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On the Universal Sequence Generated by a Class of Unimodal Functions

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The universality of the Metropolis, Stein, and Stein (MSS) sequence (*J. Combin. Theory* **15** (1973), 25–44) is established for a wide class of unimodal functions. The standard value of an *LR*-sequence is defined and a computational formula for it is established. An order on all finite *LR*-sequences is defined. It is shown that this order is equivalent to the order of Collet and Eckman (CE) ("Iterated Maps on the Interval as Dynamical Systems," Birkhauser, Boston, 1980), Louck and Metropolis ("Symbolic Dynamics of Trapezoidal Maps," Reidel–Kluwer, Hingham, Ma, 1986) and Beyer, Mauldin, and Stein (BMS), (*J. Math.Anal. Appl.* **115** (1986), 305–362). The contiguity of harmonics is proved for any finite *LR*-sequence. Finally using an important result of BMS, it is shown that a pattern is legal if and only if it is a pattern associated with a positive solution λ of the sequence of equations $[\lambda f]^k(y_0) = y_0$ $(k = 1, 2,...)$. © 1987 Academic Press, Inc.

INTRODUCTION

In this paper we study patterns that are finite sequences on two symbols, say, L and R. In a famous paper [1] Metropolis *et al.* (MSS) presented many novel ideas and suggestive numerical results. They defined legal inverse paths and their coordinates. They also implicitly defined an order on legal inverse paths in their table of legal patterns at the end of [1]. However, they did not extend this order to all inverse paths, and so they did not investigate their properties further. Therefore, they did not prove some important facts stated in their paper [1] (e.g., Lemma 1 [1, p. 35], the last two sentences of the second paragraph below Lemma 1 on harmonics of a legal inverse path, and Lemma 2 [1, p. 36]).

Collet and Eckmann (CE) [5] and Beyer *et al.* (BMS) [4] formally defined an order on the set of all patterns on two symbols. BMS successfully proved for the class of round-top, concave, unimodal functions

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that a pattern is an element of the MSS-sequence if and only if it is shift-maximal. But the inherent meaning of their order is not immediately evident.

Here we define an order on the set $\mathscr{P}$ of all patterns of finite length on two symbols by ordering the patterns according to the order on the real line of their last points; see Section 3. This order is equivalent to the order used by CE and by BMS. In other words, we give a concrete meaning to their order. Having done this, we are able to establish a (non-order preserving) 1–1 correspondence between the elements of $\mathscr{P}$ and the set of all terminating binary numbers. We also give four equivalent statements of this order; see Theorem 2 of Section 3. In addition, we directly generalize and prove, as Theorems 4, 5, and 6 in Sections 5 and 6, the three results of MSS referred to above. In Theorem 1 we prove a result that implies the universality of the MSS sequence of patterns in $\mathscr{P}$. Finally, after establishing the equivalence of our order and that of CE in Section 6, we easily show that for the class of round-top, concave, unimodal functions a pattern is an element of the MSS sequence if and only if it is legal; see Theorem 6 of Section 6.

1. Basic Definitions and Assumptions

We denote the closed interval $[-1, 1]$ by I. The symbol $\mathscr{F}$ denotes the class of unimodal functions f that satisfy the following conditions:

(1) f is continuous on I, and $f(-1) = f(1) = -1$;

(2) f achieves its maximum $f_{\max} \leqslant 1$ on some proper, closed subinterval $[a_m, b_m]$ of I $(a_m \leqslant b_m)$; and

(3) f is strictly increasing on $[-1, a_m]$ and is strictly decreasing on $[b_m, 1]$.

Our interval I, the domain of functions in $\mathscr{F}$, corresponds to the unit interval used by MSS in [1]. However, our conditions (1)–(3) are less restrictive than the conditions (A) and (B) of [1], and we do not require MSSs hypothesis (C) at all. Further, we do not assume that $f \in C^1(I)$, nor do we assume differentiability of f with respect to λ. Thus, our class $\mathscr{F}$ is wider than the class of unimodal functions investigated by MSS in [1]. We shall also consider a subclass $\mathscr{F}_0$ of $\mathscr{F}$,

$$\mathscr{F}_0 = \{f \mid f \in \mathscr{F} \text{ and } f_{\max} = 1\}.$$

This subclass is also wider than the subclass $\{\lambda_{\max} f\}$ in [1].

Let $P = A_1 A_2 \cdots A_k$, $A_i = L$ or R $(1 \leqslant i \leqslant k < \infty)$, which is a finite, formal sequence on the two symbols L and R. We call such a P a *pattern* or

LR-sequence. In accordance with [1, p. 29], we define the length of a pattern P to be $k+1$, not k. We denote the set of all patterns P with finite length on two symbols by $\mathscr{P}$.

Recall that $[a_m, b_m]$ is a subinterval of I, where $f(x) = f_{\max}$ as above, and set $y_0 = (a_m + b_m)/2$. For any $f \in \mathscr{F}$ and any $P \in \mathscr{P}$, we construct maps f^L, f^R, and f^P as

$$f^L(x) = f^{-1}(x) \cap ([-1, a_m) \cup \{y_0\}) \qquad (\forall\, x \in f(I)),$$

$$f^R(x) = f^{-1}(x) \cap ((b_m, 1] \cup \{y_0\}) \qquad (\forall\, x \in f(I)),$$

$$f^P(x) = f^{A_1} f^{A_2} \cdots f^{A_k}(x),$$

if $\{x, f^{A_k}(x), f^{A_{k-1}} f^{A_k}(x), ..., f^{A_2} f^{A_3} \cdots f^{A_k}(x)\} \subset f(I)$. It is obvious that f^L and f^R are two branches of the inverse function f^{-1}, while $f^P(x)$ is one point in $f^{-k}(x)$ (if $f^P(x)$ is well defined).

If $f^P(y_0)$ is well defined, we define $y_1 = f^{A_k}(y_0)$, $y_2 = f^{A_{k-1}}(y_1), ...,$ $y_k = f^{A_1}(y_{k-1})$. The sequence $\{y_0, y_1, ..., y_k\}$ of points in I is said to be the *generalized inverse path with respect to* f *and* P or simply the *generalized inverse path of* P. We denote this path by $W_{P,f}$. We say that y_k is the *final* or *last point* of $W_{P,f}$, and we write $\mathscr{L}(W_{P,f}) = y_k$. If $y_i \neq y_0$ $(1 \leqslant i \leqslant k)$, we call $W_{P,f}$ *an inverse path.* If $f \in \mathscr{F}_0$, each pattern (LR-sequence) has an inverse path.

2. The Universality of the MSS Sequence

We shall prove that the order in $\mathbb{R}$ of final points for different patterns is independent of the choice of $f \in \mathscr{F}$. This property underlies the universality of the MSS sequence. Universality was stated by MSS [1, pp. 30–31] without proof. We prove

Theorem 1. *Let* $f, g \in \mathscr{F}$, *and let* $P, Q \in \mathscr{P}$ *with* $P = A_1 A_2 \cdots A_k$ *and* $Q = B_1 B_2 \cdots B_n$. *Assume* $f^P(y_0)$, $f^Q(y_0)$, $g^P(y_0)$, *and* $g^Q(y_0)$ *are well defined. Then*

$$\mathscr{L}(W_{P,f}) < \mathscr{L}(W_{Q,f}) \tag{2.1}$$

if and only if

$$\mathscr{L}(W_{P,g}) < \mathscr{L}(W_{Q,g}). \tag{2.2}$$

Proof. By symmetry we need only to prove that (2.1) implies (2.2). The proof is divided into five cases.

Case (i). $A_1 \neq B_1$. Then (2.1) implies $A_1 = L$ and $B_1 = R$. Using the monotonicity properties shared by all functions in $\mathscr{F}$, we get (2.2).

Case (ii). $A_1 = B_1 = L$ and either $k = 1$ or $n = 1$. If $k = 1$, then (2.1) implies that $n > 1$ and $B_2 = R$. For if $n = 1$, $P = Q = L$ contradicting (2.1); and if $B_2 = L$, then $f^{LL}(y_{n-2}) < f(y_0)$, that is, $\mathscr{L}(W_{Q,f}) < \mathscr{L}(W_{P,f})$, which also contradicts (2.1). By the monotonicity properties shared by f and g, this implies (2.2). Similarly, if $n = 1$, then (2.1) implies that $k > 1$ and $A_2 = L$, so that (2.2) holds.

Case (iii). $A_1 = B_1 = R$ and either $k = 1$ or $n = 1$. If $k = 1$, then (2.1) implies that $n > 1$ and $B_2 = L$, which implies (2.2). If $n = 1$, then (2.1) implies $k > 1$ and $A_2 = R$, so that (2.2) holds.

One may now proceed by induction on the lengths of the patterns P and Q. The induction hypothesis is $\mathscr{L}(W_{P',f}) < \mathscr{L}(W_{Q',f})$ implies that $\mathscr{L}(W_{P',g}) < \mathscr{L}(W_{Q',g})$ if P' and Q' have lengths $k-1$ and $n-1$, respectively. We proceed to carry out the induction in Cases (iv–v).

Case (iv). $P = LA_2 \cdots A_k \equiv LP'$ and $Q = LB_2 \cdots B_n \equiv LQ'$. By the monotonicity properties of f, we see from (2.1) that

$$\mathscr{L}(W_{P',f}) < \mathscr{L}(W_{Q',f}). \tag{2.3}$$

By the induction hypothesis, we find that (2.3) implies

$$\mathscr{L}(W_{P',g}) < \mathscr{L}(W_{Q',g}). \tag{2.4}$$

From (2.4), using the monotonicity properties of g, we obtain (2.2).

Case (v). $P = RA_2 \cdots A_k \equiv RP'$ and $Q = RB_2 \cdots B_n \equiv RQ'$. By the monotonicity properties of f, we see from (2.1) that

$$\mathscr{L}(W_{P',f}) > \mathscr{L}(W_{Q',f}). \tag{2.3$'$}$$

By induction hypothesis, we obtain the inequality

$$\mathscr{L}(W_{P',g}) > \mathscr{L}(W_{Q',g}). \tag{2.4$'$}$$

From (2.4′) and the monotonicity properties of g, we obtain (2.2). The proof of Theorem 1 is now complete.

3. The Order of Patterns and Their Standard Values

We now use Theorem 1 to define a linear order "<" on the set $\mathscr{P}$ of all patterns with finite length on two symbols.

Definition 1. Given P and Q in $\mathscr{P}$, we say P *is less than* Q (or P

precedes Q) if and only if $\mathscr{L}(W_{P,f}) < \mathscr{L}(W_{Q,f})$ for each $f \in \mathscr{F}_0$. If P is less than Q, we write $P < Q$.

In order to recognize the order of patterns in terms of their symbols we use the simple triangle function $f_0 \in \mathscr{F}_0$: $f_0 = 2x + 1$ if $-1 \leqslant x \leqslant 0$; $f_0 = -2x + 1$ if $0 \leqslant x \leqslant 1$. For each $P \in \mathscr{P}$, we call the inverse path W_{P,f_0} with respect to f_0 and P the *standard inverse path of* P; we call the final point $\mathscr{L}(W_{P,f_0})$ the *standard value of* P.

Let $\mathscr{L}_P = \mathscr{L}(W_{P,f_0})$, and we define the set of real numbers

$$E = \{\mathscr{L}_P : P \in \mathscr{P}\}.$$

The ordered set $\{\mathscr{P}, <\}$ is isomorphic to E. Define $\delta_L = \frac{1}{2}$ and $\delta_R = -\frac{1}{2}$. Observe that for $x \in I$ and A either L or R, $f_0^A(x) = \delta_A(x-1)$. Thus, for $P = A_1 A_2 \cdots A_{k-1} A_k$, we have $f_0^{A_k}(0) = -\delta_{A_k}$,

$$f_0^{A_{k-1}} f_0^{A_k}(0) = f_0^{A_{k-1}}(-\delta_{A_k}) = -\delta_{A_{k-1}} - \delta_{A_{k-1}}\,\delta_{A_k},$$

and finally,

$$\mathscr{L}_P = f_0^P(0) = -\delta_{A_1} - \delta_{A_1}\,\delta_{A_2} - \cdots - \delta_{A_1}\,\delta_{A_2} \cdots \delta_{A_k} = \sum_{i=1}^{k} \prod_{j=1}^{i} \delta_{A_j}. \tag{3.1}$$

It is easy to see that there exists a 1–1 correspondence between E and the set of all terminating binary numbers. From (3.1), we immediately obtain

PROPOSITION 1. *For any two patterns P and Q in $\mathscr{P}$, there exists a pattern between them.*

If P is any pattern in $\mathscr{P}$, we may write

$$P = L^{i(1)}\, RL^{i(2)}\, R \cdots L^{i(m-1)}\, RL^{i(m)},$$

where $i(1),..., i(m)$ are nonnegative integers and $m \geqslant 1$. We call the sequence $\{i(k)\}_{k=1}^{m}$ of m elements the *power sequence of P relative to L*. Let

$$\mathscr{S} = \{\{i(1),..., i(m)\} \mid 1 \leqslant m < \infty,\ i(k) \text{ is a nonnegative integer}\}.$$

There is an obvious 1–1 correspondence between the sets $\mathscr{P}$ and $\mathscr{L}$. By (3.1) we have

$$\mathscr{L}_P = -\sum_{1}^{i(1)} 2^{-j} + \sum_{r=2}^{m} (-1)^r \sum_{K(r-1)+1}^{K(r)} 2^{-j}, \tag{3.2}$$

where $K(r) = i(1) + \cdots + i(r) + r - 1$.

We now define a lexical order $<_l$ on $\mathscr{S}$ as follows: for any two power sequences $\{i(k)\}$ $(k=1,..., m)$ and $\{j(p)\}$ $(p=1,..., n)$ in $\mathscr{S}$.

$$\{i(1),..., i(m)\} <_l \{j(1),..., j(n)\}$$

if and only if one of the following three conditions holds:

(i) There exists an r with $1 \leqslant r \leqslant \min(m, n)$ such that $i(\beta)=j(\beta)$ for $\beta=1,..., r-1$ and $(-1)^r\, i(r) < (-1)^r\, j(r)$;

(ii) $m<n$, $i(\beta)=j(\beta)$ for $\beta=1,..., m$, and m is odd;

(iii) $m>n$, $i(\beta)=j(\beta)$ for $\beta=1,..., n$, and n is even.

PROPOSITION 2. $\{i(1),..., i(m)\} <_l \{j(1),..., j(n)\}$ *if and only if*

$$\sum_{\beta=1}^{m} (-1)^{\beta}[i(\beta)+1]/x^{\beta} < \sum_{\beta=1}^{n} (-1)^{\beta}[j(\beta)+1]/x^{\beta}, \tag{3.3}$$

where $x \geqslant 2+\max\{i(1),..., i(m), j(1),..., j(n)\}$.

Proof. First suppose $i(1)=j(1),..., i(\beta-1)=j(\beta-1)$, but $i(\beta) \neq j(\beta))$ and $\beta \leqslant \min(m, n)$. We carry out the proof for even β; the proof for odd β is similar. By, condition (i) $\{i(1),..., i(m)\} <_l \{j(1),..., j(n)\}$ if and only if $j(\beta) > i(\beta)$. Choose $x \geqslant 2+\max\{i(1),..., i(m), j(1),..., j(n)\}$. Recall that each $P \in \mathscr{P}$ has finite length. Since $j(\beta) > i(\beta)$ and

$$\begin{aligned}[i(\beta)+1]x^{-\beta}+(x-1)x^{-\beta-1}+(x-1)x^{-\beta-2}+\cdots &< [i(\beta)+1]x^{-\beta}+x^{-\beta}\\ &= [i(\beta)+2]x^{-\beta}\\ &\leqslant [j(\beta)+1]x^{-\beta},\end{aligned}$$

it follows that

$$\begin{aligned}&[j(\beta)+1]x^{-\beta}-(x-1)x^{-\beta-1}-(x-1)x^{-\beta-3}\cdots\\ &\quad > [i(\beta)+1]x^{-\beta}+(x-1)x^{-\beta-2}+(x-1)x^{-\beta-4}+\cdots\end{aligned}$$

so that (3.3) holds. Conversely, note that (3.3) implies

$$\begin{aligned}&[j(\beta)+1]x^{-\beta}+(x-1)x^{-\beta-2}+(x-1)x^{-\beta-4}+\cdots\\ &\quad > [i(\beta)+1]x^{-\beta}-(x-1)x^{-\beta-1}-(x-1)x^{-\beta-3}-\cdots.\end{aligned}$$

Then

$$\begin{aligned}[j(\beta)+1]x^{-\beta} &> [i(\beta)+1]x^{-\beta}-\{(x-1)x^{-\beta-1}\\ &\qquad +(x-1)x^{-\beta-2}+(x-1)x^{-\beta-3}+\cdots\}\\ &> i(\beta)x^{-\beta}\end{aligned}$$

implies $j(\beta)+1 > i(\beta)$, so that $j(\beta) > i(\beta)$.

Second, we suppose $i(\beta)=j(\beta)$, $\beta=1,2,\ldots,n$ and n is even. By condition (iii) $\{i(1),\ldots,i(m)\}<_l\{j(1),\ldots,j(n)\}$ if and only if $m>n$. It is easy to see that if $m>n$, then

$$-[1+i(n+1)]x^{-n-1}+[1+i(n+2)]x^{-n-2}-\cdots<0$$

implies (3.3) and *vice versa*. The same argument applied to $i(\beta)=j(\beta)$ ($\beta=1,2,\ldots,m$) and m odd gives $m<n$ if and only if (3.3) holds. The proof of the proposition is complete.

Definition 1, the definition of $<_l$, (3.2), and Proposition 2 imply

THEOREM 2. *Let P and Q be any patterns in $\mathscr{P}$ with standard values $\mathscr{L}_P$ and $\mathscr{L}_Q$, respectively, and whose power sequences relative to L are $\{i(p)\}_1^m$ and $\{j(p)\}_1^n$, respectively. Then the following four statements are equivalent*:

(i) $P<Q$;

(ii) $\mathscr{L}_P<\mathscr{L}_Q$;

(iii) $\{i(1),\ldots,i(m)\}<_l\{j(1),\ldots,j(n)\}$;

(iv) $\sum_{\beta=1}^{m}(-1)^\beta[i(\beta)+1]/x^\beta<\sum_{\beta=1}^{n}(-1)^\beta[j(\beta)+1]/x^\beta$,

where $x\geqslant 2+\max\{i(1),\ldots,i(m),j(1),\ldots,j(n)\}$.

Since finding the power sequence relative to L of a pattern P in $\mathscr{P}$ is simple, Theorem 2 makes it easy to determine the order of any two patterns.

4. LEGAL PATTERNS AND LEGAL INVERSE PATHS

Consider any $P=A_1A_2\cdots A_k\in\mathscr{P}$. We call $A_1A_2\cdots A_i$ ($1\leqslant i\leqslant k$) a *left subpattern of P* and $A_iA_{i+1}\cdots A_k$ a *right subpattern of P*. We also consider the empy set $\varnothing$ to be a (left or right) subpattern of P with length 1. For $f\in\mathscr{F}$ the inverse path $W_{\varnothing,f}$ and last point $\mathscr{L}(W_{\varnothing,f})$ of $\varnothing$ are both the point y_0, where y_0 is defined in Section 1. As do MSS [1, p. 34] we adopt

DEFINITION 2. Given $P\in\mathscr{P}$ and any $f\in\mathscr{F}_0$, if the last point $\mathscr{L}(W_{P,f})$ of the path $W_{P,f}$ is greater than each other point of $W_{P,f}$, then we call P a *legal* pattern (lp); otherwise, we call P an *illegal* pattern. We call $W_{P,f}$ a *legal* inverse path (lip) if and only if P is a lp.

Using Definition 1 (in Sect. 3) of order $<$ on $\mathscr{P}$, one may easily prove

PROPOSITION 3. *A pattern P is an lp if and only if every proper right subpattern (including $\varnothing$) of P, precedes P.*

From Proposition 3 and Theorem 2, using P for Q and a proper right subpattern of P for P, we immediately obtain

THEOREM 3. *Suppose* $P = L^{i(1)} RL^{i(2)} R \cdots L^{i(m-1)} RL^{i(m)}$. *Then P is an lp if and only if*

(i) $i(1) = 0$,

(ii) $i(2) \geqslant \max\{i(3),..., i(m)\}$, *and*

(iii) *if* $i(\beta) = i(2)$ $(2 < \beta \leqslant m)$, *then*

$$\{i(\beta+1),..., i(m)\} <_l \{i(3),..., i(m)\}.$$

One may apply Theorem 3 to judge directly whether a pattern $P \in \mathscr{P}$ is legal or illegal. One does not need to choose, via Definition 2, a function $f \in \mathscr{F}_0$ and then to calculate all points of $W_{P,f}$. One only needs to know the power sequence $\{i(k)\}$ of P relative to L.

5. HARMONICS AND CONTIQUITY OF HARMONICS

Now recall (see [1, p. 32]) that the (first) harmonic H of a pattern P is defined to be PAP, where $A = L$ if P contains an odd number of R's and $A = R$ if P contains an even number of R's. Also recall that $I = [-1, 1]$. For $x_1, x_2 \in I$ let $I(x_1, x_2)$ be the open interval (x_1, x_2) or (x_2, x_1), and let $I[x_1, x_2]$ be the closed interval $[x_1, x_2]$ or $[x_2, x_1]$. No order of the endpoints x_1 and x_2 is implied by this notation. We next prove

THEOREM 4 [MSS, p. 35]. *If P is an lp, its first harmonic H is also an lp; hence, all its higher harmonics are lp's.*

Remark. Collet and Eckman's Corollary II.2.4 [5, p. 75] implies this theorem. Here we give a proof directly from the definition of an lp.

Proof. We suppose $P = A_1 A_2 \cdots A_k$ and $H = PAP$. Choose $f \in \mathscr{F}_0$, and let the inverse path $W_{H,f} = \{y_0,..., y_k, y_{k+1},..., y_{2k+1}\}$. MSS give [1, p. 35] a one sentence proof of "$x < x_H$" or, in our notation, $y_k < y_{2k+1}$. Thus, each point in $W_{P,f} = \{y_0,..., y_k\}$ is less than y_{2k+1} because P is an lp. But, whether or not each point of $\{y_{k+1},..., y_{2k}\}$ is less than y_{2k+1} is unclear. To provide an answer, we proceed as follows.

Since P is an lp, $y_k > y_i$ $(i = 0, 1,..., k-1)$. Thus, there exist no points of $\{y_1,..., y_k\}$ between y_0 and $y_{k+1} = f^A(y_k)$. If $y_i \in I(y_0, y_{k+1})$ for some i with $1 \leqslant i \leqslant k$, then $y_{i-1} > y_k$, which is a contradiction. Let $I_0 = I[y_0, y_{k+1}]$. We claim that for $2 \leqslant i \leqslant k$,

$$y_k \notin f^{A_i \cdots A_k}(I_0).$$

If it is in this interval, then

$$y_{i-1} = f^{k+1-i}(y_k) \in f^{k+1-i}(f^{A_i \cdots A_k}(I_0)) = I_0,$$

which is a contradiction. Since $y_{k-i+1} = f^{A_i \cdots A_k}(y_0) < y_k$, i.e., $f^{A_i \cdots A_k}(y_0)$ is to the left of y_k, $y_{2k+2-i} = f^{A_i \cdots A_k}(y_{k+1})$ also lies to the left of y_k. Hence, $y_{2k+2-i} < y_{2k+1}$. This completes the proof of Theorem 4.

THEOREM 5. *For any* $P = A_1 A_2 \cdots A_k \in \mathscr{P}$ *let* $Q = PAP$, *where* A *is either* L *or* R. *Relative to some* $f \in \mathscr{F}_0$, *let* $W_{Q,f} = \{y_0, ..., y_k, y_{k+1}, ..., y_{2k+1}\}$. *Then there exists no last point of any legal inverse path in the open interval* $I_1 = I(y_k, y_{2k+1})$.

Remark. We point out that Theorem 5 generalizes MSSs Lemma 2 [1, p. 36] in that their P is an lp, while the P in Theorem 5 may be either legal or illegal. In addition, the statement [1, p. 36], "Thus no inverse path of $\frac{1}{2}$ can have a coordinate x^* satisfying $x_1 < x^* < x_H$" is not appropriate, unless "no inverse path" is replaced by "no legal inverse path." (Note that the definitions of inverse path and legal inverse path in [1, p. 34] are different.) By Proposition 1, an illegal inverse path does always exist between P and PAP. For example, let $P = R$, and let x_1 be the coordinate of the pattern P. The first harmonic of P is $H = RLR$. Let x_H be the coordinate of H. To see that $Q = HL$ is a pattern whose coordinate x^* satisfies the inequalities $x_1 < x^* < x_H$, we use (3.1) or (3.2) to compute $x_1 = \mathscr{L}_P = \frac{1}{2}$, $x_H = \mathscr{L}_P = \frac{5}{8}$, $x^* = \mathscr{L}_Q = \frac{9}{16}$. Of course, there are infinitely many patterns Q with this property. Moreover, there is no proof for the statement of Theorem 5 above in MSSs proof of Lemma 2. However, Beyer and Stein [3, pp. 278–280] gave a proof for symmetric trapezoidal maps, and in 1986 Louck and Metropolis [6, Lemma B11, Appendix B] gave a correct proof for symmetric trapezoidal maps. Here we give a proof of Theorem 5 for general unimodal maps directly from our definition of an inverse path.

Proof of Theorem 5. Assume $x_n \in I_1$ is the last point of the inverse path $W_{S,f} = \{x_0, ..., x_n\}$, corresponding to some pattern $S = B_1 B_2 \cdots B_n \in \mathscr{P}$. Recall that $x_0 = y_0$. We need to prove that S is not a legal pattern.

If $n \leqslant k$, then

$$\begin{aligned}
x_{n-1} &= f(x_n) \in f(I_1) = I(y_{k-1}, y_{2k}),\\
x_{n-2} &= f^2(x_n) \in f^2(I_1) = I(y_{k-2}, y_{2k-1}),\\
&\vdots \qquad\qquad \vdots\\
x_0 &= f^n(x_n) \in f^n(I_1) = I(y_{k-n}, y_{2k+1-n}).
\end{aligned}$$

On the other hand, by definition,

$$I(y_{k-n}, y_{2k+1-n}) = f^{A_{n+1} \cdots A_k}(I(y_0, y_{k+1})),$$

and the whole open interval $I(y_{k-n}, y_{2k+1-n})$ lies to the left or to the right side of $x_0 = y_0$, that is, it cannot contain x_0. Therefore, $n \leqslant k$ is false. Next we consider $n > k$. In this case,

$$\begin{aligned} x_{n-k-1} &= f^{k+1}(x_n) \in f^{k+1}(I_1) \\ &= ff^k(I_1) = f(I(y_0, y_{k+1})) \\ &= I(1, y_k). \end{aligned}$$

If $y_{2k+1} < y_k$ or $y_{2k+1} > y_k$ and $x_{n-k-1} \in [y_{2k+1}, 1)$, then $x_n < x_{n-k-1}$. Therefore, x_n is not the rightmost point of $W_{S,f}$, and S is not a legal pattern. If $y_{2k+1} > y_k$ and $x_{n-k-1} \in I(y_k, y_{2k+1})$, then we start from x_{n-k-1} as from x_n above. If $n-k-1 \leqslant k$, then we obtain a contradiction. If $n-k-1 > k$, then $x_{n-2(k+1)} \in I(1, y_k)$, etc. We repeat this procedure: we go first from $x_{n-2(k+1)}$ as from x_n, then from $x_{n-3(k+1)}$, etc. Ultimately, we find that going from x_n to x_0 on the inverse path $W_{S,f}$ we meet a point x_i such that $x_i > x_n$. This means $W_{S,f}$ is not a lip, and the proof is completed.

Theorem 5 is equivalent to

THEOREM 5′ (Contiguity of harmonics). *If $P \in \mathscr{P}$ and $A = L$ or R, there exists no legal pattern that lies between P and PAP in the ordered set $\mathscr{P}$.*

6. SOLUTIONS λ OF $[\lambda f]^{(k)}(y_0) = y_0$ AND LEGAL PATTERNS

In this section we prove that the order $<$ on $\mathscr{P}$ is equivalent to the order introduced by Beyer *et al.* in [4, p. 309], to the order of Collet and Eckman [5, pp. 64–67], and to the order of Louck and Metropolis (LM) [6]. (LM [6] showed that their order is equivalent to BMSs.) These authors studied the set S of all finite LR-sequences terminating with a symbol C, and infinite LR-sequences as well. They defined a parity-lexicographic ordering of S from the formal structure of patterns. We denote their order by $<_S$, and we call ours $<_{\mathscr{P}}$.

PROPOSITION 4. *The orders $<_S$ and $<_{\mathscr{P}}$ are equivalent on $\mathscr{P}$.*

Proof. By Theorem 2, we need only to prove that, given any P and Q in $\mathscr{P}$, $P <_S Q$ if and only if $\mathscr{L}_P < \mathscr{L}_Q$. Note that $I^{f_0}(\mathscr{L}_P) = PC$ and $I^{f_0}(\mathscr{L}_Q) = QC$, where the definition of an itinerary $I^f(x)$ is given by Beyer *et al.* in [4, p. 311] and where f_0 is defined in Section 2. Following the proof of BMSs Lemma 4.1 [4, p. 311] (we only need to delete λ), we obtain $P <_S Q$ implies $\mathscr{L}_P < \mathscr{L}_Q$; namely, $P <_S Q$ implies $P <_{\mathscr{P}} Q$.

Conversely, given any P and Q in $\mathscr{P}$ such that $\mathscr{L}_P < \mathscr{L}_Q$, if $Q <_S P$, then the above proof implies that $\mathscr{L}_Q < \mathscr{L}_P$, which is a contradiction. If $P =_S Q$, that is, P and Q are identical, then, by the definition of $\mathscr{L}_P$, we have $\mathscr{L}_P = f_0^P(0) = f_0^Q(0) = \mathscr{L}_Q$, which is again a contradiction. Therefore $P <_{\mathscr{P}} Q$ implies $P <_S Q$, and the proof is complete.

BMS proved (see Theorem 4.2 [4, p. 312] and Theorem 5.3 [4, p. 317] that if f is a unimodal, round-top, concave function, then a finite LR-sequence is shift-maximal if and only if it is a functional MSS-sequence. Using this important result of BMS and our Propositions 3 and 4, we obtain the result that, for the class of round-top, concave, unimodal functions, a pattern is legal if and only if it is a functional MSS-sequence; in other words:

THEOREM 6. *For the class of round-top, concave unimodal functions, a pattern is associated with a solution* λ *of* $[\lambda f]^{(k)}(y_0) = y_0$ *for some* k *if and only if it is legal.*

Remarks. Theorem 6 is in part the same as Lemma 1 of MSS [1, p. 35]. However, MSSs proof of their Lemma 1 is incomplete; both BMS [4] and LM [6] confirm this. Further, Theorem 6 does not imply that a 1–1 correspondence between the set of legal patterns and the set of all solutions λ of the sequence of equations $[\lambda f]^k(y_0) = y_0$ $(k = 1, 2,...)$ exists. MSS conjectured that this correspondence is 1–1, but this has not been proved, to our knowledge, for unimodal functions in general. LM [6] proved that the correspondence is 1–1 for symmetric trapezoidal maps. Their proof is long and difficult. Beyond LMs result little is known. To prove 1–1'ness for all λ's and sequences of legal patterns in general or to prove in general that kneading sequences are monotone increasing with λ appears to be very hard.

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