) = \sum_{n\geqslant 1} L_{-j}(P_n^n)z^n$. In (A.5.9), we replace $2h(\frac{\partial\phi_j}{\partial f} - \frac{\phi_j}{f})$ by its asymptotic expansion given by (A.5.8). This proves (A.5.4).

The proposition (A.5.8) extends as follows:

PROPOSITION. – *If* $0 \leqslant u$, *for any* $k \in Z$, *we have*

$$\begin{aligned} z^2\left(\frac{f'}{f}\right)^2 \sum_{0\leqslant s<u} \frac{(k-2s+u)}{f^{u-s}} P_s^k &= \sum_{-u\leqslant n} L_{-n}\big(P_u^k\big)z^n \\ &= \frac{L_u(P_u^k)}{z^u} + \frac{L_{u-1}(P_u^k)}{z^{u-1}} + \cdots + \sum_{n\geqslant 0} L_{-n}(P_u^k)z^n. \end{aligned} \tag{A.5.14}$$

With (A.3.3) and (A.1.7), we compute the first term in the expansion:

$$L_u\big(P_u^k\big) = (k+u), \qquad L_{u-1}(P_u^k) = (k+u-1)(k-u+2)c_1.$$

Proof. – Let

$$G_{k,u} = z^2\left(\frac{f'}{f}\right)^2 \sum_{0\leqslant s<u} \frac{(k-2s+u)}{f^{u-s}} P_s^k = \sum_{n\geqslant -u} H_n^{k,u} z^n. \tag{A.5.15}$$

For $s \geqslant 1$,

$$L_s(G_{k,u}) = z^{s+1}(G_{k,u})' + 2sz^s G_{k,u} + (s+k)G_{k-s,u-s} \tag{A.5.16}$$

with the convention that $G_{k,u} = 0$ if $u < 0$. In (A.5.16), we replace the function $G_{k,u}$ by its asymptotic expansion $\sum_{n\geqslant -u} H_n^{k,u} z^n$. It gives the following conditions on the homogeneous polynomials $H_n^{k,u}$. For $p \geqslant -u$,

$$L_s\big(H_p^{k,u}\big) = (s+p)H_{p-s}^{k,u} + (s+k)H_p^{k-s,u-s}. \tag{A.5.17}$$

Since for $n \geqslant 0$, $H_{-u+n}^{k,u}$ is a homogeneous polynomial of degree n, the $H_{-u+n}^{k,u}$ are uniquely determined by (A.5.17) and the initial condition $H_{-u}^{k,u} = (k+u)$.

With (A.5.17), we find $H^{k,u}_{-u+1} = [(2-u)(k+u) + (1+k)(k+u-2)]c_1, \ldots$.

We put for $0 \leqslant u \leqslant k$, $K^{k,u}_n = L_{-n}(P^k_u)$. For $s \geqslant 1$, we use (A.4.6) and (A.3.3) to compute $L_s(K^{k,u}_n)$. The $K^{k,u}_n$ satisfy the system of equations (A.5.17) with the same initial condition $K^{k,u}_n = (k+u)$. Thus $H^{k,u}_n = L_{-n}(P^k_u)$. This proves (A.5.14).

COROLLARY. – *Let* $h_k = z^2(\frac{f'}{f})^2(\frac{f}{z})^k$, *then for* $0 \leqslant u,\ k \in Z$,

$$L_{-p}\big(P^k_u\big) = \sum_{0 \leqslant j < u} (k-2j+u) P^k_j \int\limits_{\partial D} h_{-(u-j)} \frac{\mathrm{d}t}{t^{1+p+u-j}}. \tag{A.5.18}$$

Proof. – In (A.5.15), the coefficient $H^{k,u}_n$ is given by:

$$H^{k,u}_n = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} G_{k,u} \frac{\mathrm{d}t}{t^{n+1}}.$$

We have

$$\frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} G_{k,u} \frac{\mathrm{d}t}{t^{n+1}} = \sum_{0 \leqslant s < u} (k-2s+u) P^k_s \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} t^2 \left(\frac{f'}{f}\right)^2 \frac{1}{f^{u-s}} \frac{\mathrm{d}t}{t^{n+1}}$$

and we find the right side of (A.5.18).

For $0 \leqslant u \leqslant k$, the asymptotic expansion (A.5.14) is a consequence of (A.4.15)–(A.5.1) as is proved in the following lemma:

LEMMA. – *The relation* (A.5.3) *is equivalent to* (A.5.18).

Proof. – We shall use the residue form of P^p_{r-k}. For $0 \leqslant r \leqslant p+k$,

$$P^p_{r-k} = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} t^2 \left(\frac{f'}{f}\right)^2 \frac{f^{p-r+k}}{t^{p+1}} \,\mathrm{d}t. \tag{A.5.19}$$

Thus

$$\begin{aligned} L_{-k}\big(P^p_{r-k}\big) &= \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} L_{-k}\left[t^2 \left(\frac{f'}{f}\right)^2 \frac{f^{p-r+k}}{t^{p-r+k}}\right] \frac{\mathrm{d}t}{t^{r-k+1}} \\ &= \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} L_{-k}[h_{p-r+k}] \frac{\mathrm{d}t}{t^{r-k+1}}. \end{aligned} \tag{A.5.20}$$

We calculate $L_{-k}[h_{p-r+k}]$ with (A.5.5)

$$\begin{aligned} L_{-k}\big(P^p_{r-k}\big) &= \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} t^{1-k} h'_{p+k-r} + (p-r+k-2k) t^{-k} h_{p+k-r} \frac{\mathrm{d}t}{t^{r-k+1}} \\ &\quad + \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} \left[(p-r+k-2)\frac{\phi_k}{f} + 2\frac{\partial \phi_k}{\partial f}\right] h_{p+k-r} \frac{\mathrm{d}t}{t^{r-k+1}} \\ &= (p-k) P^{r+k}_r + \sum_{j=0}^{k} (k-2j-p+r) P^k_j \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial D} t^2 \left(\frac{f'}{f}\right)^2 \frac{f^{p-r+j}}{t^{p+1}} \,\mathrm{d}t \end{aligned}$$

$$= (p-k)P_r^{r+k} + \sum_{j=0}^{k}(k-2j-p+r)P_j^k \frac{1}{2\mathrm{i}\pi}\int_{\partial D} h_{p-r+j}\frac{\mathrm{d}t}{t^{r-j+1}}$$
$$= (p-k)P_r^{r+k} + K,$$

where at the second step, we replace h_{p+k-r} by its asymptotic expansion in the first term and ϕ_k by $\phi_k = -\sum_{j=0}^{k} P_j^k \frac{1}{f^{k-j-1}}$ in the second term. If $j > r$, there is no residue for the integral

$$\frac{1}{2\mathrm{i}\pi}\int_{\partial D} h_{p-r+j}\frac{\mathrm{d}t}{t^{r-j+1}}.$$

Thus

$$K = \sum_{j=0}^{\inf(k,r)} (k-2j-p+r)P_j^k \frac{1}{2\mathrm{i}\pi}\int_{\partial D} h_{p-r+j}\frac{\mathrm{d}t}{t^{r-j+1}}$$
$$= \sum_{0\leqslant j<r-p} (k-2j-p+r)P_j^k \frac{1}{2\mathrm{i}\pi}\int_{\partial D} h_{p-r+j}\frac{\mathrm{d}t}{t^{r-j+1}}$$
$$+ \sum_{r-p\leqslant j\leqslant \inf(k,r)} (k-2j-p+r)P_j^k P_{r-j}^p.$$

With the convention $P_j^p = 0$ if $j < 0$ or if $j > p$ as in (A.5.3), we have for the last term:

$$\sum_{r-p\leqslant j\leqslant \inf(k,r)} (k-2j-p+r)P_j^k P_{r-j}^p = \sum_{0\leqslant j\leqslant k} (k-2j-p+r)P_j^k P_{r-j}^p$$

and to prove (A.5.3) is the same as to prove

$$L_{-p}\big(P_{r-p}^k\big) = \sum_{0\leqslant j\leqslant r-p} (k-2j-p+r)P_j^k \frac{1}{2\mathrm{i}\pi}\int_{\partial D} h_{p-r+j}\frac{\mathrm{d}t}{t^{1+r-j}}.$$

We put $r = p+u$, this gives (A.5.18) and proves the lemma.

A.6. More asymptotic expansions and polynomials found with the $(L_k)_{k\geqslant 1}$

We consider functions $\phi(z, c_1, c_2, \ldots, c_n, \ldots)$ which satisfy for any $k \geqslant 1$, the partial differential equation:

$$L_k(\phi) = z^{k+1}\phi' + \alpha_k z^k \phi' + \beta_k(z), \tag{A.6.1}$$

where α_k is a constant, $\beta_k(z) = \sum_{p\geqslant 0} b_{p,k} z^p$ is a polynomial in z independent of the $(c_i)_{i\geqslant 1}$, and $b_{p,k}$ are constants. Thus the linear operator $\mathcal{Z}_k$ defined by

$$\mathcal{Z}_k \phi = z^{k+1}\partial_z \phi + \alpha_k z^k \phi \tag{A.6.2}$$

is independent of the $(c_i)_{i\geqslant 1}$.

PROPOSITION. – *Consider $\phi(z) = \sum_{n\geqslant 0} P_n z^n$ where for any $n \geqslant 0$, P_n is a function of the $(c_i)_{i\geqslant 1}$. Assume that $\phi(z)$ is a solution of* (A.6.1) *for any $k \geqslant 1$, then the sequence $(P_n)_{n\geqslant 0}$ is* *such that*

$$L_k(P_n)=\begin{cases} b_{n,k} & \textit{for } n\leqslant k-1,\\ (n-k+\alpha_k)P_{n-k}+b_{n,k} & \textit{for } n\geqslant k.\end{cases} \tag{A.6.3}$$

Proof. – We identify the coefficients of z^n to obtain $L_k(P_n)$. The operator L_k is a derivation and we obtain $L_k(z\frac{f'}{f})=z(\frac{L_k(f)}{f})'$. Then we match the coefficients of z^n in the asymptotic expansions.

For the following asymptotic expansions:

$$\frac{1}{f},\qquad \frac{z^p}{f^p},\qquad u=z\frac{f'}{f},\qquad u^p,\qquad v=\frac{u'}{u},\qquad z\frac{f''}{f'} \tag{A.6.4}$$

the coefficient of z^n for $n\geqslant 1$ is a homogeneous polynomial in the variables $c_1,c_2,\dots,c_n$, and we have an equation of the type (A.6.1). For example:

$$\begin{gathered} u=z\frac{f'}{f}=1+c_1z+\cdots=1+\sum_{n=1}^{\infty}Q_nz^n,\\ L_k\left(z\frac{f'}{f}\right)=z\left(z^{k+1}\frac{f'}{f}\right)'=z^{k+1}\left(z\frac{f'}{f}\right)'+kz^k\left(z\frac{f'}{f}\right)\\ \text{thus } L_k(Q_n)=nQ_{n-k}\quad \text{for } n\geqslant k+1,\\ L_k(Q_k)=k\quad\text{and}\quad L_k(Q_n)=0\quad \text{for } n\leqslant k-1, \end{gathered} \tag{A.6.5}$$

$$\begin{gathered} \frac{u'}{u}=\frac{1}{z}+\frac{f''}{f'}-\frac{f'}{f}=c_1+\sum_{n\geqslant 1}T_nz^n,\\ z\frac{f''}{f'}=2c_1z+\left(6c_2-4c_1^2\right)z^2+\cdots=\sum_{n\geqslant 1}R_nz^n,\\ L_k\left(\frac{f''}{f'}\right)=k(k+1)z^{k-1}+z^{k+1}\left(\frac{f''}{f'}\right)'+(k+1)z^k\frac{f''}{f'}\\ \text{thus } L_k(R_n)=nR_{n-k}\quad \text{for } n\geqslant k+1,\\ L_k(R_k)=k(k+1)\quad\text{and}\quad L_k(R_n)=0\quad\text{for } n\leqslant k-1. \end{gathered} \tag{A.6.6}$$

PROPOSITION. – *Let*

$$\mathcal{P}(z)=hz\frac{f'}{f}+cz\frac{f''}{f'}=\sum_{n\geqslant 0}\mathcal{P}_nz^n, \tag{A.6.7}$$

where h and c are two constants, then

$$L_k(\mathcal{P}_n)=n\mathcal{P}_{n-k}+\left(ck(k+1)+hk\right)\delta_{k,n}. \tag{A.6.8}$$

Proof. – We use (A.6.5)–(A.6.6).

PROPOSITION. – *Assume that ϕ is a solution of*:

$$L_k(\phi)=z^{k+1}\phi'+(k+1)z^k\phi+k(k+1)z^{k-1}, \tag{A.6.8}$$

then $w=\phi'-\frac{1}{2}\phi^2$ *is a solution of*:

$$L_k(w)=z^{k+1}w'+2(k+1)z^k w+k\big(k^2-1\big)z^{k-2}. \tag{A.6.9}$$

Proof. –

$$L_k(\phi')=z^{k+1}\phi''+2(k+1)z^k\phi'+k(k+1)z^{k-1}\phi+k\big(k^2-1\big)z^{k-2},$$
$$L_k\big(\phi^2\big)=z^{k+1}\big(\phi^2\big)'+2(k+1)z^k\phi^2+2k(k+1)z^{k-1}\phi.$$

From these two equalities, we deduce immediately (A.6.9).

COROLLARY. – *Consider the Schwarzian derivative* $S(f)=(\frac{f''}{f'})'-\frac{1}{2}(\frac{f''}{f'})^2$. *Let*

$$\begin{aligned}\mathcal{P}_{h,c}(z)&=hz^2\frac{(f')^2}{f^2}+cz^2S(f)=\sum_{n\geqslant 0}\mathcal{P}_n^{h,c}z^n\\ &=h+2c_1hz+\big[h\big(4c_2-c_1^2\big)+6c\big(c_2-c_1^2\big)\big]z^2\\ &\quad+\big[h(6c_3-2c_1c_2)+c\cdots\big]z^3+\cdots,\end{aligned} \tag{A.6.10}$$

where h *and* c *are two constants*; *then*

$$L_k(\mathcal{P}_n)=(n+k)\mathcal{P}_{n-k}+\big(2hk+ck\big(k^2-1\big)\big)\delta_{k,n}. \tag{A.6.11}$$

Proof. – The function $z\frac{f''}{f'}$ satisfies (A.6.8), see (A.6.6). Thus the Schwarzian derivative $w_1=S(f)$ satisfies (A.6.9). On the other hand, from (A.2.7), we see that $w_2=(\frac{f'}{f})^2$ satisfies:

$$L_k(w_2)=z^{k+1}w_2'+2(k+1)z^kw_2.$$

A.7. The relation $L_{-k}(P_p)-L_{-p}(P_k)=(p-k)P_{p+k}$, for $p\geqslant 0,\ k\geqslant 0$

We consider the identity for $k\geqslant 0,\ p\geqslant 0$,

$$L_{-k}(P_p)-L_{-p}(P_k)=(p-k)P_{p+k} \tag{A.7.1}$$

and sequences of homogeneous polynomials which satisfy (A.7.1). If $(U_n)_{n\geqslant 0}$ and $(V_n)_{n\geqslant 0}$ satisfy (A.7.1), then for any constants h and c, $(hU_n+cV_n)_{n\geqslant 0}$ also satisfy (A.7.1). For the function $\mathcal{P}_{h,c}(z)$ of (A.6.10), we shall prove that the polynomials $\mathcal{P}_n^{h,c}$ satisfy (A.7.1). When $c=0$ and $h\neq 0$, the relation (A.7.1) has been proved for $\mathcal{P}_p=P_p^p$ in (A.5.4) (see the corollary of (A.5.8)). In the following, we prove (A.7.1) when $h=0$ and $c\neq 0$.

THEOREM. – *The polynomials* $\mathcal{P}_n$ *defined by*:

$$z^2S(f)=\sum_{n\geqslant 0}\mathcal{P}_nz^n \tag{A.7.2}$$

satisfy (A.7.1).

Proof. – We have $\mathcal{P}_0=\mathcal{P}_1=0$. We first consider the case where $k=1$ in (A.7.1) and we prove that for any $p\geqslant 1$, the sequence of polynomials $\mathcal{P}_n$ satisfy:

$$L_{-1}(\mathcal{P}_p)=(p-1)\mathcal{P}_{p+1} \tag{A.7.3}$$

we compute $L_{-1}(\frac{f''}{f'}) = (\frac{f''}{f'})'$, thus

$$L_{-1}S(f) = \big(S(f)\big)' \tag{A.7.4}$$

and we replace in (A.7.4) the functions by their asymptotic expansions.

To prove the theorem for any k, we use the following lemma:

LEMMA. – *Let $\phi = f''/f'$, we have*:

$$L_{-k}(\phi) = k(k-1)z^{-(k+1)} + (1-k)z^{-k}\phi + z^{1-k}\phi' + \frac{\partial^2\phi_k}{\partial f^2}f', \tag{A.7.5}$$

$$L_{-k}\big(S(f)\big) = -k\big(k^2-1\big)z^{-(k+2)} + 2(1-k)z^{-k}S(f) + z^{1-k}\big(S(f)\big)' + \frac{\partial^3\phi_k}{\partial f^3}(f')^2. \tag{A.7.6}$$

Proof. – We have $L_{-k}(S(f)) = L_{-k}(\phi') - \phi L_{-k}(\phi) = \cdots$.

We prove (A.7.1) when $k = 2$.

LEMMA. – *Let $z^2S(f) = \sum_{n\geqslant 0} P_n z^n = 6(c_2 - c_1^2)z^2 + \cdots$, we have*:

$$L_{-2}(P_p) - L_{-p}(P_2) = (p-2)P_{p+2}. \tag{A.7.7}$$

Proof. – For the polynomials in (A.7.2), we have

$$L_{-p}(\mathcal{P}_2) = A_1^p\frac{\partial}{\partial c_1}(\mathcal{P}_2) + A_2^p\frac{\partial}{\partial c_2}(\mathcal{P}_2) = -12c_1A_1^p + 6A_2^p,$$

thus we have to prove that

$$L_{-2}(P_p) = (p-2)P_{p+2} + 6A_2^p - 12c_1A_1^p. \tag{A.7.8}$$

We compute

$$L_{-2}\big(z^2S(f)\big) = -6\frac{1}{z^2} + z\big(S(f)\big)' - 2S(f) + 6z^2\frac{(f')^2}{f^4}.$$

We replace $z^2S(f)$ and $z^2\frac{(f')^2}{f^4}$ by their expansions, and we obtain (A.7.8).

LEMMA. – *Let $z^2S(f) = \sum_{n\geqslant 0}\mathcal{P}_{n+2}z^{n+2}$, we have*:

$$L_{-3}(\mathcal{P}_p) - L_{-p}(\mathcal{P}_3) = (p-3)\mathcal{P}_{p+3}. \tag{A.7.9}$$

Proof. – Since

$$\frac{\partial^3\phi_3}{\partial f^3} = \frac{24}{f^5}(1 + c_1 f),$$

we obtain

$$L_{-3}\big(S(f)\big) = -\frac{24}{z^5} - \frac{4}{z^3}S(f) + \frac{1}{z^2}\big(S(f)\big)' + \frac{24}{f^5}(f')^2(1 + c_1 f). \tag{A.7.10}$$

In (A.7.10), we replace $S(f)$ by $\sum_{n\geqslant 0}\mathcal{P}_{n+2}z^n$,

$$\text{(A.7.11)}\qquad \begin{aligned}&\sum_{p\geqslant 0}L_{-3}(P_{p+2})z^p\\&\quad=-\frac{24}{z^5}-4\sum_{p\geqslant 0}P_{p+2}z^{p-3}+\sum_{p\geqslant 0}pP_{p+2}z^{p-3}+\frac{24}{f^5}(f')^2(1+c_1f).\end{aligned}$$

We compute

$$\frac{(f')^2}{f^5}(1+c_1f)=\frac{1}{z^5}+\frac{(c_2-c_1^2)}{z^3}+\sum_{p\geqslant 0}M_pz^{p-2}+\sum_{p\geqslant 0}c_1\big(A_2^p-2c_1A_1^p\big)z^{p-2}.$$

We let

$$\text{(A.7.12)}\qquad G_p=M_p+c_1A_2^p-2c_1^2A_1^p=A_3^p-2c_1A_2^p-2c_2A_1^p+3c_1^2A_1^p.$$

From (A.7.11), we obtain

$$\text{(A.7.13)}\qquad \sum_{p\geqslant 0}L_{-3}(P_{p+2})z^p=\sum_{p\geqslant 0}(p-1)P_{p+5}z^p+24\sum_{p\geqslant 0}G_{p+2}z^p.$$

By matching equal powers of z, we get for any $p\geqslant 0$,

$$\text{(A.7.14)}\qquad L_{-3}(P_{p+2})=(p-1)P_{p+5}+24G_{p+2}.$$

To prove (A.7.9), it is enough to show that

$$\text{(A.7.15)}\qquad L_{-(p+2)}(\mathcal{P}_3)=24G_{p+2}.$$

Since

$$\mathcal{P}_3=24\big(c_3-2c_1c_2+c_1^3\big)$$

the relation (A.7.15) is satisfied.

PROPOSITION. – *Let* $z^2S(f)=\sum_{n\geqslant 0}\mathcal{P}_{n+2}z^{n+2}$, *we have*:

$$\text{(A.7.16)}\qquad \begin{aligned}\frac{\partial^3\phi_k}{\partial f^3}(f')^2&=\frac{k(k^2-1)}{z^{k+2}}-0\times\frac{c_1}{z^{k+1}}+\frac{12(k-1)(c_2-c_1^2)}{z^k}+\cdots+\sum_{n\geqslant 0}L_{-(n+2)}(\mathcal{P}_k)z^n\\&=\frac{L_k(\mathcal{P}_k)}{z^{k+2}}+\frac{L_{k-1}(\mathcal{P}_k)}{z^{k+1}}+\frac{L_{k-2}(\mathcal{P}_k)}{z^k}+\cdots+\frac{L_{k-j}(\mathcal{P}_k)}{z^{k+2-j}}+\cdots\end{aligned}$$

and for $k\geqslant 0,\ p\geqslant 0$

$$\text{(A.7.17)}\qquad L_{-k}(\mathcal{P}_p)-L_{-p}(\mathcal{P}_k)=(p-k)\mathcal{P}_{p+k}.$$

Proof. – $S(f)=\sum_{n\geqslant 0}\mathcal{P}_{n+2}z^n$, we replace in the expression of $L_{-k}(S(f))$,

$$\sum_{n\geqslant 0}L_{-k}(\mathcal{P}_{n+2})z^n=2(1-k)\sum_{m\geqslant k}\mathcal{P}_{m+2}z^{m-k}+\sum_{m\geqslant k}m\mathcal{P}_{m+2}z^{m-k}+\sum_{n\geqslant 0}H_n^kz^n,$$

where H_n^k is the coefficient of z^n in the asymptotic expansion of $\frac{\partial^3\phi_k}{\partial f^3}(f')^2$, i.e.

$$\frac{\partial^3\phi_k}{\partial f^3}(f')^2 = \frac{k(k^2-1)}{z^{k+2}} - 0\times\frac{c_1}{z^{k+1}} + \cdots + \sum_{n\geqslant 0} H_n z^n. \tag{A.7.18}$$

By matching equal powers of z^n in (A.7.18), we obtain:

$$L_{-k}(\mathcal{P}_{n+2}) = (n+2-k)\mathcal{P}_{n+k+2} + H_n^k. \tag{A.7.19}$$

To prove (A.7.17), it is enough to prove that for $k\geqslant 2$,

$$H_n^k = L_{-(n+2)}(\mathcal{P}_k). \tag{A.7.20}$$

We use the following lemmas:

LEMMA. – *For any $j\geqslant 1$,*

$$L_j\left(\frac{\partial^p\phi_k}{\partial f^p}\right) = z^{j+1}\left(\frac{\partial^p\phi_k}{\partial f^p}\right)' + (j+k)\frac{\partial^p\phi_{k-j}}{\partial f^p} \tag{A.7.21}$$

with the convention that $\phi_p = 0$ if $p<0$.

Proof. – We prove the relation by recurrence on p.

LEMMA. – *Consider the asymptotic expansion* (A.7.18), *where k is given*; *we have for $j\geqslant 1$*:

$$L_j\big(H_n^k\big) = (n+j+2)H_{n-j}^k + (j+k)H_n^{k-j} \tag{A.7.22}$$

with the conventional $H_n^p = 0$ if $p<0$.

Proof. – We match the coefficients of the asymptotic expansion in (A.7.21) with $p=3$.

LEMMA. – *We put*

$$K_n^k = L_{-(n+2)}(\mathcal{P}_k)$$

then for $j\geqslant 1$,

$$L_j\big(K_n^k\big) = (n+j+2)K_{n-j}^k + (j+k)K_n^{k-j}. \tag{A.7.23}$$

Proof. –

$$L_j\big(L_{-(n+2)}(\mathcal{P}_k)\big) = L_{-(n+2)}\big(L_j(\mathcal{P}_k)\big) + (j+n+2)L_{j-(n+2)}(\mathcal{P}_k)$$

thus

$$L_j\big(K_n^k\big) = (n+j+2)K_{n-j}^k + L_{-(n+2)}\big(L_{-j}(\mathcal{P}_k)\big).$$

Since $L_j(\mathcal{P}_k) = (j+k)\mathcal{P}_{k-j} + \text{constant}$, we obtain (A.7.23).

Proof of (A.7.20). – For k fixed, the two sequences H_n^k and $K_n^k = L_{-(n+2)}(\mathcal{P}_k)$ follow the same recurrence formulas (A.7.22) and (A.7.23). Since we know that H_n^k is a homogeneous polynomial in the variables (c_i), the relations (A.7.22)–(A.7.23) determine completely the sequences H_n^k

with the condition $H^k_{-(k+2)} = k(k^2-1)$. Thus the two sequences are equal. This proves (A.7.20) and thus (A.7.16) and (A.7.17)

With the relations (A.7.22)–(A.7.23) and the condition

$$H^k_{-(k+2)} = k\big(k^2-1\big) \tag{A.7.24}$$

we shall compute the first terms in the asymptotic expansion (A.7.16). We have $H^k_{-(k+1)} = \alpha c_1$ thus $L_1(H^k_{-(k+1)}) = \frac{\partial}{\partial c_1}(H^k_{-(k+1)}) = \alpha$. By (A.7.22)–(A.7.23), we know that:

$$L_1\big(H^k_{-(k+1)}\big) = \big(1+2-(k+1)\big)H^k_{-(k+2)} + (1+k)H^{(k-1)}_{-(k+1)},$$

we replace $H^k_{-(k+2)}$ by (A.7.24) and also using (A.7.24) with $k-1$ instead of k, we obtain $H^{(k-1)}_{-(k+1)} = (k-1)((k-1)^2-1)$, we find $\alpha = 0$,

$$H^k_{-(k+1)} = 0.$$

For $H^k_{-k} = \alpha c_1^2 + \beta c_2$, we have $L_1(H^k_{-k}) = (\frac{\partial}{\partial c_1} + 2c_1\frac{\partial}{\partial c_2})(\alpha c_1^2 + \beta c_2) = 2\alpha c_1 + 2\beta c_1$. On the other hand, from (A.7.22)–(A.7.23),

$$L_1\big(H^k_{-k}\big) = (-k+1+2)H^k_{-(k+1)} + (1+k)H^{k-1}_{-k} = 0$$

this gives $\alpha + \beta = 0$. Then, we compute $L_2(H^k_{-k}) = \frac{\partial}{\partial c_2}(\alpha c_1^2 + \beta c_2) = \beta$; from (A.7.22)–(A.7.23), $L_2(H^k_{-k}) = (-k+2+2)H^k_{-(k+2)} + (k+2)H^{k-2}_{-k}$, this gives $\beta = 12(k-1)$,

$$H^k_{-k} = 12(k-1)\big(c_2 - c_1^2\big).$$

COROLLARY. – *The residue at $z=0$ of $\frac{\partial^3\phi_k}{\partial f^3}(f')^2\frac{1}{z^{n+1}}$ is equal to $L_{-(n+2)}(\mathcal{P}_k)$.*

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