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COMPLEX AND REAL K-THEORY AND LOCALIZATION

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Introduction

The main purpose of this note is to prove some facts about localization (in the sense of Bousfield [8]) of a space X with respect to real and complex K-Theory. In particular we compare the spaces which are acyclic for the real and complex K-homologies $\widetilde{KOA}_*$ and $\widetilde{KA}_*$ with coefficients in an arbitrary abelian group A (Theorem 1.7):

If X is an arbitrary CW-complex, then $\widetilde{KA}_*(X)=0$ if and only if $\widetilde{KOA}_*(X)=0$.

The proof of this theorem is based on methods of Anderson's thesis [3] and a suggestion of the referee (proof of Proposition 1.5). Theorem 1.7 implies that localization with respect to complex KA-homology is equivalent to localization with respect to real KOA-homology (Corollary 1.8). Moreover it turns out that the coefficient group A can be restricted to special classes of groups [6] (Corollary 1.10):

For an arbitrary abelian group A there exists a set of primes J such that the localizations X_{KA}, X_{KS}, X_{KOS} and X_{KOA} are equivalent, where either $S=\mathbb{Z}[J^{-1}]$ or $S=\bigoplus_{p\in J}\mathbb{Z}/p$ depending on A).

As a consequence of Corollary 1.8 we obtain many new examples of K-local spaces. In particular the spaces of the spectrum of real K-Theory are K-local as well as BSO, all homogeneous spaces obtained from the stable classical groups O, U, SO, SU, Spin, and loop spaces of such (Proposition 2.3).

In [11] Mislin has shown that the localization X_E with respect to a generalized homology E_* can be constructed out of rational and mod p information in a similar way as in the case of ordinary homology provided all spaces involved are 1-connected (these spaces are X_E, $\check{X}_E=\prod_p X_{E\mathbb{Z}_{(p)}}$, $X_{E\mathbb{Q}}$, $(X_E)_{E\mathbb{Q}}$, or $\hat{X}_E=\prod_p X_{E\mathbb{Z}/p}$, $(\hat{X}_E)_{E\mathbb{Q}}$ respectively).

For K-Theory this is the case if X is 1-connected [11]. We show that this condition can be replaced by a weaker one (Theorem 3.1):

For a space X with finite fundamental group the diagrams

$$\begin{array}{ccc} X_K & \longrightarrow & \check{X}_K \\ \downarrow & & \downarrow \\ X_{K\mathbb{Q}} & \longrightarrow & (\check{X}_K)_{K\mathbb{Q}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_K & \longrightarrow & \hat{X}_K \\ \downarrow & & \downarrow \\ X_{K\mathbb{Q}} & \longrightarrow & (\hat{X}_K)_{K\mathbb{Q}} \end{array}$$

are (up to homotopy) fibre squares. Furthermore $X_{K\mathbb{Z}_{(p)}} = (X_K)_{H\mathbb{Z}_{(p)}}$ and $X_{K\mathbb{Z}/p} = (X_K)_{H\mathbb{Z}/p}$ for all primes p (Corollary 3.4).

As an application of Theorem 3.1 we compute the homotopy groups of $BU[2n]_K$, $n \geqslant 2$, the K-localization of the $(2n-1)$-connective coverings of BU. In this way we obtain further examples of the periodicity theorem in [11]. I am very much indebted to G. Mislin, E. Dror and M. Huber for ideas and helpful discussions. Finally I would like to thank the referee for his valuable suggestions.

1. Acyclic spaces in K-Theory

1.1

Let E_* denote a homology theory (defined by a spectrum E) on ΠCW, the pointed homotopy category of CW-complexes. A space $X \in \Pi\mathrm{CW}$ is called *E-local* in the sense of [8], if for every E_*-isomorphism $f: A \to B \in \Pi\mathrm{CW}$ the induced map

$$f^*: [B, X] \to [A, X]$$

is a bijection. A *E-localization* of $X \in \Pi\mathrm{CW}$ is defined to be a E_*-isomorphism $X \to X_E \in \Pi\mathrm{CW}$ such that X_E is E-local. Such a E-localization of X is unique up to equivalence. The main theorem of [8] asserts its existence. The E-localization functor is characterized by the acylic spaces of the homology E_* as follows: Two homology theories F_* and G_* are said to have "the same acyclic spaces", if the following equivalent conditions hold [6]:

(i) For $X \in \Pi\mathrm{CW}$, $\tilde{F}_*(X) = 0$ if and only if $\tilde{G}_*(X) = 0$.

(ii) For a map $f: X \to Y \in \Pi\mathrm{CW}$, $f_*: F_*(X) \to F_*(Y)$ is an isomorphism if and only if $f_*: G_*(X) \to G_*(Y)$ is an isomorphism.

Proposition 1.1. *If the homology theories F_* and G_* have the same acyclic spaces they define equivalent localization functors on* ΠCW.

The proof is immediate from the universal property of localization [8].

1.2

A homology F_* is called *connective*, if $F_i(pt) = 0$ for i sufficiently small. In [6] it is shown that any connective homology F_* has the same acyclic spaces as $H_*(-: S)$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p\in J} \mathbb{Z}/p$ for some set of primes J. The class of acyclic spaces in K-Theory is not so restricted, since it contains the spaces $K(G, n)$, G a torsion group, $n \geq 2$ [5] and the cofibres of the maps $A(p): M(\mathbb{Z}/p, 2p+1) \to M(\mathbb{Z}/p, 3)$, p an odd prime, and $A(2): M(\mathbb{Z}/2, 13) \to M(\mathbb{Z}/2, 5)$ (see [1] and [11]).

In this section we will show however, that real and complex K-homology $\widetilde{KOA}_*$ and $\widetilde{KA}_*$ with coefficients in an arbitrary abelian group have the same acyclic spaces as KS_*, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p\in J} \mathbb{Z}/p$ for a suitable set of primes J. For this we need the methods of Anderson [3], relating various K-(co)homology theories by long exact sequences.

Let $\eta: S^3 \to S^2$ be the Hopf map. We set $h = H, 1 \in K^0(P)$, where H denotes the (complex) Hopf bundle over the projective plane $P = S^2 \cup_\eta e^4$. As in [3] one defines a transformation of cohomology theories

$$W: \tilde{K}^n(X) \to \widetilde{KO}^{n+4}(X \wedge P)$$

by

$$W(x) = r(\pi_U^{-2}(x) \wedge \tilde{h}),$$

where the notations are as follows:

$r: K \to KO$ is the map of spectra [12, p. 303] induced by realification of complex vector bundles. $\pi_U: \Sigma^2 K \to K$ is the equivalence induced by Bott periodicity in complex K-Theory. The (exterior) product

$$\wedge: \tilde{K}^m(X) \otimes \tilde{K}^n(Y) \to \tilde{K}^{m+n}(X \wedge Y)$$

is induced by the tensor product of complex vector bundles and can be extended to arbitrary complexes X and Y in ΠCW (or spectra) [12, Chapter 13]. With this product K is a ring spectrum, that is, $\wedge$ commutes with suspensions [12, Proposition 13.55]. Hence the map W can be defined on arbitrary complexes (or spectra) and is a natural transformation of cohomology theories on ΠCW.

Theorem 1.2. *The transformation W induces an equivalence of spectra $K \simeq KO \wedge E$, where $E = S^0 \cup_\eta e^2$ is the suspension spectrum whose second term is P* [2, *p.* 206].

Proof The cofibre sequence of suspension spectra

$$\Sigma S^0 \xrightarrow{\eta} S^0 \to E \to \Sigma^2 S^0 \xrightarrow{\Sigma\eta} \Sigma S^0$$

is by S-duality converted into the cofibre sequence

$$\Sigma^{-1} S^0 \xleftarrow{\eta^*} S^0 \leftarrow E^* \leftarrow \Sigma^{-2} S^0 \xleftarrow{\Sigma^{-1}(\eta)^*} \Sigma^{-1} S^0$$

and for arbitrary spectra X there is a commutative diagram [12, Corollary 14.33]

$$\begin{array}{ccccccccccc}
\cdots\leftarrow & [X, KO\wedge\Sigma^{-1}S^0] & \leftarrow & [X, KO\wedge S^0] & \leftarrow & [X, KO\wedge E^*] & \leftarrow & [X, KO\wedge\Sigma^{-2}S^0] & \leftarrow & [X, KO\wedge\Sigma^{-1}S^0] \\
 & \downarrow\cong & & \downarrow\cong & & \cong\downarrow D & & \downarrow\cong & & \downarrow\cong \\
\cdots\leftarrow & [X\wedge\Sigma S^0, KO] & \leftarrow & [X\wedge S^0, KO] & \leftarrow & [X\wedge E, KO] & \leftarrow & [X\wedge\Sigma^2S^0, KO] & \leftarrow & [X\wedge\Sigma S^0, KO]
\end{array}$$

The transformation

$$D^{-1}\circ W:[X, K]\xrightarrow{W}[X\wedge\Sigma^2E, \Sigma^4KO]\xrightarrow{D^{-1}}[X, \Sigma^4KO\wedge\Sigma^{-2}E^*]$$

defines a map of spectra

$$W:K\to KO\wedge\Sigma^{-2}E^*$$

which by the relation $E^*=\Sigma^{-2}E$ reads

$$W:K\to KO\wedge E.$$

By a theorem of R. Wood the map W induces isomorphisms in the coefficients and is therefore an equivalence.

Corollary 1.3. *For any $X\in\Pi\mathrm{CW}$ (or for any spectrum X) the cofibration of spectra*

$$KO\wedge E\to KO\wedge\Sigma^2S^0\to KO\wedge\Sigma S^0$$

induces long exact sequences

$$\cdots\to\widetilde{KA}_n(X)\to\widetilde{KOA}_{n-2}(X)\to\widetilde{KOA}_{n-1}(X)\to\widetilde{KA}_{n-1}(X)\to\cdots$$

and

$$\cdots\to\widetilde{KA}^n(X)\to\widetilde{KOA}^{n+2}(X)\to\widetilde{KOA}^{n+1}(X)\to\widetilde{KA}^{n+1}(X)\to\cdots$$

where A denotes an arbitrary abelian group, and $KA=K\wedge SA$ (or $KOA=KO\wedge SA$) is the spectrum K (or KO) with coefficients in A [2, *p.* 200].

Corollary 1.4. *Let $X\in\Pi\mathrm{CW}$. Then*

$$\widetilde{KA}_*(X)=0\quad \textit{if}\ \widetilde{KOA}_*(X)=0,$$

and

$$\widetilde{KA}^*(X)=0\quad \textit{if}\ \widetilde{KOA}^*(X)=0.$$

The next result shows that the statements of Corollary 1.4 are also valid in the opposite direction.

Proposition 1.5. *Let $X \in \Pi\mathrm{CW}$. Then*

$$\widetilde{KOA}_*(X)=0 \quad \textit{if } \widetilde{KA}_*(X)=0,$$

and

$$\widetilde{KOA}^*(X)=0 \quad \textit{if } \widetilde{KA}^*(X)=0.$$

Proof. Let $E^{(n)}$, $n \in \mathbb{N}$, denote the CW-spectrum $S^0 \cup_{\eta^n} e^{n+1}$, where $\eta^n = \Sigma^{n-1}\eta^{n-1}$ and $\eta = \eta^1 : S^1 \to S^0$ is the (stable) Hopf map. For every $n \in \mathbb{N}$ there is a homotopy-commutative diagram of cofibrations:

$$\begin{array}{ccccc}
S^n & \xrightarrow{\eta^n} & S^0 & \longrightarrow & E^{(n)} \\
\downarrow{\scriptstyle \Sigma^{n-1}\eta} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
S^{n-1} & \xrightarrow{\eta^{n-1}} & S^0 & \longrightarrow & E^{(n-1)} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{n-1}E^{(1)} & \longrightarrow & * & \longrightarrow & \Sigma^n E^{(1)}
\end{array}$$

In particular we have constructed cofibrations

$$E^{(2)} \to E^{(1)} \to \Sigma^2 E^{(1)},$$
$$E^{(3)} \to E^{(2)} \to \Sigma^3 E^{(1)},$$
$$E^{(4)} \to E^{(3)} \to \Sigma^4 E^{(1)}.$$

But $E^{(4)} \simeq S^0 \vee S^5$ since $\eta^4 = 0$. This follows from the fact that the 2-component of the 4-stem of the stable homotopy groups of spheres is trivial. Smashing the above cofibrations and equivalence with KOA we obtain long exact sequences similar to Corollary 1.3. Proposition 1.5 follows now by Theorem 1.2. Proposition 1.5 together with Corollary 1.4 imply

Theorem 1.7. *Let $X \in \Pi\mathrm{CW}$. Then for an arbitrary abelian group A*

(i) $\widetilde{KA}_*(X)$ *if and only if* $\widetilde{KOA}_*(X)=0$
(ii) $\widetilde{KA}^*(X)=0$ *if and only if* $\widetilde{KOA}^*(X)=0$.

By Theorem 1.7 and Proposition 1.1 we get

Corollary 1.8. *Let $X \in \Pi\mathrm{CW}$. Then for any abelian group A the localization X_{KA} with respect to KA-homology is equivalent to the localization X_{KOA} with respect to KOA-homology.*

1.3

A class $\mathfrak{M}$ of abelian groups is special in the sense of Bousfield [6], if it satisfies the following conditions:

(i) $0 \in \mathfrak{M}$.

(ii) $\mathfrak{M}$ is closed under arbitrary direct sums.

(iii) If $A_1 \to A_2 \to A_3 \to A_4 \to A_5$ is an exact sequence of abelian groups with A_1, A_2, A_4, $A_5 \in \mathfrak{M}$, then $A_3 \in \mathfrak{M}$.

For an abelian group A, one can obtain a special class $\mathfrak{M}_A$ by taking all G with $G \otimes A = 0 = \operatorname{Tor}(G, A)$, or by taking all G with $\operatorname{Hom}(G, A) = 0 = \operatorname{Ext}(G, A)$ [6]. We call this class $\mathfrak{M}^A$.

By Proposition 2.3 of [6] the only special classes of abelian groups are the J-primary torsion groups and the uniquely J-divisible groups, where J is a set of primes.

Proposition 1.9. *Let F_* be a homology theory on* ΠCW, *and let $\mathfrak{N}_{FA}$ denote the class of acyclic spaces of $\widetilde{FA}_*$ for an arbitrary abelian group A. Then there exists an abelian group S such that $\mathfrak{N}_{FA} = \mathfrak{N}_{FS}$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p \in J} \mathbb{Z}/p$ for some set of primes J.*

Corollary 1.10. *Let A be any abelian group. Then there exists a set of primes J such that $X_{KA} = X_{KS}$ and $X_{KOA} = X_{KOS}$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p \in J} \mathbb{Z}/p$.*

It follows that investigating localization with respect to real and complex K-homology with coefficients we can restrict attention to complex K-homology with coefficients in a special class of groups.

Proof of Proposition 1.9. Let $\mathfrak{M}_A$ be the class of all abelian groups G with $G \otimes A = 0 = \operatorname{Tor}(G, A)$. We show first that there exists a group S with $\mathfrak{M}_A = \mathfrak{M}_S$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p \in J} \mathbb{Z}/p$ for some set of primes J.
For this purpose we apply the universal coefficient theorem

$$0 \to \tilde{H}_n(X) \otimes A \to \tilde{H}_n(X, A) \to \operatorname{Tor}(\tilde{H}_{n-1}(X), A) \to 0.$$

Hence $\tilde{H}_n(X) \in \mathfrak{M}_A$ for all $n \in \mathbb{N}$ if $X \in \mathfrak{N}_{HA}$. Conversely for any $G \in \mathfrak{M}_A$ one can find a $X \in \mathfrak{N}_{HA}$ such that $\tilde{H}_n(X) = G$. (Take e.g. the Moore space $X = M(G, n)$, $n \geq 2$.) Therefore we get the relation

$$\mathfrak{M}_A = \{\tilde{H}_n(X),\ n \in \mathbb{N} \mid X \in \mathfrak{N}_{HA}\}.$$

By Theorem 4.5 of [6] there is a group S in a special class such that

$$\mathfrak{N}_{HA} = \mathfrak{N}_{HS}$$

i.e.

$$\mathfrak{M}_A = \mathfrak{M}_S.$$

This relation together with the universal coefficient theorem [2, p. 201]

$$0 \to \tilde{F}_n(X) \otimes A \to \widetilde{FA}_n(X) \to \mathrm{Tor}(\tilde{F}_{n-1}(X), A) \to 0$$

implies that

$$X \in \mathfrak{N}_{FA} \quad \text{if and only if} \quad X \in \mathfrak{N}_{FS}.$$

With regard to the acyclic spaces of a cohomology $\widetilde{FA}^*$, we can deduce a similar result if there exists a homology E_* and a universal coefficient theorem

$$0 \to \mathrm{Ext}(E_{n-1}(X), A) \to FA^n(X) \to \mathrm{Hom}(E_n(X), A) \to 0$$

which is valid for any $X \in \Pi\mathrm{CW}$ and any abelian group A. It is known [4], [14], that besides ordinary cohomology there are universal coefficient theorems in real and complex K-Theory:

$$0 \to \mathrm{Ext}(KSp_{n-1}(X), A) \to KOA^n(X) \to \mathrm{Hom}(KSp_n(X), A) \to 0$$

where $n \in \mathbb{Z}/8$ and $KSp = \Sigma^4 KO$ and

$$0 \to \mathrm{Ext}(K_{n-1}(X), A) \to KA^n(X) \to \mathrm{Hom}(K_n(X), A) \to 0$$

where $n \in \mathbb{Z}/2$.

Proposition 1.11. *Let F be either H, K or KO and let $\mathfrak{N}^{FA}$ denote the class of acyclic spaces of $\widetilde{FA}^*$. Then there exists a set of primes J such that $\mathfrak{N}^{FA} = \mathfrak{N}^{FS}$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p \in J} \mathbb{Z}/p$.*

The proof is completely analogous to the proof of Proposition 1.9. In general, in the "reductions" $\mathfrak{N}_{FA} = \mathfrak{N}_{FS_1}$ and $\mathfrak{N}^{FA} = \mathfrak{N}^{FS_2}$ the groups S_1 and S_2 are not the same, as the following example may show:

Example. Let F be either H, K or KO and let $A = \mathbb{Q}/\mathbb{Z}$. Then $\mathfrak{N}^{FA} = \mathfrak{N}^{F\mathbb{Z}} = \mathfrak{N}^F$, but $\mathfrak{N}_{FA} = \mathfrak{N}_{FS}$, where $S = \bigoplus_{p \in P} \mathbb{Z}/p$. If however A itself is special i.e. $A = S$, where either $S = \mathbb{Z}[J^{-1}]$ or $S = \bigoplus_{p \in J} \mathbb{Z}/p$ for some set of primes J, then we have

Proposition 1.12. *Let F be either H, K or KO and let $A = S$ be a special group. Then $\mathfrak{N}_{FS} = \mathfrak{N}^{FS}$.*

This follows from the fact that $\mathfrak{M}_S = \mathfrak{M}^S$, if S is special.

Corollary 1.13. *Let F be either H, K or KO and let S be a special group. Then $f: X \to Y \in \Pi\mathrm{CW}$ is a FS_*-isomorphism if and only if it is a FS^*-isomorphism. We therefore call f in this case just a FS-isomorphism.*

2. Some examples of K-local spaces

In [11] Mislin has given several examples of K-local spaces. In particular the representing spaces of a theory $\widetilde{EA}^*$ are E-local for an arbitrary abelian group A, if E is a ring spectrum [11, Theorem 1.11]. From this result and Corollary 1.9 we immediately deduce

Proposition 2.1. *Let A be any abelian group. Then the representing spaces of $\widetilde{KOA}^*$ are K-local.*

From the results in [9] we see that any H-local space can be built up from Eilenberg-Mac Lane Spaces $K(A, n)$. Proposition 2.1 gives us nontrivial examples of K-local spaces which can be built up from the spaces U and BU. In this way we can obtain $BO \times BSp$ as a "two stage Postnikov system" with U and BU as fibres. For this we use the fibre sequences [3]

$$U \to C^0 \to BU \xrightarrow{\psi} BU$$

and

$$BU \to BO \times BSp \to C^0 \to U.$$

Thereby ψ denotes the operation $\psi^1 - \psi^{-1}$ and C^0 is the representing space of selfconjugate K-Theory $\widetilde{KC}^0$. We then obtain a diagram

$$\begin{array}{ccccc}
BU & \longrightarrow & BO \times B\mathrm{Spin} & & \\
 & & \downarrow & & \\
U & \longrightarrow & C^0 & \longrightarrow & U \\
 & & \downarrow & & \\
 & & BU & \xrightarrow{\psi} & BU
\end{array}$$

Proposition 2.2. *BSO and* Spin *are K-local, but $B\mathrm{Spin}_K = BSO$.*

Proof. BSO is the universal covering space of BO. Hence there is a fibration

$$BSO \to BO \to \mathbb{R}P^\infty$$

which is trivial since it has a section and BO is a H-space. Hence BSO (and $\mathbb{R}P^\infty$) are factors of the K-local space BO (Proposition 2.1) and therefore K-local.

Spin is the universal covering of SO. From the fibration

$$\mathrm{Spin} \xrightarrow{\pi} SO \to \mathbb{R}P^{\infty}$$

we deduce, that Spin is K-local, since $SO = \Omega BSO$ is K-local. For the third assertion we need the fact that $B\pi : B\mathrm{Spin} \to BSO$ is a K-isomorphism. As we see from a Serre spectral sequence argument, $B\pi$ is a $H\mathbb{Q}$-isomorphism and a $K\mathbb{Z}/p$-isomorphism for p any odd prime, and a result of Snaith [13] tells, that $B\pi$ is a $K\mathbb{Z}/2$-isomorphism.

Proposition 2.3. *All homogeneous spaces obtained from the stable classical groups O, SO, SU, Sp and all loop spaces of such are K-local.*

Proof. For any pair $G \subset H$ of topological groups there is a fibration

$$H/G \to B_G \to B_H.$$

If B_G and B_H are K-local, then also H/G is K-local. From Propositions 2.1 and 2.2 and [11] it follows, that the classifying spaces of the stable groups O, U, SO, SU and Sp are K-local.

3. Fibre squares

It has been shown in [11], that the localization X_E with respect to a homology E_* can be constructed out of rational and mod p information quite in a way as in the case $E = H$. Let $\mathbb{Z}_{(p)}$ denote the integers localized at p and define

$$\check{X}_E = \prod_{p \in P} X_{E\mathbb{Z}_{(p)}}, \qquad \hat{X}_E = \prod_{p \in P} X_{E\mathbb{Z}/p}.$$

Then the following two squares are fibre squares, if the spaces involved are all 1-connected [11, Proposition 1.9]:

$$\begin{array}{ccc} X_E & \longrightarrow & \check{X}_E \\ \downarrow & & \downarrow \\ X_{E\mathbb{Q}} & \longrightarrow & (\check{X}_E)_{E\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X_E & \longrightarrow & \hat{X}_E \\ \downarrow & & \downarrow \\ X_{E\mathbb{Q}} & \longrightarrow & (\hat{X}_E)_{E\mathbb{Q}}. \end{array} \tag{3.1}$$

For $E = KR$, R a subring of the rationals or $R = \mathbb{Z}/p$, p a prime, one has [11] that X_{KR} is 1-connected if X is 1-connected. Hence in this case the diagrams (3.1) are fibre squares. In the next theorem we generalize this statement to spaces with a finite fundamental group:

Theorem 3.1. *Let $X \in \Pi\mathrm{CW}$ be a space with a finite fundamental group. Then the digrams*

$$\begin{array}{ccc} X_K & \longrightarrow & \check{X}_K \\ \downarrow & & \downarrow \\ X_{K\mathbb{Q}} & \longrightarrow & (\check{X}_K)_{K\mathbb{Q}} \end{array} \qquad\qquad \begin{array}{ccc} X_K & \longrightarrow & \hat{X}_K \\ \downarrow & & \downarrow \\ X_{K\mathbb{Q}} & \longrightarrow & (\hat{X}_K)_{K\mathbb{Q}} \end{array}$$

are fibre squares.

To prove this theorem we apply some methods of [10]. First we need the following Proposition and its Corollary. We can assume X to be connected.

Proposition 3.2. *Let $X \in \Pi\mathrm{CW}$ and let R be a subring of the rationals or $R = \mathbb{Z}/p$, p a prime. Then the KR-localization can: $X_{HR} \to X_{KR}$ induces an epimorphism in the fundamental groups.*

Proof. $\mathrm{can}: X_{HR} \to X_{KR}$ is a KR-isomorphism. Hence by [11, Theorem 2.6] $\mathrm{can}_*: H_1(X_{HR}, R) \to H_1(X_{KR}, R)$ is an isomorphism. Since X_{KR} is HR-local, we can apply the generalized Whitehead theorem [7] and we get an epimorphism $\pi_1(X_{HR}) \to \pi_1(X_{KR})$.

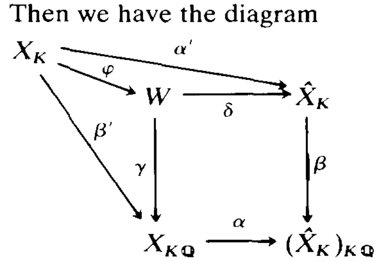
Corollary 3.3. *Let $X \in \Pi\mathrm{CW}$ be a space whose fundamental group is HR-local. Then $\mathrm{can}_*: \pi_1(X) \to \pi_1(X_{KR})$ is an epimorphism.*

This follows directly from the factorization [11]

$$\begin{array}{ccc} X & \longrightarrow & X_{HR} \\ & \searrow & \downarrow \\ & & X_{KR} \end{array}$$

and the relation $\pi_1(X_{HR}) = \pi_1(X)_{HR} = \pi_1(X)$.

Proof of Theorem 3.1. Since the proofs for the two squares are similar, we consider only the second square. Then we have the diagram



where W is the (homotopy theoretic) pullback. By [8] W is K-local. It is therefore enough to show that φ is a K-isomorphism. For this purpose we show that φ is a $K\mathbb{Q}$-isomorphism and a $K\mathbb{Z}/p$-isomorphism for any prime p.

Since $\pi_1(X)$ is finite, Corollary 3.3 implies that $X_{K\mathbb{Q}} = X_{H\mathbb{Q}}$ and $(\hat{X}_K)_{K\mathbb{Q}} = (\hat{X}_K)_{H\mathbb{Q}}$ are 1-connected.

Let F be the fibre of the map $\alpha : X_{K\mathbb{Q}} \to (\hat{X}_K)_{K\mathbb{Q}}$. F is also the fibre of the map δ. In addition F is connected and $\mathbb{Q}$-local. By Lemma 1.13 of [10] $\gamma : W \to X_{K\mathbb{Q}}$ is a $H\mathbb{Q}$-isomorphism since this is the case for β. The map β' is also a $H\mathbb{Q}$-isomorphism and hence $\varphi : X_K \to W$ is a $H\mathbb{Q}$-isomorphism. This immediately implies that φ is a $K\mathbb{Q}$-isomorphism. We now show that $\alpha' : X_K \to \hat{X}_K$ is a $K\mathbb{Z}/p$-isomorphism for any prime p. For this we consider the projection

$$\pi : \hat{X}_K \to X_{K\mathbb{Z}/p}$$

with the fibre $F' = \prod_{q \neq p} X_{K\mathbb{Z}/q}$. By Corollary 3.3 $\pi_1(F')$ is finite and by Lemma 7.5 of [8] even nilpotent. $X_{K\mathbb{Z}/q}$ is $H\mathbb{Z}/q$-local. Hence the homotopy groups of $X_{K\mathbb{Z}/q}$ are uniquely s-divisible for any prime $s \neq q$, [10, Lemma 1.24]. It follows that the homotopy groups of F' are uniquely p-divisible. By Lemma 1.26 of [10] the homology groups of F' are as well uniquely p-divisible, and by [10, Lemma 1.27] the groups $\tilde{H}_*(F', \mathbb{Z}/p)$ vanish. By a mod-p Serre spectral sequence argument $\pi_* : H_*(\hat{X}_K, \mathbb{Z}/p) \to H_*(X_{K\mathbb{Z}/p}, \mathbb{Z}/p)$ is an isomorphism.

Since $\alpha'_p : X_K \to X_{K\mathbb{Z}/p}$ is a $K\mathbb{Z}/p$-isomorphism, the diagram

$$\begin{array}{ccc} X_K & \xrightarrow{\alpha'} & \hat{X}_K \\ & \searrow^{\alpha'_p} & \downarrow{\scriptstyle \pi} \\ & & X_{K\mathbb{Z}/p} \end{array}$$

implies, that α' is a $K\mathbb{Z}/p$-isomorphism for any prime p. By [10, Lemma 1.13] $\delta : W \to \hat{X}_K$ is a $H\mathbb{Z}/p$-isomorphism for any p, since $\alpha : X_{K\mathbb{Q}} \to (\hat{X}_K)_{K\mathbb{Q}}$ is a $H\mathbb{Z}/p$-isomorphism for any p. Therefore $\varphi : X_K \to W$ is a $K\mathbb{Z}/p$-isomorphism for any prime p.

Corollary 3.4. *Let $X \in \Pi\mathrm{CW}$ be a space with a finite fundamental group. Then*

$$(X_K)_{H\mathbb{Z}_{(p)}} = X_{K\mathbb{Z}_{(p)}} \quad \text{and} \quad (X_K)_{H\mathbb{Z}/p} = X_{K\mathbb{Z}/p}.$$

Proof. We consider the diagram

$$\begin{array}{ccc} (X_K)_{H\mathbb{Z}/p} & \xrightarrow{\varepsilon} & X_{K\mathbb{Z}/p} \\ \uparrow & \nearrow^{\alpha'_p} & \uparrow{\scriptstyle \pi} \\ X_K & \xrightarrow{\alpha'} & \hat{X}_K \end{array}$$

From the proof of Theorem 3.1 it follows, that α' is a $H\mathbb{Z}/p$-isomorphism for any p, since $\delta: W \to \hat{X}_K$ is. Therefore α'_p and ε are $H\mathbb{Z}/p$-isomorphisms for any p. Since $X_{K\mathbb{Z}/p}$ is $H\mathbb{Z}/p$-local, the map ε is an equivalence. The first statement is proved similarly.

Corollary 3.5. *Let A be a finite abelian group. Then*

$$K(A,1)_{K\mathbb{Z}_{(p)}} = K(A \otimes \mathbb{Z}_{(p)}, 1) \quad \textit{and} \quad K(A,1)_{K\mathbb{Z}/p} = K(A \otimes \mathbb{Z}/p, 1).$$

As an application of Theorem 3.1 and Corollary 3.4 we compute the homotopy groups of $BU[2n]_K$, the K-localization of the $(2n-1)$-connective coverings of BU.

Proposition 3.6. *Let $X = BU[2n]$, $n \geq 2$. Then*

(i) $X_{K\mathbb{Z}/p} = BSU_{H\mathbb{Z}/p}$

(ii) $$\pi_i(X_K) = \begin{cases} \pi_i BU & i \geq 2n, \\ 0 & i = 2n-1, \\ 0 & i < 2n,\ i\ \textit{even}, \\ \mathbb{Q}/\mathbb{Z} & 5 \leq i < 2n-1,\ i\ \textit{odd}, \\ 0 & 0 \leq i \leq 3. \end{cases}$$

Similar results are valid for $BO[n]$, $n \geq 2$.

Proof. The first assertion follows from [11, Corollary 2.3] and from Corollary 3.4 (or see [11]). For the computation of $\pi_i(X_K)$ we use the Mayer-Vietoris sequence for the homotopy groups of the fibre square (Theorem 3.1 or [11])

$$\begin{array}{ccc} X_K & \longrightarrow & \hat{X}_K \\ \downarrow & & \downarrow \\ X_{K\mathbb{Q}} & \longrightarrow & (\hat{X}_K)_{K\mathbb{Q}} \end{array}$$

$$\cdots \to \pi_{i+1}\hat{X}_K \oplus \pi_{i+1}X_{K\mathbb{Q}} \to \pi_{i+1}(\hat{X}_K)_{K\mathbb{Q}} \to \pi_i X_K \to \pi_i \hat{X}_K \oplus \pi_i X_{K\mathbb{Q}} \to \pi_i(\hat{X}_K)_{K\mathbb{Q}} \to \cdots$$

From this sequence we can immediately read off the groups

$\pi_i X_K = \pi_i BU, \quad i \geq 2n.$

$\pi_{2n-1} X_K = 0.$

$\pi_i X_K = \ker(\hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \otimes \mathbb{Q}) = 0, \quad$ if $i < 2n$, i even.

$\pi_i X_K = \operatorname{coker}(\hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \otimes \mathbb{Q}) = \mathbb{Q}/\mathbb{Z}, \quad$ if $5 \leq i < 2n-1$, i odd.

$\pi_i X_K = 0, \quad$ if $0 \leq i \leq 3$.

Remark. The groups $\pi_i(X_K)$ of the form $\mathbb{Q}/\mathbb{Z}$ effect the periodicity in the homotopy groups $\pi_i(\mathbb{Z}/p, X_K)$ [11].

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