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On the Adomian (Decomposition) Method and Comparisons with Picard's Method

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This paper shows that the Adomian decomposition method ("Stochastic Systems," Academic Press, New York, 1983) is not to be identified with any of the earlier methods since it solves a wide class of nonlinear and stochastic equations without the often unphysical assumptions which have become customary. Similarities between the Picard method and the decomposition method are purely superficial. © 1987 Academic Press, Inc.

The decomposition method (Adomain [1]) has some significant advantages over numerical methods in common use. It is an analytic, continuous, verifiable, rapidly convergent approximation which yields insight into the character and behavior of the solution similar to that obtained from so-called exact solutions. The latter change the problem by linearizing it or making sometimes unjustified and unphysical assumptions on stochastic behavior. The decomposition provides a readily computed accurate solution when numerical evaluation is needed. It eliminates restrictive assumptions such as white noise, "smallness" assumptions, linearization, as well as cumbersome integrations. These procedures are not merely formal. The inverses are not only evaluated; they lead generally to specific solutions which are easily verified. An example is illuminating for comparison of solutions by the Picard method and by the decomposition method [1, 2].

Consider $y' = -(\beta + \alpha y^2)^{1/2}$ with $y(0) = 1$. In the standard form $Ly + Ny = x$ discussed in [1], we have $x = 0$ and $Ny = (\beta + \alpha y^2)^{1/2}$. The Ny term is written in terms of Adomian's A_n polynomials [1]. For this Ny, the A_n are given by

$$A_0 = (\beta + \alpha y^2)^{1/2}$$

$$A_1 = -\alpha(y_0 y_1)(\beta + \alpha y_0^2)^{-1/2}$$

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$$A_2 = -(\alpha/2)[(y_1^2 + 2y_0 y_2)(\beta + \alpha y_0^2)^{-1/2} - (y_0^2 y_1^2)(\beta + \alpha y_0^2)^{-3/2}]$$

$$\vdots$$

The solution $y = \sum_{n=0}^{\infty} y_n$ with $y_0 = k$ and

$$y_1 = -L^{-1}A_0 = -L^{-1}(\beta + \alpha y_0^2)^{1/2}$$

$$y_2 = -L^{-1}A_1 = -\alpha L^{-1}(y_0 y_1)(\beta + \alpha y_0^2)^{-1/2}$$

$$y_3 = -L^{-1}A_2 = -(\alpha/2)\, L^{-1}[(y_1^2 + 2y_0 y_2)(\beta + \alpha y_0^2)^{-1/2} - (y_0^2 y_1^2)(\beta + \alpha y_0^2)^{-3/2}]$$

$$\vdots$$

Thus the approximate solution is

$$y = \sum_{n=0}^{\infty} y_n = k - (\beta + \alpha k^2)^{1/2} x + akx^2/2 - \alpha(\beta + \alpha k^2)^{1/2}(x^3/6) + \cdots.$$

For a numerical computation, assume, e.g., that $k = 5$, $\alpha = \frac{1}{4}$, $\beta = 1$ (values chosen at random). Then $y_0 = 5.0$, $y_1 = -2.692x$, $y_2 = 0.625x^2$, $y_3 = -0.112x^3, \ldots$. Let ϕ_n represent the n-term approximate solution $\sum_{i=0}^{n-1} y_i$, then, if $x = 1$, $\phi_1 = 5, 0$, $\phi_2 = 2.31$, $\phi_3 = 2.93$, $\phi_4 = 2.82$. The n-term solution is a rapidly damped oscillating function which stabilizes rapidly to a number between 2.8 and 2.9. Actually ϕ_4 is within 1% of the correct value. The true solution ϕ_∞ is 2.8319457.... If $x = 0, 1$, $\phi_1 = 5.0$, $\phi_2 = 4.73$ and this is already sufficient to two decimal place accuracy since further change is in the third decimal place. Since the equation is separable, one can also write $dy(\beta + \alpha y^2)^{1/2} = dx$ and integrate, which yields the same answer but involves more work since one gets an unwieldy computation in the form

$$y + (y^2 + \beta/\alpha)^{1/2} = [k + (k^2 + \beta/\alpha)^{1/2}]\, e^{-\sqrt{\alpha}\, x}$$

$$y + (y^2 + 4)^{1/2} = [5 + \sqrt{29}]\, e^{-(1/2)x}$$

and after some rearrangements

$$y = \frac{27 + 5\sqrt{29} - 2e^x}{(e^x)^{1/2}(5 + \sqrt{24})}$$

and if $x = 1$, $y \simeq 2.832$.

The Picard procedure for the same example yields

$$y_1 = k - \int_0^x (\beta + \alpha k^2)^{1/2}\, dx$$

which is the same decomposition so far. However, it then becomes too difficult as

$$y_2 = k - \int_0^x (\beta + \alpha y_1^2)^{1/2}\, dx.$$

Substituting y_1 above makes it clear that Picard does not deal adequately with such problems while decomposition does.[1]

When composite nonlinearities involving products, radicals, trigonometric functions, etc., are involved, the decomposition is superior. If stochasticity is involved as well as nonlinear functions, the Picard method becomes useless while the Adomian method solves wide classes of equations which can be nonlinear and stochastic. It solves these without linearization, perturbation, or unrealistic assumptions on stochastic processes and provides accurate solutions. With nth-order differential equations with stochastic process coefficients, the Picard method is useless.

The advantage of Adomian's method over the Picard method is the ease of computation of successive term. In the linear case, each terms is a simple integral of the preceding term with an elementary Green's function even in high-order equations. In the nonlinear case we only add an integral of another A_n. Thus the last term is transformed rather than the entire preceding solution, and it is transformed linearly while Picard, in general, requires a nonlinear transformation (of the entire preceding solution). In the stochastic case, Picard fails completely as discussed in [1]. The Picard method is useful for existence proofs for first-order differential equations which are deterministic. The decomposition method, on the other hand, is a global procedure demonstrated to solve very wide classes of equations from algebraic equations to systems of nonlinear stochastic partial differential equations [2–11].

The decomposition method provides an analytic, continuous, verifiable, rapidly convergent approximation yielding insight into character and behavior of the solution similar to that obtained from exact closed form solutions. The latter are not necessarily optimal since they linearize or often make unphysical assumptions on stochastic processes. If nonlinear or stochastic effects are significant, this means the solutions will be physically more realistic using decomposition. The latter method requires no "smallness" assumptions, but where perturbation theory would be ade-

[1] Thus the comment by L. Arnold in *Math. Rev.* 84f:60091b is inappropriate and incorrect.

quate, decomposition gives the same results [1, 2]. If numerical evaluation is needed, decomposition provides a readily computed accurate solution. Specific solutions are obtained which are readily verified. Resemblances between the Picard method and Adomain's decomposition are purely superficial [12]. Recent applications to frontier physical problems [13] make the point even more strongly.

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