Pencils of Real Symmetric Matrices and Real Algebraic Curves

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ABSTRACT

Let a rational real algebraic curve \( \Gamma \) of degree \( n \) in the real projective plane \((x, y, z)\) be given by its parametric equations; let \( \Gamma \), not containing the point \( O = (0, 0, 1) \), have the property that (I) every line through the point \( O \) intersects \( \Gamma \) in \( n \) real points (counting multiplicities). An explicit form is given for real symmetric matrices \( A \) and \( B \) of order \( n \) such that \( \det(xA + yB + zI) = 0 \) is an equation for \( \Gamma \). For \( A \) diagonal, \( B \) is a diagonal multiple of a Loewner matrix. The result partly proves a previous conjecture of the author about general real algebraic curves of degree \( n \) having the property (I). It is also shown that in this general case, the conjecture is not true.

INTRODUCTION

In [1], the author formulated the following

CONJECTURE (C). In a real projective plane let \( \varphi(x, y, z) = 0 \) be a homogeneous equation of degree \( n \) of a curve \( \Gamma \) satisfying:

(a) \( \varphi(0, 0, 1) = 1 \);  
(b) every real line passing through the point \( (0, 0, 1) \) intersects \( \Gamma \) in \( n \) real points (counting multiplicities).

Then there exist real symmetric matrices \( A \) and \( B \) of order \( n \) such that
identically
\[ \varphi(x, y, z) = \det(xA + yB + zI), \tag{1} \]

It being the identity matrix of order \( n \).

[It is trivial that \( \varphi(x, y, z) \) defined by (1) satisfies both (a) and (b).]

In the present paper, we prove the conjecture for the case that every irreducible component of \( \Gamma \) is a rational curve [i.e. essentially a curve with parametric equations \( x = u(t), y = v(t), z = w(t) \) where \( u, v, w \) are polynomials], and disprove it in the general case.

RESULTS

**Theorem 1.** Let \( u(t), v(t), w(t) \) be real polynomials of degree at most \( n \) such that \( v(t) \) has \( n \) distinct real roots \( b_1, \ldots, b_n \) and for \( \varepsilon = 1 \) or \( -1 \),

\[ \varepsilon u(b_k)v'(b_k) > 0, \quad k = 1, \ldots, n. \]

Then the matrices

\[ A = \text{diag}(\alpha_k), \quad \alpha_k = -\frac{w(b_k)}{u(b_k)}, \]

\[ B = (b_{ik}), \]

where

\[ b_{ik} = -\varepsilon \frac{w(b_i) - w(b_k)}{u(b_i) - u(b_k)} \sqrt{\frac{u(b_i)}{v'(b_i)}} \sqrt{\frac{u(b_k)}{v'(b_k)}} \]

for \( i \neq k, \quad i, k = 1, \ldots, n, \)

\[ b_{ii} = -\left(\frac{w(t)}{u(t)}\right)'_{t=b_i} \frac{u(b_i)}{v'(b_i)}, \quad i = 1, \ldots, n, \]
and the column vector \( y(t) = (y_k) \),

\[
y_k = \sqrt{\frac{u(b_k)}{v'(b_k)}} \prod_{j \neq k} (t - b_j),
\]

satisfy

\[
[u(t)A + v(t)B + w(t)I]y(t) = 0 \tag{2}
\]

for all real \( t \).

Proof. Observe first that for \( t = b_k, \; k = 1, \ldots, n \), (2) is true, since \( u(b_k)A + w(b_k)I \) is a diagonal matrix the \( k \)th diagonal entry of which is \( u(b_k) \alpha_k + w(b_k) \), i.e. zero, while \( y(b_k) \) has all coordinates except the \( k \)th equal to zero.

Thus let \( t \neq b_k \) for all \( k = 1, \ldots, n \). Denote by \( D(t) \) the matrix

\[
D(t) = \frac{u(t)}{v(t)} A + \frac{w(t)}{v(t)} I,
\]

which can also be written as \( \text{diag}(d_i(t)) \), where

\[
d_i(t) = \frac{1}{v(t)} \left( w(t) - \frac{u(t)}{u(b_i)} w(b_i) \right).
\]

To compute \( D(t)y(t) \), write

\[
\frac{d_i(t)y_i}{v(t)} = \sqrt{\frac{u(b_i)}{v'(b_i)}} \frac{1}{t - b_i} \left( w(t) - \frac{u(t)}{u(b_i)} w(b_i) \right)
\]

and decompose the right-hand side into partial fractions:

\[
\frac{d_i(t)y_i}{v(t)} = \sum_{k=1}^{n} \frac{c_{ik}}{t - b_k},
\]
where for \( i \neq k, \)

\[
c_{ik} = \sqrt{\frac{\varepsilon}{v'(b_i)}} \frac{1}{v'(b_k)} \left[ \frac{1}{b_k - b_i} \left( w(b_k) - u(b_k) \frac{w(b_i)}{u(b_i)} w(b_i) \right) \right]
\]

\[
= \sqrt{\frac{\varepsilon}{v'(b_i)}} \frac{u(b_i)}{v'(b_i)} \frac{w(b_i) - w(b_k)}{b_i - b_k},
\]

\[
c_{ii} = \sqrt{\frac{\varepsilon}{v'(b_i)}} \frac{u(b_i)}{v'(b_i)} \lim_{t \to b_i} \frac{w(t) - w(b_i)}{u(t) - u(b_i)}
\]

\[
= \sqrt{\frac{\varepsilon}{v'(b_i)}} \frac{u(b_i)}{v'(b_i)} \left[ \frac{w(t)}{u(t)} \right]_{t = b_i}.
\]

On the other hand, \( B y(t) \) equals \( s(t) = (s_i), \) where

\[
s_i = -\varepsilon v(t) \sqrt{\frac{\varepsilon}{v'(b_i)}}
\]

\[
\times \left[ \sum_{k=1}^{n} \frac{w(b_i) - w(b_k)}{u(b_i) - u(b_k)} \frac{u(b_k)}{v'(b_k)} \frac{1}{t - b_k} \right.
\]

\[
+ \varepsilon \frac{u(b_i)}{v'(b_i)} \left[ \frac{w(t)}{u(t)} \right]_{t = b_i} \frac{1}{t - b_i} \right],
\]

which is equal to

\[
- v(t) \sum_{k=1}^{n} \frac{c_{ik}}{t - b_k},
\]
i.e. to \(-d_i(t)y_i\). Consequently,

\[ D(t)y(t) + By(t) = 0, \]

so that by the definition of \(D(t)\),

\[ [u(t)A + v(t)B + w(t)I]y(t) = 0. \]

**Theorem 2.** Conjecture (C) is true if the curve \(\Gamma\) is rational or if every irreducible component of \(\Gamma\) is rational. In the general case, Conjecture (C) is false for every \(n \geq 3\).

**Proof.** Let \(\Gamma\) be a real algebraic curve satisfying (a) and (b) of Conjecture (C). To prove the first part, we can suppose that \(\Gamma\) itself is an irreducible rational curve, since if the assertion is true for this case and \(\Gamma\) is reducible with rational components, then both \(A\) and \(B\) can be taken as direct sums of matrices corresponding to individual components.

Thus let \(x = u(t), y = v(t), z = w(t)\) be parametric equations of the irreducible rational curve \(\Gamma\), where \(u, v, w\) are relatively prime polynomials with maximum degree \(n\) and such that [4, p. 209] the three determinants of order two in the matrix

\[
\begin{pmatrix}
u(t_1) & w(t_1) \\
u(t_2) & w(t_2)
\end{pmatrix}
\]

have (as polynomials in \(t_1\) and \(t_2\)) the greatest common divisor \(t_1 - t_2\).

Assumption (a) of Conjecture (C) also implies that \(u(t)\) and \(v(t)\) are relatively prime polynomials with maximum degree \(n\), since otherwise the point \((0,0,1)\) would belong to \(\Gamma\).

Let us show that (a) and (b) imply that the degrees of \(u\) and \(v\) differ at most by one. Suppose the contrary, say, \(\deg u(t) = n, \deg v(t) < n - 2\). Setting

\[
\tilde{u}(t) = t^n u(t^{-1}), \quad \tilde{v}(t) = t^n v(t^{-1}), \quad \tilde{w}(t) = t^n w(t^{-1}),
\]

it is clear that \(\tilde{u}, \tilde{v}, \tilde{w}\) are again polynomials and

\[
x = \tilde{u}(t), \quad y = \tilde{v}(t), \quad z = \tilde{w}(t)
\]
are also parametric equations for $\Gamma$. These have the form

\[
x = u_0 + u_1 t + \cdots + u_n t^n, \quad u_0 \neq 0,
\]
\[
y = t^k (v_k + v_{k+1} t + \cdots + v_n t^{n-k}), \quad v_k \neq 0, \quad k \geq 2,
\]
\[
z = \tilde{w}(t).
\]

For $\varepsilon > 0$ sufficiently small, the line

\[
y = -\varepsilon \frac{v_k}{u_0} x
\]

intersects $\Gamma$ in points the new parameters $t$ of which satisfy

\[
v_k (t^k + \varepsilon) + t^{k+1} W_\varepsilon(t) = 0, \quad (3)
\]

where $W_\varepsilon(t)$ is a polynomial in $t$ and $\varepsilon$. Since $k \geq 2$, the equation $t^k + \varepsilon = 0$ has at least two nonreal roots. However, the same is true (for $\varepsilon$ sufficiently small) for the smallest (in modulus) roots of (3), a contradiction to (b).

We can now apply a theorem of Obreschkoff [2, p. 12] stating that for real relatively prime polynomials $u$, $v$ of degrees differing at most by one, the following conditions are equivalent:

(i) the equation

\[
F(t) = \lambda u(t) + \mu v(t) = 0
\]

has for all $(\lambda, \mu) \neq (0, 0)$ only real roots;

(ii) the polynomials $u$, $v$ both have only real roots, and these roots interlace, i.e., between any two neighboring roots of one polynomial is exactly one root of the other polynomial.

Thus the geometric condition (b) of Conjecture (C) is equivalent with (i) in Obreschkoff's theorem. Consequently, (ii) holds. Let—say—$v(t)$ have degree $n$, and let $b_1 < b_2 < \cdots < b_n$ be its roots. Let $\varepsilon = \text{sgn} u(b_1) v'(b_1)$. Since $u(b_2) u(b_1) < 0$ as well as $v'(b_1) v'(b_2) < 0$, etc., it follows that

\[
\varepsilon u(b_k) v'(b_k) > 0, \quad k = 1, \ldots, n.
\]
By Theorem 1, there exist real symmetric constant $n \times n$ matrices $A$, $B$ such that

$$\det[u(t)A + v(t)B + w(t)I] = 0$$

identically, since the vector $y(t)$ in (2) is never a zero vector. Since $\Gamma$ is irreducible, $\det(xA + yB + zI)$ is the (unique) homogeneous polynomial $\varphi(x, y, z)$ of degree $n$ for which

$$\varphi(x, y, z) = 0$$

is an equation for $\Gamma$, and $\varphi(0, 0, 1) = 1$.

To prove the second part, we shall follow a suggestion of V. Vilhelm [3]. Choose, for $n = 3$,

$$(4) \quad \varphi(x, y, z) = x^3 + x^2y - 2x^2z - y^2z + z^3,$$

and consider the corresponding curve $\Gamma$ given by $\varphi(x, y, z) = 0$. Then (a) in Conjecture (C) is true. To show that also (b) holds, observe that the line $x = 0$ intersects $\Gamma$ in the points $(0, 1, 1)$, $(0, -1, 1)$, and $(0, 1, 0)$. For $y = kx$, $k$ fixed, we obtain as intersection points those $(x, kx, z)$ for which

$$\left(1 + k\right)x^3 - \left(k^2 + 2\right)x^2z + z^3 = 0. \quad (5)$$

Set $x = 1$; for $z$, we obtain three real roots of (5), since the derivative of the left-hand side is $3z^2 - (k^2 + 2)$ with two real roots $z_1 < 0$, $z_2 > 0$ such that the left-hand side of (5) is positive for $z_1$ and negative for $z_2$.

Suppose now that

$$\varphi(x, y, z) = \det(xA + yB + zI) \quad (6)$$

for $A$, $B$ real symmetric (eventually Hermitian) $3 \times 3$ matrices. We can assume that $B$ is diagonal. Comparing the coefficients in (5) and (6), one sees easily that

(i) the diagonal entries of $B$ are 1, $-1$, and 0;
(ii) the diagonal entries of $A$ are all zero;
(iii) the off-diagonal entries of $A$ have to satisfy a system of equations which does not have a real solution (nor even a solution corresponding to the Hermitian case), a contradiction.
To prove the general case for $n > 3$, one shows similarly that multiplying the right-hand side of (4) by $z^{n-3}$ leads to a form which again cannot be written as det$(xA + yB + zI)$.

**Remark.** Adding a parameter in the right-hand side of (4), one can show that there are intervals for which the curve satisfies Conjecture (C) and intervals for which it does not. This leaves open the problem of a geometric description of the situation, even in the case $n = 3$.

**REFERENCES**

3. V. Vilhelm, Private communication.

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