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Lévy driven moving averages and semimartingales

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Abstract

The aim of the present paper is to study the semimartingale property of continuous time moving averages driven by Lévy processes. We provide necessary and sufficient conditions on the kernel for the moving average to be a semimartingale in the natural filtration of the Lévy process, and when this is the case we also provide a useful representation. Assuming that the driving Lévy process is of unbounded variation, we show that the moving average is a semimartingale if and only if the kernel is absolutely continuous with a density satisfying an integrability condition.

MSC: 60G48; 60H05; 60G51; 60G17

Keywords: Semimartingales; Moving averages; Lévy processes; Bounded variation; Absolutely continuity; Stable processes; Fractional processes

1. Introduction

The present paper is concerned with the semimartingale property of moving averages (also known as stochastic convolutions) which are driven by Lévy processes. More precisely, let $(X_t)_{t\geq 0}$ be a moving average of the form

$$X_t = \int_0^t \phi(t-s)\mathrm{d}Z_s, \quad t \geq 0, \tag{1.1}$$

where $(Z_t)_{t\geq 0}$ is a Lévy process and $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function for which the integral exists. We are interested in the question whether $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale, where $(\mathcal{F}_t^Z)_{t\geq 0}$ denotes the natural filtration of $(Z_t)_{t\geq 0}$. In addition, two-sided moving averages (see (1.6)) are studied as well.

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According to [1, page 533], a stationary process is a moving average if and only if its spectral measure is absolutely continuous. Key examples of moving averages are the Ornstein–Uhlenbeck process, the fractional Brownian motion, and their generalizations, the Ornstein–Uhlenbeck type process (see [2]) and the linear fractional stable motion (see [3]). Moving averages occur naturally in many different contexts, e.g. in stochastic Volterra equations (see [4]), in stochastic delay equations (see [5]), and in turbulence (see [6]). Moreover, to capture the long-range dependence of log-returns in financial markets it is natural to consider the fractional Brownian motion instead of the Brownian motion in the Black–Scholes model (see [7, Part III]), and to capture also heavy tails one is often led to more general moving averages.

It is often important that the process of interest is a semimartingale, and in particular the following two properties are crucial: Firstly, if $(X_t)_{t\geq 0}$ models an asset price which is locally bounded and satisfies the No Free Lunch with Vanishing Risk condition then $(X_t)_{t\geq 0}$ has to be an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale (see [8, Theorem 7.2]). Secondly, it is possible to define a "reasonable" stochastic integral $\int_0^t H_s \mathrm{d}X_s$ for all locally bounded $(\mathcal{F}_t^Z)_{t\geq 0}$-predictable processes $(H_t)_{t\geq 0}$ if and only if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale due to the Bichteler–Dellacherie Theorem (see [9, Theorem 7.6]). In view of the numerous applications of moving averages it is thus natural to study the semimartingale property of these processes.

Let $(Z_t)_{t\geq 0}$ denote a general semimartingale, $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be absolutely continuous with a bounded density and let $(X_t)_{t\geq 0}$ be given by (1.1). Then by a stochastic Fubini result it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale, see e.g. [4, Theorem 3.3] or [5, Theorem 5.2]. In the case where $(Z_t)_{t\in\mathbb{R}}$ is a two-sided Wiener process, $\phi \in L^2(\mathbb{R}_+, \lambda)$ (λ denotes the Lebesgue measure) and $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^{t} \phi(t-s)\mathrm{d}Z_s, \quad t \geq 0, \tag{1.2}$$

Knight [10, Theorem 6.5] shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous with a square integrable density ($\mathcal{F}_t^{Z,\infty} := \sigma(Z_s : -\infty < s \leq t)$). Related results can be found in [11–13]. Moreover, results characterizing when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale are given in [14,15].

The above presented results only provide sufficient conditions on ϕ or are only concerned with the Brownian case. In the present paper we study the case where $(Z_t)_{t\geq 0}$ is a Lévy process and we provide necessary and sufficient conditions on ϕ for $(X_t)_{t\geq 0}$, given by (1.1), to be an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Assume that $(Z_t)_{t\geq 0}$ is of unbounded variation and has characteristic triplet (γ, σ^2, ν). Our main result is the following:

$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' satisfying

$$\int_0^t \int_{[-1,1]} \left(\left|x\phi'(s)\right|^2 \wedge \left|x\phi'(s)\right| \right) \nu(\mathrm{d}x)\mathrm{d}s < \infty, \quad \forall t > 0, \text{ if } \sigma^2 = 0, \tag{1.3}$$

$$\int_0^t \left|\phi'(s)\right|^2 \mathrm{d}s < \infty, \quad \forall t > 0, \text{ if } \sigma^2 > 0. \tag{1.4}$$

In the case where $(Z_t)_{t\geq 0}$ is a symmetric α-stable Lévy process, (1.3) corresponds to $\phi' \in L^\alpha([0,t], \lambda)$ for all $t > 0$ when $\alpha \in (1,2)$ and to $\left|\phi'\right| \log^+(\left|\phi'\right|) \in L^1([0,t], \lambda)$ for all $t > 0$ when $\alpha = 1$.

Assume that $(Z_t)_{t\geq 0}$ is of unbounded variation. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale it can be decomposed as

$$X_t = \phi(0)Z_t + \int_0^t \left(\int_0^u \phi'(u-s)\mathrm{d}Z_s \right) \mathrm{d}u, \quad t \geq 0. \tag{1.5}$$

As a corollary of (1.5) it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if it is absolutely continuous, which is also equivalent to ϕ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.3)–(1.4) and $\phi(0) = 0$.

Finally we study two-sided moving averages, i.e. where $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^t (\phi(t-s) - \psi(-s))\mathrm{d}Z_s, \quad t \geq 0, \tag{1.6}$$

$(Z_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process and $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are deterministic functions for which the integral exists. Note that in this case $(X_t)_{t\geq 0}$ has stationary increments, and when $\psi = 0$ it is a stationary process. Several examples, including fractional Lévy processes and hence also the linear fractional stable motion, are given in Section 5.

The conditions on ϕ from the one-sided case translate into necessary conditions in the two-sided case. That is, if $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale then ϕ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.3)–(1.4). Moreover, [10, Theorem 6.5] is extended from the Gaussian case to the α-stable case with $\alpha \in (1, 2]$.

The paper is organized as follows. In Section 2 we collect some preliminary results. The main results are presented in Section 3. All proofs are given in Section 4. The two-sided case is considered in Section 5.

2. Preliminaries

Throughout the paper $(\Omega, \mathcal{F}, P)$ denotes a complete probability space. Let $(Z_t)_{t\geq 0}$ denote a Lévy process with characteristic triplet (γ, σ^2, ν), that is for $t \geq 0$, $E[\mathrm{e}^{\mathrm{i}\theta Z_t}] = \mathrm{e}^{t\kappa(\theta)}$ for all $\theta \in \mathbb{R}$, where

$$\kappa(\theta) = \mathrm{i}\gamma\theta - \sigma^2\theta^2/2 + \int \left(\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s 1_{\{|s|\leq 1\}} \right) \nu(\mathrm{d}s), \quad \theta \in \mathbb{R}. \tag{2.1}$$

For a general treatment of Lévy processes we refer to [16,17] or [18]. Let $f : \mathbb{R} \to \mathbb{R}$ denote a measurable function. Following [19, page 460] we say that f is Z-integrable if there exists a sequence of simple functions $(f_n)_{n\geq 1}$ such that $f_n \to f$ λ-a.s. and $\lim_n \int_A f_n(s)\mathrm{d}Z_s$ exists in probability for all $A \in \mathcal{B}([0, t])$ and all $t > 0$ (recall that λ denotes the Lebesgue measure). In this case we define $\int_0^t f(s)\mathrm{d}Z_s$ as the limit in probability of $\int_0^t f_n(s)\mathrm{d}Z_s$. By [19, Theorem 2.7], f is Z-integrable if and only if the following three conditions are satisfied for all $t > 0$:

$$\int_0^t f(s)^2\sigma^2\mathrm{d}s < \infty, \tag{2.2}$$

$$\int_0^t \int \left(|xf(s)|^2 \wedge 1 \right) \nu(\mathrm{d}x)\mathrm{d}s < \infty, \tag{2.3}$$

$$\int_0^t \left| f(s) \left(\gamma + \int x(1_{\{|xf(s)|\leq 1\}} - 1_{\{|x|\leq 1\}})\, \nu(\mathrm{d}x) \right) \right| \mathrm{d}s < \infty. \tag{2.4}$$

In this case $\int_0^t f(s)\mathrm{d}Z_s$ is infinitely divisible with characteristic triplet $(\gamma_f, \sigma_f^2, \nu_f)$ given by

$$\gamma_f = \int_0^t f(s)\left(\gamma + \int x(1_{\{|xf(s)|\le 1\}} - 1_{\{|x|\le 1\}})\,\nu(\mathrm{d}x)\right)\mathrm{d}s, \tag{2.5}$$

$$\sigma_f^2 = \int_0^t f(s)^2\sigma^2\mathrm{d}s, \tag{2.6}$$

$$\nu_f(A) = (\nu \times \lambda)((x,s) \in \mathbb{R} \times [0,t] : xf(s) \in A \setminus \{0\}), \quad A \in \mathcal{B}(\mathbb{R}). \tag{2.7}$$

If f is locally square integrable it is easily shown that (2.2)–(2.4) are satisfied and hence $\int_0^t f(s)\mathrm{d}Z_s$ is well-defined for all $t \ge 0$. Note also that (2.4) is satisfied if $(Z_t)_{t\ge 0}$ is symmetric. Recall that $(Z_t)_{t\ge 0}$ is a symmetric α-stable Lévy process with $\alpha \in (0,2]$ if $\gamma = \sigma^2 = 0$ and ν has density $s \mapsto c\,|s|^{-1-\alpha}$ for some $c > 0$ when $\alpha \in (0,2)$, and $\nu = 0$ and $\gamma = 0$ when $\alpha = 2$. In this case (2.2)–(2.4) reduce to $f \in L^\alpha([0,t],\lambda)$ for all $t > 0$.

A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be of bounded variation if on each finite interval $[0,t]$ the total variation of f is finite, that is

$$\mathrm{V}_t(f) := \sup \sum_{i=1}^n |f(t_i) - f(t_{i-1})| < \infty, \tag{2.8}$$

where the sup is taken over all partitions $0 = t_0 < \cdots < t_n = t$, $n \ge 1$ of $[0,t]$. Note that a Lévy process $(Z_t)_{t\ge 0}$ is of bounded variation if and only if $\int_{[-1,1]} |s|\ \nu(\mathrm{d}s) < \infty$ and $\sigma^2 = 0$ (see e.g. [16, Theorem 21.9]). Let I denote an interval and $f : I \to \mathbb{R}$. Then f is said to be absolutely continuous if there exists a locally integrable function h such that

$$f(t) - f(u) = \int_u^t h(s)\mathrm{d}s, \quad u,t \in I,\ u \le t, \tag{2.9}$$

and in this case h is called the density of f. If $f : I \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are two measurable functions, then f is said to have locally g-moment if

$$\int_u^t g(|f(s)|)\mathrm{d}s < \infty, \quad u,t \in I,\ u \le t. \tag{2.10}$$

If (2.10) is satisfied with $g(x) = x^\alpha$ for some $\alpha > 0$ then f is said to have locally α-moment.

An increasing family of σ-algebras $(\mathcal{F}_t)_{t\ge 0}$ is called a filtration if it satisfies the usual conditions of right-continuity and completeness. For each process $(Y_t)_{t\ge 0}$ we let $(\mathcal{F}_t^Y)_{t\ge 0}$ denote its natural filtration, i.e. $(\mathcal{F}_t^Y)_{t\ge 0}$ is the least filtration for which $(Y_t)_{t\ge 0}$ is $(\mathcal{F}_t^Y)_{t\ge 0}$-adapted. Let $(\mathcal{F}_t)_{t\ge 0}$ denote a filtration. We say that $(X_t)_{t\ge 0}$ is an $(\mathcal{F}_t)_{t\ge 0}$-semimartingale if it admits the following representation

$$X_t = X_0 + M_t + A_t, \quad t \ge 0, \tag{2.11}$$

where $(M_t)_{t\ge 0}$ is a càdlàg local $(\mathcal{F}_t)_{t\ge 0}$-martingale starting at 0 and $(A_t)_{t\ge 0}$ is $(\mathcal{F}_t)_{t\ge 0}$-adapted, càdlàg, of bounded variation and starting at 0, and X_0 is $\mathcal{F}_0$-measurable. (Recall that càdlàg means right-continuous with left-hand limits.)

We need the following standard notation: For functions $f, g : \mathbb{R} \to (0,\infty)$ we write $f(x) \approx g(x)$ as $x \to \infty$ if f/g is bounded above and below on some interval (K,∞), where $K > 0$. Furthermore we write $f(x) = o(g(x))$ as $x \to \infty$ if $f(x)/g(x) \to 0$ as $x \to \infty$. A similar notation is used as $x \to 0$.

Assume that ν has positive mass on $[-1,1]$. Similar to [20] we let $\xi\colon[0,\infty)\to[0,\infty)$ be given by

$$\xi(x)=\int_{[-1,1]}\left(|sx|^2\wedge|sx|\right)\,\nu(\mathrm{d}s),\quad x\ge 0. \tag{2.12}$$

Note that ξ is 0 at 0, continuous and increasing and satisfies:

(i) $\xi(x)/x\to\int_{[-1,1]}|s|\,\nu(\mathrm{d}s)\in(0,\infty]$ as $x\to\infty$,
(ii) If $\int_{[-1,1]}|s|^\alpha\,\nu(\mathrm{d}s)<\infty$ for $\alpha\in(1,2]$ then $\xi(x)=o(x^\alpha)$ as $x\to\infty$.

To show (i)–(ii) let

$$H(x)=x\int_{x^{-1}\le|s|\le 1}|s|\,\nu(\mathrm{d}s)\quad\text{and}\quad K(x)=x^2\int_{|s|<x^{-1}}s^2\,\nu(\mathrm{d}s), \tag{2.13}$$

and note that $\xi(x)=H(x)+K(x)$ for $x>1$. We have

$$\int_{x^{-1}\le|s|\le 1}|s|\,\nu(\mathrm{d}s)\le\xi(x)x^{-1}\le\int_{[-1,1]}|s|\,\nu(\mathrm{d}s),\quad x>1, \tag{2.14}$$

where the first inequality follows from $H\le\xi$ and the second from (2.12) since $|xs|^2\wedge|xs|\le|xs|$. Hence by (2.14) and monotone convergence (i) follows. To show (ii) assume $\int_{[-1,1]}|s|^\alpha\,\nu(\mathrm{d}s)<\infty$ for some $\alpha\in(1,2]$. For all $\epsilon>0$ we have

$$\limsup_{x\to\infty}H(x)x^{-\alpha}\le\int_{[-\epsilon,\epsilon]}|s|^\alpha\,\nu(\mathrm{d}s), \tag{2.15}$$

and

$$K(x)x^{-\alpha}\le\int_{|s|<x^{-1}}|s|^\alpha\,\nu(\mathrm{d}s), \tag{2.16}$$

which shows $\xi(x)x^{-\alpha}\to 0$ as $x\to\infty$ and completes the proof of (ii).

Assume that ν is absolutely continuous in a neighborhood of zero with a density f satisfying $f(x)\approx|x|^{-\alpha-1}$ as $x\to 0$ for some $\alpha\in(0,2)$ (this is satisfied in the α-stable case). An easy calculation shows:

(1) $\xi(x)\approx x^\alpha$ as $x\to\infty$ if $\alpha\in(1,2)$,
(2) $\xi(x)\approx x\log(x)$ as $x\to\infty$ if $\alpha=1$,
(3) $\int_{[-1,1]}|s|\,\nu(\mathrm{d}s)<\infty$ if $\alpha\in(0,1)$.

3. Main results

Let $(Z_t)_{t\ge 0}$ denote a nondeterministic Lévy process with characteristic triplet (γ,σ^2,ν) and $\phi\colon\mathbb{R}_+\to\mathbb{R}$ be a measurable function which is Z-integrable (see (2.2)–(2.4)). Throughout this section we let $(X_t)_{t\ge 0}$ be the moving average

$$X_t=\int_0^t\phi(t-s)\mathrm{d}Z_s,\quad t\ge 0. \tag{3.1}$$

Theorem 3.1 below is the main result of the paper. It provides a complete characterization of when $(X_t)_{t\ge 0}$ is an $(\mathcal{F}_t^Z)_{t\ge 0}$-semimartingale. Recall the definition of the function ξ in (2.12).

Theorem 3.1. *Assume that $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' which is locally square integrable when $\sigma^2 > 0$ and has locally ξ-moment when $\sigma^2 = 0$ (that is, ϕ' satisfies (1.3)–(1.4)).*

Assume that $(Z_t)_{t\geq 0}$ is of bounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if it is of bounded variation which is also equivalent to ϕ is of bounded variation.

In particular, if $\sigma^2 = 0$, $\int_{[-1,1]} |x|^\alpha \, \nu(\mathrm{d}x) < \infty$ for some $\alpha \in (1, 2]$ and ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally α-moment, then it follows by (ii) and the above theorem that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a semimartingale the next proposition provides a useful representation of this process.

Proposition 3.2. *Assume that $(Z_t)_{t\geq 0}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Then*

$$X_t = \phi(0)Z_t + \int_0^t \left(\int_0^u \phi'(u-s)\mathrm{d}Z_s \right) \mathrm{d}u, \quad t \geq 0, \tag{3.2}$$

where ϕ' denotes the density of ϕ and $(\int_0^u \phi'(u-s)\mathrm{d}Z_s)_{u\geq 0}$ is chosen measurable.

Hence we obtain the following corollary.

Corollary 3.3. *Assume that $(Z_t)_{t\geq 0}$ is of unbounded variation. Then the following four statements are equivalent:*

(a) *$(X_t)_{t\geq 0}$ is càdlàg and of bounded variation,*
(b) *$(X_t)_{t\geq 0}$ is absolutely continuous,*
(c) *$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale and $\phi(0) = 0$,*
(d) *ϕ is absolutely continuous with a density satisfying (1.3)–(1.4) and $\phi(0) = 0$.*

In the symmetric α-stable case with $\alpha \in (1, 2)$ the equivalence between (b) and (d) follows by [21, Theorem 6.1]. Braverman and Samorodnitsky [22] study, among other things, processes $(Y_t)_{t\geq 0}$ on the form $Y_t = \int_0^t f(t,s)\mathrm{d}Z_s$, where $(Z_t)_{t\geq 0}$ is a symmetric Lévy process and f is a deterministic function. Their Theorem 5.1 provides necessary and sufficient conditions on $f(t,s)$ for $(X_t)_{t\geq 0}$ to be absolutely continuous. In [23,24] necessary and sufficient conditions on ϕ are obtained for $(X_t)_{t\geq 0}$ to have locally bounded or continuous sample paths.

The next corollary follows by Theorem 3.1 and the estimates on ξ given in (1)–(3).

Corollary 3.4. *Assume that $\sigma^2 = 0$ and ν is absolutely continuous in a neighborhood of zero with a density f satisfying $f(x) \approx |x|^{-\alpha-1}$ as $x \to 0$ for some $\alpha \in (0, 2)$ (this is satisfied in the α-stable case with $\alpha \in (0, 2)$). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if*

(i) *ϕ is absolutely continuous with a density having locally α-moment when $\alpha \in (1, 2)$,*
(ii) *ϕ is absolutely continuous with a density having locally $x \log^+(x)$-moment when $\alpha = 1$,*
(iii) *ϕ is of bounded variation when $\alpha \in (0, 1)$.*

Here $\log^+$ denotes the positive part of log, i.e. $\log^+(x) = \log(x)$ for $x \geq 1$ and 0 otherwise.

In the following let $(X_t)_{t\geq 0}$ be the Riemann–Liouville fractional integral given by

$$X_t = \int_0^t (t-s)^\tau \mathrm{d}Z_s, \quad t \geq 0, \tag{3.3}$$

where τ is such that the integral exists. If $(Z_t)_{t\geq 0}$ is a Wiener process and $\tau > -1/2$, $(X_t)_{t\geq 0}$ is called a Lévy fractional Brownian motion (see [25, page 424]). Assume that $(Z_t)_{t\geq 0}$ has no Brownian component (i.e. $\sigma^2 = 0$). Using (2.2)–(2.4) it follows that for $(X_t)_{t\geq 0}$ to be well-defined one of the following (I)–(III) must be satisfied:

(I) $\tau > -1/2$,
(II) $\tau = -1/2$ and $\int_{[-1,1]} x^2 |\log|x|| \, \nu(\mathrm{d}x) < \infty$,
(III) $\tau < -1/2$ and $\int_{[-1,1]} |x|^{-1/\tau} \, \nu(\mathrm{d}x) < \infty$.

Condition (I) is also sufficient for $(X_t)_{t\geq 0}$ to be well-defined and when $(Z_t)_{t\geq 0}$ is symmetric, the conditions (I)–(III) are both necessary and sufficient for $(X_t)_{t\geq 0}$ to be well-defined. When $\tau = 0$, $(X_t)_{t\geq 0} = (Z_t)_{t\geq 0}$; thus let us assume $\tau \neq 0$. As a consequence of Theorem 3.1 we have the following.

Corollary 3.5. *Let $(X_t)_{t\geq 0}$ be given by (3.3) and assume that $(Z_t)_{t\geq 0}$ has no Brownian component. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if one of the following (1)–(3) is satisfied:*

(1) $\tau > 1/2$,
(2) $\tau = 1/2$ *and* $\int_{[-1,1]} x^2 |\log|x|| \, \nu(\mathrm{d}x) < \infty$,
(3) $\tau \in (0, 1/2)$ *and* $\int_{[-1,1]} |x|^{1/(1-\tau)} \, \nu(\mathrm{d}x) < \infty$.

Note that $1/(1-\tau) \in (1, 2)$ when $\tau \in (0, 1/2)$. Let us in particular consider

$$X_t = \int_0^t (t-s)^{H-1/\alpha} \mathrm{d}Z_s, \quad t \geq 0, \tag{3.4}$$

where $(Z_t)_{t\geq 0}$ is a symmetric α-stable Lévy process with $\alpha \in (0, 2]$ and $H > 0$ (note that $(X_t)_{t\geq 0}$ is well-defined). To avoid trivialities assume $H \neq 1/\alpha$. As a consequence of Corollary 3.5 ($\alpha \in (0, 2)$) and Theorem 3.1 ($\alpha = 2$) it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if $H > 1$ when $\alpha \in [1, 2]$ or $H > 1/\alpha$ when $\alpha \in (0, 1)$.

4. Proofs

Throughout this section $(X_t)_{t\geq 0}$ is given by (3.1). We extend ϕ to a function from $\mathbb{R}$ into $\mathbb{R}$ by setting $\phi(s) = 0$ for $s \in (-\infty, 0)$. For any function $f: \mathbb{R} \to \mathbb{R}$, let $\Delta_t f$ denote the function $s \mapsto t(f(1/t+s) - f(s))$ for all $t > 0$. We start by the following extension of [26, Theorem 24].

Lemma 4.1. *Let I be either $\mathbb{R}_+$ or $\mathbb{R}$, $f: I \to \mathbb{R}$ be locally integrable and $g: \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function satisfying $g(x)/x \to \infty$ as $x \to \infty$ and let $(r_k)_{k\geq 1}$ be a sequence satisfying $r_k \to \infty$. Then f is absolutely continuous with a density having locally g-moment if and only if $(g(|\Delta_{r_k} f|))_{k\geq 1}$ is bounded in $L^1([a, b], \lambda)$ for all $a, b \in I$ with $a < b$. In this case $\{g(|\Delta_t f|) : t > \epsilon\}$ is bounded in $L^1([a, b], \lambda)$ for all $a, b \in I$ with $a < b$ and all $\epsilon > 0$.*

If $(Z_t)_{t\geq 0}$ is of unbounded variation the above lemma can be applied with ξ playing the role of g (ξ is given by (2.12)), since in this case ξ satisfies all the conditions imposed on g except ξ is not convex. But h, defined by $h(x) = x^2 1_{\{x\leq 1\}} + (2x-1)1_{\{x>1\}}$ for all $x \geq 0$, is convex and if we let

$$g(x) = \int_{[-1,1]} h(|xs|) \, \nu(\mathrm{d}s), \quad x \geq 0, \tag{4.1}$$

then g satisfies all the conditions in the lemma and $g/2 \le \xi \le g$. Thus, if $f : I \to \mathbb{R}$ is locally integrable then f is absolutely continuous with a density having locally ξ-moment if and only if $(\xi(|\Delta_{r_k} f|))_{k\ge1}$ is bounded in $L^1([a,b],\lambda)$ for all $a, b \in I$ with $a < b$.

Proof. Note that g is continuous and $x \mapsto g(|x|)$ is a convex function from $\mathbb{R}$ into $\mathbb{R}$, since g is increasing and convex. Let $a, b \in I$ satisfying $a < b$ be given and assume that $(g(|\Delta_{r_k} f|))_{k\ge1}$ is bounded in $L^1([a,b],\lambda)$. Since $g(x)/x \to \infty$ as $x \to \infty$, $\{\Delta_{r_k} f : k \ge 1\}$ is uniformly integrable and hence weakly sequentially compact in $L^1([a,b],\lambda)$ (see e.g. [27, Chapter IV.8, Corollary 11]). Choose a subsequence $(n_k)_{k\ge1}$ of $(r_k)_{k\ge1}$ and an $h \in L^1([a,b],\lambda)$ such that $\Delta_{n_k} f \to h$ in the weak $L^1([a,b],\lambda)$-topology. For all $c, d \in [a,b]$ with $c < d$ we have

$$\int_c^d \Delta_{n_k} f \,\mathrm{d}\lambda \to \int_c^d h \,\mathrm{d}\lambda, \quad \text{as } k \to \infty. \tag{4.2}$$

Moreover,

$$\begin{aligned}\int_c^d \Delta_{n_k} f \,\mathrm{d}\lambda &= n_k\left(\int_{c+1/n_k}^{d+1/n_k} f\,\mathrm{d}\lambda - \int_c^d f\,\mathrm{d}\lambda\right)\\ &= n_k \int_d^{d+1/n_k} f\,\mathrm{d}\lambda - n_k \int_c^{c+1/n_k} f\,\mathrm{d}\lambda \to f(d) - f(c), \quad \text{as } k \to \infty,\end{aligned} \tag{4.3}$$

for $\lambda \times \lambda$-a.a. $c < d$. Thus, we conclude that f is absolutely continuous with density h. Since $\Delta_{n_k} f \to h$ in the weak $L^1([a,b],\lambda)$-topology we may choose a sequence $(\kappa_n)_{n\ge1}$ of convex combinations of $(\Delta_{n_k} f)_{k\ge1}$ such that $\kappa_n \to h$ in $L^1([a,b],\lambda)$, see [28, Theorem 3.13]. By convexity and continuity of g we have

$$\int_a^b g(|h|)\mathrm{d}\lambda \le \liminf_{n\to\infty} \int_a^b g(|\kappa_n|)\mathrm{d}\lambda \le \sup_{k\ge1} \int_a^b g(|\Delta_{n_k} f|)\mathrm{d}\lambda < \infty, \tag{4.4}$$

which shows that h has g-moment on $[a,b]$. This completes the proof of the *if*-part.

Assume conversely that f is absolutely continuous with a density, h, having locally g-moment. For all $t > \epsilon$, we have by Jensen's inequality that

$$\begin{aligned}\int_a^b g\left(\left|t\int_s^{s+1/t} h(u)\mathrm{d}u\right|\right)\mathrm{d}s &\le \int_a^b \left(t\int_0^{1/t} g(|h(u+s)|)\mathrm{d}u\right)\mathrm{d}s\\ &= t\int_0^{1/t}\int_a^b g(|h(u+s)|)\mathrm{d}s\mathrm{d}u \le \int_a^{b+1/\epsilon} g(|h(s)|)\mathrm{d}s < \infty,\end{aligned} \tag{4.5}$$

which shows that $\{g(|\Delta_t f|) : t > \epsilon\}$ is bounded in $L^1([a,b],\lambda)$ and completes the proof. □

In what follows, we are going to use two Lévy–Itô decompositions of $(Z_t)_{t\ge0}$ (see e.g. [16, Theorem 19.2]).

(a) Decompose $(Z_t)_{t\ge0}$ as $Z_t = Z_t^1 + Z_t^2$, where $(Z_t^1)_{t\ge0}$ and $(Z_t^2)_{t\ge0}$ are two independent Lévy processes with characteristic triplets $(0, \sigma^2, \nu_1)$ respectively $(\gamma, 0, \nu_2)$, where $\nu_1 = \nu|_{[-1,1]}$ and $\nu_2 = \nu|_{[-1,1]^c}$. $(Z_t^1)_{t\ge0}$ and $(Z_t^2)_{t\ge0}$ are $(\mathcal{F}_t^Z)_{t\ge0}$-adapted. Moreover, when ϕ is locally bounded we let

$$X_t^1 = \int_0^t \phi(t-s)\mathrm{d}Z_s^1, \quad \text{and} \quad X_t^2 = \int_0^t \phi(t-s)\mathrm{d}Z_s^2, \quad t \ge 0. \tag{4.6}$$

(b) Decompose $(Z_t)_{t\geq 0}$ as $Z_t = W_t + Y_t$, where $(W_t)_{t\geq 0}$ is a Wiener process with variance parameter σ^2 and $(Y_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma, 0, \nu)$. $(W_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent and $(\mathcal{F}_t^Z)_{t\geq 0}$-adapted. Moreover, let

$$X_t^W = \int_0^t \phi(t-s)\mathrm{d}W_s, \quad \text{and} \quad X_t^Y = \int_0^t \phi(t-s)\mathrm{d}Y_s, \quad t \geq 0. \tag{4.7}$$

If $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg it follows by [29, Theorem 4] and a symmetrization argument that by modification on a set of Lebesgue measure 0, we may and do choose ϕ càdlàg.

The following lemma is closely related to [10, Theorem 6.5].

Lemma 4.2. *We have the following:*

(i) $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^Z)_{t\geq 0}$*-semimartingale if* ϕ *is absolutely continuous on* $\mathbb{R}_+$ *with a locally square integrable density.*
(ii) *Assume that* $(Z_t)_{t\geq 0}$ *is a Wiener process. Then* ϕ *is absolutely continuous on* $\mathbb{R}_+$ *with a locally square integrable density if* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^Z)_{t\geq 0}$*-semimartingale.*

Proof. (i): Decompose $(Z_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ as in (a) above. Since both ϕ and $(Z_t^2)_{t\geq 0}$ are càdlàg and of bounded variation, $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation as well. Hence, it is enough to show that $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Since

$$X_t^1 = \int_0^t (\phi(t-s) - \phi(0))\mathrm{d}Z_s^1 + \phi(0)Z_t^1, \quad t \geq 0, \tag{4.8}$$

we may and do assume $\phi(0) = 0$. Then, ϕ is absolutely continuous on $\mathbb{R}$ with locally square integrable density and hence for all $T > 0$, $\|\Delta_t\phi\|_{L^2([-T,T],\lambda)} \leq K$ for some constant $K > 0$ and all $t > 1/T$ by Lemma 4.1 with $g(x) = x^2$. By letting $c = E[|Z_1^1|^2]$ we have (recall that ϕ is zero on $(-\infty, 0)$)

$$\begin{aligned} E[(X_t^1 - X_u^1)^2] &= c\,\|\phi(t-\cdot) - \phi(u-\cdot)\|^2_{L^2([0,t],\lambda)} \\ &\leq cK^2(t-u)^2, \quad \forall\, 0 \leq u \leq t \leq T, \end{aligned} \tag{4.9}$$

which by the Kolmogorov–Čentsov Theorem (see [30, Chapter 2, Theorem 2.8]) shows that $(X_t^1)_{t\geq 0}$ has a continuous modification (also to be denoted $(X_t^1)_{t\geq 0}$). Moreover, for all $0 = t_0 < \cdots < t_n = T$ we have

$$E\left[\sum_{i=1}^n \left|X_{t_i}^1 - X_{t_{i-1}}^1\right|\right] \leq \sum_{i=1}^n \left\|X_{t_i}^1 - X_{t_{i-1}}^1\right\|_{L^2(P)} \leq \sqrt{c}KT, \tag{4.10}$$

which shows that $(X_t^1)_{t\geq 0}$ is of integrable variation and hence an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale.

To show (ii) assume that $(Z_t)_{t\geq 0}$ is a standard Wiener process and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Since $(X_t)_{t\geq 0}$ is a Gaussian process, Stricker [31, Proposition 4+5] entails that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-quasimartingale on each compact interval $[0, N]$. For $0 \leq u \leq t$ we have

$$\begin{aligned} E\left[\left|E[X_t - X_u|\mathcal{F}_u^Z]\right|\right] &= E\left[\left|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\mathrm{d}Z_s\right|\right] \\ &= \sqrt{\frac{2}{\pi}}\left\|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\mathrm{d}Z_s\right\|_{L^2(P)} \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^u \big(\phi(t-s) - \phi(u-s)\big)^2 \mathrm{d}s \right)^{1/2}$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^u \big(\phi(t-u+s) - \phi(s)\big)^2 \mathrm{d}s \right)^{1/2}, \tag{4.11}$$

where the second equality follows by Gaussianity, which implies that

$$\sum_{i=1}^{nN} E\left[\left|E[X_{i/n} - X_{(i-1)/n} | \mathcal{F}^Z_{(i-1)/n}]\right|\right] \geq \frac{Nn}{\sqrt{\pi 2}} \left(\int_0^{N/2} \big(\phi(1/n+s) - \phi(s)\big)^2 \mathrm{d}s \right)^{1/2}. \tag{4.12}$$

Since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-quasimartingale on $[0, N]$, the left-hand side of (4.12) is bounded in n (see [32, Chapter VI, Definition 38]), showing that $(\Delta_n \phi)_{n\geq 1}$ is bounded in $L^2([0, N/2], \lambda)$. By Lemma 4.1 with $g(x) = x^2$ this shows that ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density. □

Lemma 4.3. *If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale then $(X^1_t)_{t\geq 0}$ is an $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale.*

Proof. Assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale, fix $T > 0$ and let

$$A := \{\Delta Z^2_t = 0 \;\forall t \in [0, T]\}. \tag{4.13}$$

Note that $P(A) > 0$ and $(Z^1_t)_{t\geq 0}$ is P-independent of A. Let Q^A denote the probability measure given by $Q^A(B) := P(B \cap A)/P(A)$.$(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale under Q^A, since Q^A is absolutely continuous with respect to P. Moreover, since $(Z_t)_{t\geq 0}$ and $(Z^1_t)_{t\geq 0}$ are Q^A-indistinguishable it follows that $(X^1_t)_{t\geq 0}$ is an $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale under Q^A and since A is independent of $(Z^1_t)_{t\geq 0}$ this is also true under P. □

In the next lemma we study the jump structure of $(X_t)_{t\geq 0}$.

Lemma 4.4. *Assume that $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg. Then $(\Delta X_t 1_{\{\Delta Z_t \neq 0\}})_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable.*

Before proving the lemma we note the following:

Remark 4.5. (a) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ denote two independent càdlàg processes such that $P(\Delta X_t = 0) = P(\Delta Y_t = 0) = 1$ for all $t \geq 0$. Then as a consequence of Tonelli's Theorem we have $P(\Delta X_t \Delta Y_t = 0, \; \forall t \geq 0) = 1$.
(b) If ν is concentrated on $[-1, 1]$ then the mapping $t \mapsto \int_0^t \phi(t-s)\mathrm{d}Z_s$ is continuous from $\mathbb{R}_+$ into $L^1(P)$. This follows by approximating ϕ with continuous functions.

Proof of Lemma 4.4. Since $X_t = \int_0^t (\phi(t-s) - 1)\mathrm{d}Z_s + Z_t$ we may and do assume $\phi(0) \neq 0$. Recall that ϕ is chosen càdlàg; moreover, $\Delta\phi(0) = \phi(0)$.

First we show the lemma in the case where ν is a finite measure. Let τ_n denote the time of the nth jump of $(Z_t)_{t\geq 0}$ ($(\tau_{n+1} - \tau_n)_{n\geq 1}$ is thus an i.i.d. sequence of exponential distributions) and let $(\sigma_n)_{n\geq 1} \subseteq [0, \infty)$ denote the jump times of ϕ. Note that the event

$$B := \{\exists\, (j, k) \neq (j', k') : \tau_j + \sigma_k = \tau_{j'} + \sigma_{k'}\}, \tag{4.14}$$

has probability zero. Since $(Z_t)_{t\geq 0}$ only has finitely many jumps on each compact interval we may regard $(X_t)_{t\geq 0}$ as a pathwise Lebesgue–Stieltjes integral and hence it follows that

$$(\Delta X_t)_{t\geq 0} = \left(\sum_{k\geq 1} \Delta Z_{t-\sigma_k} \Delta\phi(\sigma_k) \right)_{t\geq 0}. \tag{4.15}$$

Let us show that on B^c the series $\sum_{k\geq 1} \Delta Z_{t-\sigma_k} \Delta\phi(\sigma_k)$ has at most one term which differs from zero for all $t \geq 0$. Indeed, to see this assume that $\Delta Z_{t-\sigma_k} \Delta\phi(\sigma_k)$ and $\Delta Z_{t-\sigma_{k'}} \Delta\phi(\sigma_{k'})$ both differ from zero, where $k \neq k'$. Then there exist $n, n' \geq 1$ such that $\tau_n = t - \sigma_k$ and $\tau_{n'} = t - \sigma_{k'}$ which implies $\tau_n + \sigma_k = \tau_{n'} + \sigma_{k'}$, and hence we have a contradiction. In particular, if $\Delta Z_t \neq 0$ then $\Delta Z_t \Delta\phi(0) \neq 0$ and thus $\Delta X_t = \Delta Z_t \Delta\phi(0) = \phi(0)\Delta Z_t$.

Now let $(Z_t)_{t\geq 0}$ be a general Lévy process for which $\sigma^2 = 0$. For each $n \geq 1$, decompose $(Z_t)_{t\geq 0}$ as $Z_t = Y_t^n + U_t^n$, where $(Y_t^n)_{t\geq 0}$ and $(U_t^n)_{t\geq 0}$ are two independent Lévy processes with characteristic triplets $(0, 0, \nu|_{[-1/n,1/n]})$ respectively $(0, 0, \nu|_{[-1/n,1/n]^c})$. Moreover, set

$$X_t^{Y^n} = \int_0^t \phi(t-s)\mathrm{d}Y_s^n \quad \text{and} \quad X_t^{U^n} = \int_0^t \phi(t-s)\mathrm{d}U_s^n. \tag{4.16}$$

Since $(U_t^n)_{t\geq 0}$ has piecewise constant sample paths the second integral is a pathwise Lebesgue–Stieltjes integral. Hence $(X_t^{U^n})_{t\geq 0}$ is càdlàg and it follows that $(X_t^{Y^n})_{t\geq 0}$ is càdlàg as well. Set

$$C := \bigcap_{n\geq 1} \{\Delta X_t^{Y^n} \Delta U_t^n = 0,\ \forall t \geq 0\}, \tag{4.17}$$

$$D := \bigcap_{n\geq 1} \{\Delta X_t^{U^n} 1_{\{\Delta U_t^n \neq 0\}} = \phi(0)\Delta U_t^n,\ \forall t \geq 0\}. \tag{4.18}$$

From Remark 4.5(b) it follows that $P(\Delta X_t^{Y^n} = 0) = 1$ for all $t \geq 0$ which together with Remark 4.5(a) shows that C has probability one. Moreover, from the first part of the proof it follows that D has probability one. When $\Delta Z_t \neq 0$, choose $n \geq 1$ such that $|\Delta Z_t| > 1/n$. Thus, $\Delta U_t^n \neq 0$, and hence $\Delta X_t^{Y^n} = 0$ on C, which shows $\Delta X_t = \Delta X_t^{U^n} = \phi(0)\Delta U_t^n = \phi(0)\Delta Z_t$ on $C \cap D$ and completes the proof. □

Lemma 4.6. *Assume that $\sigma^2 = \gamma = 0$, ν is concentrated on $[-1, 1]$ and $(X_t)_{t\geq 0}$ is a special $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Then $(\phi(0)Z_t)_{t\geq 0}$ is the martingale component of $(X_t)_{t\geq 0}$.*

Proof. Let $X_t = M_t + A_t$ denote the canonical decomposition of $(X_t)_{t\geq 0}$. Since $(Z_t)_{t\geq 0}$ is a Lévy process, it is quasi-left-continuous (see [33, Chapter II, Corollary 4.18]) and thus there exists a sequence of totally inaccessible stopping times $(\tau_n)_{n\geq 1}$ which exhausts the jumps of $(Z_t)_{t\geq 0}$. On the other hand, since $(A_t)_{t\geq 0}$ is predictable there exists a sequence of predictable times $(\sigma_n)_{n\geq 1}$ which exhausts the jumps of $(A_t)_{t\geq 0}$. From the martingale representation theorem for Lévy processes (see [33, Chapter III, Theorem 4.34]) it follows that $(M_t)_{t\geq 0}$ is a purely discontinuous martingale which jumps only when $(Z_t)_{t\geq 0}$ does. Furthermore, since

$$P(\exists\, n, k \geq 1 : \tau_n = \sigma_k < \infty) = 0, \tag{4.19}$$

Lemma 4.4 shows

$$\phi(0)\Delta Z_{\tau_n} = \Delta X_{\tau_n} = \Delta M_{\tau_n} + \Delta A_{\tau_n} = \Delta M_{\tau_n}, \quad P\text{-a.s. on } \{\tau_n < \infty\}\ \forall n \geq 1. \tag{4.20}$$

Hence $(\Delta M_t)_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable which implies that $(M_t)_{t\geq 0}$ and $(\phi(0)Z_t)_{t\geq 0}$ are indistinguishable since they both are purely discontinuous martingales (see [33, Chapter I, Corollary 4.19]). This completes the proof. □

The following lemma is concerned with the bounded variation case and it relies on an inequality by Marcus and Rosiński [20].

Lemma 4.7. *Assume that $\gamma = \sigma^2 = 0$, ν is concentrated on $[-1, 1]$ and $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally ξ-moment and $\phi(0) = 0$.*

Recall the definition of $\Delta_t\phi$ and of $\mathrm{V}_t(f)$ in (2.8).

Proof. Let $N \geq 1$ be given. We start by showing the following (i) and (ii) under the assumptions stated in the lemma:

(i) If $(X_t)_{t\geq 0}$ is of bounded variation then $E[\mathrm{V}_N(X)] < \infty$ for all $N \geq 1$.
(ii) For all $N \geq 1$,

$$\frac{N}{8}\sup_{n\geq 1}\left\{\left(\int_{-N/2}^{N/2}\xi(|\Delta_{2^n}\phi(s)|)\mathrm{d}s\right)\wedge\left(\int_{-N/2}^{N/2}\xi(|\Delta_{2^n}\phi(s)|)\mathrm{d}s\right)^{1/2}\right\}$$
$$\leq E[\mathrm{V}_N^{\mathcal{D}}(X)] \leq 3N\sup_{n\geq 1}\left\{\int_{-N}^{N}\xi(|\Delta_{2^n}\phi(s)|)\mathrm{d}s+1\right\}, \tag{4.21}$$

where for each $f:\mathbb{R}_+\to\mathbb{R}$ we let

$$\mathrm{V}_N^{\mathcal{D}}(f) = \sup_{n\geq 1}\sum_{i=1}^{2^n N}\left|f(i/2^n) - f((i-1)/2^n)\right|. \tag{4.22}$$

To show (i) assume that $(X_t)_{t\geq 0}$ is of bounded variation. By Rosiński [29, Theorem 4], $\phi(\cdot - s)$ is of bounded variation for λ-a.a. $s \in \mathbb{R}_+$; in particular there exists an $s \in \mathbb{R}_+$ such that $\phi(\cdot - s)$ is of bounded variation. Hence ϕ is of bounded variation. Let $T := [0, N] \cap \mathbb{Q}$, $\underline{X}: \Omega \to \mathbb{R}^{\mathrm{T}}$ denote the canonical random element induced by $(X_t)_{t\in T}$ and let μ be given by

$$\mu(A) = (\lambda\times\nu)\left((s,x)\in[0,t_0]\times\mathbb{R} : x\phi(\cdot - s)\in A\setminus\{\underline{0}\}\right),\quad A\in\mathcal{B}(\mathbb{R}^{\mathrm{T}}). \tag{4.23}$$

For all $t_1,\ldots,t_n \in T$, $(X_{t_1},\ldots,X_{t_n})$ is infinitely divisible with Lévy measure $\mu\circ p_{t_1,\ldots,t_n}^{-1}$, where $p_{t_1,\ldots,t_n}(f) = (f(t_1),\ldots,f(t_n))$ for all $f\in\mathbb{R}^{\mathrm{T}}$. For $f\in\mathbb{R}^{\mathrm{T}}$ let $q(f)$ denote the total variation of f on T. Then $q:\mathbb{R}^{\mathrm{T}}\to[0,\infty]$ is clearly a lower-semicontinuous pseudonorm on $\mathbb{R}^{\mathrm{T}}$ (see [34, page 998]). Since ν has compact support and ϕ is of bounded variation there exists an $r_0 > 0$ such that $\mu(f\in\mathbb{R}^{\mathrm{T}} : q(f) > r_0) = 0$ and hence by Lemma 2.2 in [34], $E[\mathrm{e}^{\epsilon q(\underline{X})}] < \infty$ for some $\epsilon > 0$. In particular $(X_t)_{t\geq 0}$ is of integrable variation on $[0, N]$.

(ii) From [20, Corollary 1.1] we have

$$1/4\min(a_{i,n}, a_{i,n}^{1/2}) \leq E\left[\left|2^n(X_{i/2^n} - X_{(i-1)/2^n})\right|\right] \leq 3\max(a_{i,n}, a_{i,n}^{1/2}), \tag{4.24}$$

where

$$a_{i,n} := \int_{-1/2^n}^{(i-1)/2^n}\xi(|\Delta_{2^n}\phi(s)|)\mathrm{d}s. \tag{4.25}$$

By monotone convergence we have

$$E[\mathrm{V}_N^{\mathcal{D}}(X)] = \sup_{n\geq 1} \frac{1}{2^n} \sum_{i=1}^{2^n N} E\left[\left|2^n (X_{i/2^n} - X_{(i-1)/2^n})\right|\right], \tag{4.26}$$

and hence

$$\begin{aligned} \frac{N}{2} \sup_{n\geq 1} \inf_{2^n N/2<i\leq 2^n N} E\left[\left|2^n (X_{i/2^n} - X_{(i-1)/2^n})\right|\right] &\leq E[\mathrm{V}_{0,N}^{\mathcal{D}}(X)] \\ \leq N \sup_{n\geq 1} \sup_{1\leq i\leq 2^n N} E\left[\left|2^n (X_{i/2^n} - X_{(i-1)/2^n})\right|\right], \end{aligned} \tag{4.27}$$

which by (4.24) shows (4.21).

Assume that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation and hence by (i) of integrable variation. From (ii) it follows that $(\xi(\Delta_{2^n}\phi))_{n\geq 1}$ is bounded in $L^1([-a, a], \lambda)$ for all $a > 0$. Conversely, if $(\xi(\Delta_{2^n}\phi))_{n\geq 1}$ is bounded in $L^1([-a, a], \lambda)$ for all $a > 0$, (ii) shows that $E[\mathrm{V}_N^{\mathcal{D}}(X)] < \infty$; in particular $\mathrm{V}_N^{\mathcal{D}}(X) < \infty$ P-a.s. Since in addition $(X_t)_{t\geq 0}$ is right-continuous in probability by Remark 4.5(b) it has a càdlàg modification (also to be denoted $(X_t)_{t\geq 0}$), which is of bounded variation since $\mathrm{V}_N(X) = \mathrm{V}_N^{\mathcal{D}}(X) < \infty$ P-a.s.

Finally, the discussion just below Lemma 4.1 completes the proof, since $(Z_t)_{t\geq 0}$ is of unbounded variation. □

We have the following consequence of the Bichteler–Dellacherie Theorem.

Lemma 4.8. *Let $(Y_t)_{t\geq 0}$, $(U_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\geq 0}$ denote four processes such that $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^U)_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}_t^{\tilde{U}})_{t\geq 0}$-adapted and $(Y_\cdot, U_\cdot) \overset{\mathcal{D}}{=} (\tilde{Y}_\cdot, \tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^U)_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}_t^{\tilde{U}})_{t\geq 0}$-semimartingale.*

Proof. Since $(Y_t)_{t\geq 0}$, by assumption, is càdlàg and $(Y_t)_{t\geq 0} \overset{\mathcal{D}}{=} (\tilde{Y}_t)_{t\geq 0}$ we may choose a càdlàg modification of $(\tilde{Y}_t)_{t\geq 0}$ (also to be denoted $(\tilde{Y}_t)_{t\geq 0}$). By the Bichteler–Dellacherie Theorem (see [32, Theorem 80]) we must show that for all $t > 0$ the set of random variables given by

$$\left\{ \sum_{i=1}^n \tilde{H}_{t_{i-1}} (\tilde{Y}_{t_i} - \tilde{Y}_{t_{i-1}}) : n \geq 1,\ 0 \leq t_0 < \cdots < t_n \leq t,\ \tilde{H}_{t_i} \in \mathcal{F}_{t_i}^{\tilde{U}},\ \left|\tilde{H}_{t_i}\right| \leq 1 \right\} \tag{4.28}$$

is bounded in $L^0(P)$. Since each $\tilde{H}_s \in \mathcal{F}_s^{\tilde{U}}$ satisfying $\left|\tilde{H}_s\right| \leq 1$ is given by

$$\tilde{H}_s = \lim_{n\to\infty} F_n((\tilde{U}_u)_{u\leq s+1/n}) \quad P\text{-a.s.}, \tag{4.29}$$

for some $F_n : \mathbb{R}^{[0,s+1/n]} \to [-1, 1]$ which is $\mathcal{B}(\mathbb{R})^{[0,s+1/n]}$-measurable, our assumptions imply that for each random variable in the above set there exist $H_{t_i} \in \mathcal{F}_{t_i}^U$ satisfying $\left|H_{t_i}\right| \leq 1$ for $i = 0, \ldots, n-1$ such that

$$\sum_{i=1}^n \tilde{H}_{t_{i-1}} (\tilde{Y}_{t_i} - \tilde{Y}_{t_{i-1}}) \overset{\mathcal{D}}{=} \sum_{i=1}^n H_{t_{i-1}} (Y_{t_i} - Y_{t_{i-1}}). \tag{4.30}$$

Thus since $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^U)_{t\geq 0}$-semimartingale, another application of the Bichteler–Dellacherie Theorem shows that the set given in (4.28) is bounded in $L^0(P)$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We prove the result in the following three steps (1)–(3). Recall the Lévy–Itô decompositions (a) and (b).

(1) Let $\sigma^2 > 0$.

Assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Let $\tilde{Z}_t = Y_t - W_t$ and $\tilde{X}_t = \int_0^t \phi(t-s)\, d\tilde{Z}_s$. We have $\mathcal{F}_t^Z = \mathcal{F}_t^W \vee \mathcal{F}_t^Y = \mathcal{F}_t^{-W} \vee \mathcal{F}_t^Y = \mathcal{F}_t^{\tilde{Z}}$ and since $(X_., Z_.) \stackrel{\mathcal{D}}{=} (\tilde{X}_., \tilde{Z}_.)$, Lemma 4.8 shows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Therefore $(X_t^W)_{t\geq 0} := ((X_t - \tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale and thus an $(\mathcal{F}_t^W)_{t\geq 0}$-semimartingale, and by Lemma 4.2(ii) we conclude that ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density.

On the other hand, if ϕ is absolutely continuous with a locally square integrable density it follows by Lemma 4.2(i) that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale.

(2) Let $\sigma^2 = 0$ and $(Z_t)_{t\geq 0}$ be of unbounded variation.

Assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. By Lemma 4.3 it follows that $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z^1})_{t\geq 0}$-semimartingale. Let $T = \mathbb{Q} \cap [0, t]$, $q(f) = \sup_{s\in T} |f(s)|$ for all $f \in \mathbb{R}^T$ and μ be given by (4.23) with ν replaced by ν_1. Since ν_1 has compact support and ϕ is locally bounded (recall that ϕ is chosen càdlàg) there exists an $r_0 > 0$ such that $\mu(f \in \mathbb{R}^{\mathrm{T}} : q(f) \geq r_0) = 0$ and hence, according to Rosiński [34, Lemma 2.2], $E[\sup_{s\in[0,t]} |X_s^1|] < \infty$. This shows that $(X_t^1)_{t\geq 0}$ is a special $(\mathcal{F}_t^{Z^1})_{t\geq 0}$-semimartingale. Let $X_t^1 = M_t + A_t$ denote the canonical $(\mathcal{F}_t^{Z^1})_{t\geq 0}$-decomposition of $(X_t^1)_{t\geq 0}$. Then Lemma 4.6 yields $(M_t)_{t\geq 0} = (\phi(0) Z_t^1)_{t\geq 0}$ and hence $(A_t)_{t\geq 0}$, given by

$$A_t = \int_0^t \psi(t-s) \mathrm{d}Z_s^1, \quad t \geq 0, \tag{4.31}$$

where $\psi(t) = \phi(t) - \phi(0)$ for $t \geq 0$, is of bounded variation. Thus, by Lemma 4.7 we conclude that ψ, and hence also ϕ, is absolutely continuous on $\mathbb{R}_+$ with a density having locally ξ-moment.

Assume conversely that ϕ is absolutely continuous with a density having locally ξ-moment. Since ϕ and $(Z_t^2)_{t\geq 0}$ are càdlàg and of bounded variation it follows that $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation as well. Let $(A_t)_{t\geq 0}$ be given by (4.31). By Lemma 4.7 it follows that $(A_t)_{t\geq 0}$ is càdlàg and of bounded and hence $(X_t^1)_{t\geq 0} = (\phi(0) Z_t^1 + A_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale and we have shown that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale.

(3) Let $(Z_t)_{t\geq 0}$ be of bounded variation.

Assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. By arguing as in (2) it follows that $(A_t)_{t\geq 0}$ given by (4.31) is of bounded variation. Hence [29, Theorem 4] and a symmetrization argument shows that ψ, and hence also ϕ, is of bounded variation.

Assume conversely that ϕ is of bounded variation. Since $(Z_t)_{t\geq 0}$ is càdlàg and of bounded variation it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation and hence an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. $\square$

To show Proposition 3.2 we need the following Fubini type result.

Lemma 4.9. *Let $T > 0$, μ denote a finite measure on $\mathbb{R}_+$ and let $f: \mathbb{R}_+^2 \to \mathbb{R}$ be a measurable function such that either* (i) *or* (ii) *is satisfied, where*

(i) $\sigma^2 = 0$, $\xi(|f(t, \cdot)|) \in L^1([0, T], \lambda)$ *for all* $t \geq 0$ *and* $\xi(|f|) \in L^1(\mathbb{R}_+ \times [0, T], \mu \times \lambda)$.

(ii) $\sigma^2 > 0$, $f(t, \cdot) \in L^2([0, T], \lambda)$ *for all* $t \geq 0$, *and* $f \in L^2(\mathbb{R}_+ \times [0, T], \mu \times \lambda)$.

Then $(\int_0^T f(t,s)\mathrm{d}Z_s)_{t\geq 0}$ *can be chosen measurable and in this case*

$$\int\left(\int_0^T f(t,s)\mathrm{d}Z_s\right)\mu(\mathrm{d}t) = \int_0^T\left(\int f(t,s)\,\mu(\mathrm{d}t)\right)\mathrm{d}Z_s \quad P\text{-a.s.} \tag{4.32}$$

Proof. Assume that (i) is satisfied. To show (4.32) we may and do assume that $(Z_t)_{t\geq 0}$ has characteristic triplet $(0,0,\nu)$ where ν is concentrated on $[-1,1]$. Let g be given by (4.1). Since g is 0 at 0, symmetric, increasing, convex, $\lim_{x\to\infty} g(x)=\infty$ and $g(2x)\leq 4g(x)$ for all $x\geq 0$, g is a Young function satisfying the Δ_2-condition (see [35, page 5+22]). Let $L^g([0,T],\lambda)$ denote the Orlicz space of measurable functions with finite g-moment on $[0,T]$ equipped with the norm

$$\|h\|_g = \inf\left\{c>0 : \int_0^T g(c^{-1}h(s))\mathrm{d}s \leq 1\right\}. \tag{4.33}$$

According to Chapter 3.3, Theorem 10, and Chapter 3.5, Theorem 1, in [35], $L^g([0,T],\lambda)$ is a separable Banach space. Let $f_t := f(t,\cdot)$ for all $t\geq 0$. Since $\xi(|f_t|)\in L^1([0,T],\lambda)$ for all $t\geq 0$, it is easy to check that f_t satisfies (2.2)–(2.4) and hence $Y_t := \int_0^T f_t(s)\mathrm{d}Z_s$ is well-defined for all $t\geq 0$. We show that $(Y_t)_{t\geq 0}$ has a measurable modification. Since $L^g([0,T],\lambda)$ is separable and $t\mapsto \|f_t-h\|_g$ is measurable for all $h\in L^g([0,T],\lambda)$ it follows that $t\mapsto f_t$ is a measurable mapping from $\mathbb{R}_+$ into $L^g([0,T],\lambda)$. Furthermore, since $L^g([0,T],\lambda)$ is separable there exists $(h_k^n)_{n,k\geq 1}\subseteq L^g([0,T],\lambda)$ and disjoint measurable sets $(A_k^n)_{k\geq 1}$ for all $n\geq 1$ such that with

$$f_t^n(s) = \sum_{k\geq 1} h_k^n(s)1_{A_k^n}(t), \tag{4.34}$$

we have $\|f_t - f_t^n\|_g \leq 2^{-n}$ for all $t\geq 0$. Set $Y_t^n = \sum_{k\geq 1}\int_0^T h_k^n(s)\mathrm{d}Z_s 1_{A_k^n}(t)$ for all $t\geq 0$ and $n\geq 1$. Then $(Y_t^n)_{t\geq 0}$ is a measurable process and by [20, Theorem 2.1] it follows that

$$\|Y_t^n - Y_t\|_{L^1(P)} \leq 3\|f_t^n - f_t\|_g \leq 3\times 2^{-n}, \quad \forall t\geq 0,\ \forall n\geq 1. \tag{4.35}$$

For all $t\geq 0$ and $\omega\in\Omega$ let $\tilde{Y}_t(\omega) = \lim_n Y_t^n(\omega)$ when the limit exists in $\mathbb{R}$ and zero otherwise. Then $(\tilde{Y}_t)_{t\geq 0}$ is measurable and for all $t\in\mathbb{R}$, $\tilde{Y}_t = Y_t$ P-a.s. by (4.35). Thus we have constructed a measurable modification of $(Y_t)_{t\geq 0}$.

Let us show that both sides of (4.32) are well-defined. Since $g/2\leq \xi\leq g$ and $\xi(ax)\leq (a+1)^2\xi(x)$ for all $x,a>0$, it follows by Jensen's inequality that

$$\int_0^T \xi\left(\int |f(t,s)|\,\mu(\mathrm{d}t)\right)\mathrm{d}s \leq \frac{2(\mu(\mathbb{R})+1)^2}{\mu(\mathbb{R})}\int_0^T\int \xi(|f(t,s)|)\,\mu(\mathrm{d}t)\mathrm{d}s < \infty. \tag{4.36}$$

Thus, the right-hand side of (4.32) is well-defined. The left-hand side is well-defined as well since

$$\begin{aligned} E\left[\int\left|\int_0^T f(t,s)\mathrm{d}Z_s\right|\mu(\mathrm{d}t)\right] &\leq 3\int\left(\int_0^T \xi(|f_t(s)|)\mathrm{d}s\right)\vee\left(\int_0^T \xi(|f_t(s)|)\mathrm{d}s\right)^{1/2}\mu(\mathrm{d}t) \\ &< \infty, \end{aligned} \tag{4.37}$$

where the first inequality follows by [20, Corollary 1.1]. Furthermore, (4.32) is obviously true for simple f on the form

$$f(t,s)=\sum_{i=1}^{n}\alpha_i 1_{(s_{i-1},s_i]}(t)1_{(t_{i-1},t_i]}(s). \tag{4.38}$$

If f is a given function satisfying (i) we can choose a sequence of simple $(f_n)_{n\geq 1}$ converging to f and satisfying $|f_n|\leq |f|$. We have

$$\int\left(\int_0^{\mathrm{T}} f_n(u,s)\mathrm{d}Z_u\right)\mu(\mathrm{d}s)=\int_0^{\mathrm{T}}\left(\int f_n(u,s)\,\mu(\mathrm{d}s)\right)\mathrm{d}Z_u, \tag{4.39}$$

and by estimates as above it follows that we can go to the limit in $L^1(P)$ in (4.39), which shows (4.32).

The case (ii) follows by a similar argument. In this case we have to work in $L^2([0,T],\lambda)$ instead of $L^g([0,T],\lambda)$. □

Proposition 3.2 is an immediate consequence of Theorem 3.1 and Lemma 4.9, since

$$\phi(t-s)=\phi(0)+\int_0^{t-s}\phi'(u)\mathrm{d}u=\phi(0)+\int_0^t 1_{\{s\leq u\}}\phi'(u-s)\mathrm{d}u,\quad s\in[0,t]. \tag{4.40}$$

5. The two-sided case

Let $(X_t)_{t\geq 0}$ be given by

$$X_t=\int_{-\infty}^{t}(\phi(t-s)-\psi(-s))\,\mathrm{d}Z_s,\quad t\geq 0, \tag{5.1}$$

where $(Z_t)_{t\in\mathbb{R}}$ is a (two-sided) nondeterministic Lévy process with characteristic triplet (γ,σ^2,ν) and $\phi,\psi:\mathbb{R}\to\mathbb{R}$ are measurable functions for which the integral exists (still in the sense of [19, page 460]). Also assume that ϕ and ψ are 0 on $(-\infty,0)$ and let $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$ denote the least filtration for which $\sigma(Z_s:-\infty<s\leq t)\subseteq\mathcal{F}_t^{Z,\infty}$ for all $t\geq 0$. From [19, Theorem 2.8] it follows that $(X_t)_{t\geq 0}$ is well-defined if and only if

$$X_t^1=\int_0^t\phi(t-s)\mathrm{d}Z_s,\quad\text{and}\quad X_t^2=\int_{-\infty}^0(\phi(t-s)-\psi(-s))\mathrm{d}Z_s, \tag{5.2}$$

are well-defined. Similar to Lemma 4.8 we have the following.

Lemma 5.1. *Let $(Y_t)_{t\geq 0}$, $(U_t)_{t\in\mathbb{R}}$, $(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\in\mathbb{R}}$ denote four processes such that $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-adapted and $(Y_\cdot,U_\cdot)\overset{\mathcal{D}}{=}(\tilde{Y}_\cdot,\tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-semimartingale.*

Lemma 5.2. *Assume that $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z})_{t\geq 0}$-semimartingale and $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation.*

Proof. The *if*-part is trivial. To show the *only if*-part assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Let $\tilde{X}_t=X_t^1-X_t^2$ and let $\tilde{Z}_t=Z_t$ for $t\geq 0$ and $\tilde{Z}_t=-Z_t$ when $t<0$.

Since $(Z_t)_{t\in\mathbb{R}}$ is symmetric $(X_\cdot, Z_\cdot) \overset{\mathcal{D}}{=} (\tilde{X}_\cdot, \tilde{Z}_\cdot)$ and from Lemma 5.1 it follows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale since $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0} = (\mathcal{F}_t^{Z,\infty})_{t\geq 0}$. Thus, $(X_t^1)_{t\geq 0} = ((X_t + \tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}_t^{Z})_{t\geq 0}$-semimartingale. Moreover, $(X_t^2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence càdlàg and of bounded variation since X_t^2 is $\mathcal{F}_0^{Z,\infty}$-measurable for all $t \geq 0$. □

We have the following consequence of Lemma 5.2 and Theorem 3.1.

Proposition 5.3. *Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume that it is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale.*

If $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation then ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' satisfying (1.3)–(1.4).

If $(Z_t)_{t\in\mathbb{R}}$ is of bounded variation then $(X_t)_{t\geq 0}$ is of bounded variation and ϕ is of bounded variation as well.

Proof. Let $\tilde{Z}_t = Z_t - Z_t'$ where $(Z_t')_{t\in\mathbb{R}}$ is an independent copy of $(Z_t)_{t\in\mathbb{R}}$ and let $(X_t')_{t\geq 0}$ be given by

$$X_t' = \int_{-\infty}^{t} (\phi(t-s) - \psi(-s))\,\mathrm{d}Z_s', \quad t \geq 0. \tag{5.3}$$

By Lemma 5.1, $(X_t')_{t\geq 0}$ is an $(\mathcal{F}_t^{Z',\infty})_{t\geq 0}$-semimartingale, which by independence of filtrations shows that $(\tilde{X}_t)_{t\geq 0} := (X_t - X_t')_{t\geq 0}$ is a semimartingale in the $(\mathcal{F}_t^{Z,\infty} \vee \mathcal{F}_t^{Z',\infty})_{t\geq 0}$-filtration and hence in the $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-filtration. Since $(\tilde{Z}_t)_{t\in\mathbb{R}}$ is symmetric Lemma 5.2 shows that $(\tilde{X}_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z}})_{t\geq 0}$-semimartingale and since $(\tilde{Z}_t)_{t\geq 0}$ has characteristic triplet $(0, 2\sigma^2, \tilde{\nu})$ where $\tilde{\nu}(A) = \nu(A) + \nu(-A)$, the proposition follows by Theorem 3.1. □

Let $(X_t)_{t\geq 0}$ denote a fractional Lévy motion, that is

$$X_t = \int_{-\infty}^{t} ((t-s)^\tau - (-s)_+^\tau)\mathrm{d}Z_s, \quad t \geq 0, \tag{5.4}$$

where τ is such that the integral exists and $x_+ := x \vee 0$ for all $x \in \mathbb{R}$. In the following let us assume that $(Z_t)_{t\in\mathbb{R}}$ has no Brownian component. Recall the definition of X_t^2 in (5.2). From [19, Theorem 2.8] it follows that it is necessary (and sufficient when $(Z_t)_{t\geq 0}$ is symmetric) that

$$\int_0^\infty \int \left(\left| x((t+s)^\tau - s^\tau) \right|^2 \wedge 1 \right) \nu(\mathrm{d}x)\mathrm{d}s < \infty \tag{5.5}$$

for X_t^2 to be well-defined. A simple calculation shows that (5.5) is satisfied if and only if

$$\tau < 1/2 \quad \text{and} \quad \int_{[-1,1]^c} |x|^{1/(1-\tau)}\ \nu(\mathrm{d}x) < \infty. \tag{5.6}$$

Thus it is necessary that (5.6) and (I)–(III) are satisfied for $(X_t)_{t\geq 0}$ to be well-defined, and when $(Z_t)_{t\in\mathbb{R}}$ is symmetric these conditions are also sufficient. Marquardt [36] studies processes of the form (5.4) under the assumptions that $\sigma^2 = 0$, $\int_{[-1,1]^c} |x|^2\ \nu(\mathrm{d}x) < \infty$, $\gamma = -\int_{[-1,1]^c} x\,\nu(\mathrm{d}x)$ and $0 < \tau < 1/2$. See also [37] for a study of the well-balanced case.

To avoid trivialities assume $\tau \neq 0$. As an application of Proposition 5.3 and Corollary 3.5 we have the following.

Corollary 5.4. *Assume that* $(Z_t)_{t\in\mathbb{R}}$ *has no Brownian component and let* $(X_t)_{t\geq 0}$ *be given by* (5.4). *If* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$*-semimartingale then* $\tau \in (0, 1/2)$ *and* $\int_{[-1,1]} |x|^{1/(1-\tau)}\,\nu(\mathrm{d}x) < \infty$.

In particular let $(X_t)_{t\geq 0}$ denote a linear fractional stable motion with indexes $\alpha \in (0, 2]$ and $H \in (0, 1)$, that is

$$X_t = \int_{-\infty}^{t} \Big((t-s)^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\Big)\,\mathrm{d}Z_s, \quad t \geq 0, \tag{5.7}$$

where $(Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process (see [3, Definition 7.4.1]). For $\alpha = 2$, $(X_t)_{t\geq 0}$ is a fractional Brownian motion (fBm) with Hurst parameter H (up to a scaling constant). From Corollary 5.4 it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $H = 1/\alpha$.

Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume that $(Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process with $\alpha \in (1, 2]$. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale it follows by Proposition 5.3 and result (1) at the end of Section 2 that ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally α-moment. The next result shows that this condition is actually necessary and sufficient for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if we delete "locally". Thus, extending [10, Theorem 6.5] from $\alpha = 2$ to $\alpha \in (1, 2]$ we have the following.

Proposition 5.5. *Let* $(X_t)_{t\geq 0}$ *be given by* (5.1) *and assume that* $(Z_t)_{t\in\mathbb{R}}$ *is a symmetric* α*-stable Lévy process with* $\alpha \in (1, 2]$. *Then* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$*-semimartingale if and only if* ϕ *is absolutely continuous on* $\mathbb{R}_+$ *with a density in* $L^\alpha(\mathbb{R}_+, \lambda)$.

Let B denote a Banach space (not necessarily separable) and assume that there exists a countable subset D of the unit ball of B' (the topological dual space of B) such that

$$\|x\| = \sup_{F\in D} |F(x)|, \quad \forall x \in B. \tag{5.8}$$

Following [38, page 133], a B-valued random element X is called α-stable if $\sum_{i=1}^n a_i F_i(X)$ is a real-valued α-stable random variable for all $n \geq 1$, $F_1, \ldots, F_n \in D$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Let T denote an interval in $\mathbb{R}_+$ and let B denote the subspace of $\mathbb{R}^{\mathrm{T}}$ containing all functions which are càdlàg and of bounded variation. Then B is a Banach space in the total variation norm (but not separable) and since the unit ball of B' consists of F of the form

$$F(f) = \sum_{i=1}^{n} a_i(f(t_i) - f(t_{i-1})), \quad f \in B, \tag{5.9}$$

where $(a_i)_{i=1}^n \subseteq [-1, 1]$ and $(t_i)_{i=0}^n$ is an increasing sequence in T, it follows that B satisfies (5.8).

Proof of Proposition 5.5. For $\alpha = 2$ the result follows by [11, Theorem 3.1]; thus let us assume $\alpha \in (1, 2)$.

Assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. According to Lemma 5.2 $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Consider $(X_t^2)_{t\geq 0}$ as an α-stable random element with values in the Banach space consisting of functions which are càdlàg and of bounded variation equipped with the total variation norm. Hence from [38, Proposition 5.6] it follows that $(X_t^2)_{t\geq 0}$ is of integrable variation on each compact interval. Moreover, by [20, Corollary 1.1] we have

$$E\left[\left|n(X_{i/n}^2 - X_{(i-1)/n}^2)\right|\right] \geq \frac{1}{4}\left(a_{i,n} \wedge \sqrt{a_{i,n}}\right), \quad i, n \geq 1, \tag{5.10}$$

where

$$a_{i,n} := \int_{(i-1)/n}^{\infty} \tilde{\xi}(|\Delta_n \phi(s)|)\mathrm{d}s, \quad \text{and} \quad \tilde{\xi}(x) := \int (|xs|^2 \wedge |xs|)\, \nu(\mathrm{d}s). \tag{5.11}$$

Since $i \mapsto a_{i,n}$ is decreasing it follows that

$$E[\mathrm{V}_1(X^2)] \geq \sup_{n\geq 1} \sum_{i=1}^{n} E\left[\left|X_{i/n}^2 - X_{(i-1)/n}^2\right|\right] \geq \sup_{n\geq 1} \frac{1}{4}\left(a_{n,n} \wedge \sqrt{a_{n,n}}\right). \tag{5.12}$$

By (5.12) we conclude that $(a_{n,n})_{n\geq 1}$ is bounded and hence $(\tilde{\xi}(|\Delta_n \phi|))_{n\geq 1}$ is bounded in $L^1([1,\infty),\lambda)$. A straightforward calculation shows $\tilde{\xi}(x) = c_1 x^\alpha$ for all $x \geq 0$ for some constant $c_1 > 0$, which implies that $(\Delta_n \phi)_{n\geq 1}$ is bounded in $L^\alpha([1,\infty),\lambda)$. Since $\alpha > 1$, a sequence in $L^\alpha([1,\infty),\lambda)$ is bounded if and only if it is weakly sequentially compact (see [27, Chapter IV.8, Corollary 4]). Thus, by arguing as in Lemma 4.1 it follows that ϕ is absolutely continuous with a density in $L^\alpha([1,\infty),\lambda)$. Furthermore, since $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale it follows by Corollary 3.4 that ϕ is absolutely continuous on $\mathbb{R}_+$ with a density locally in $L^\alpha(\mathbb{R}_+,\lambda)$. This shows the *only if*-part.

Assume conversely that ϕ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$. By Corollary 3.4 $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Thus it is enough to show that $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Since ϕ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$ it follows by arguing as in Lemma 4.1 that $\|\phi(t-\cdot) - \phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq c(t-u)$ for some $c > 0$ and all $0 \leq u \leq t$. For all $p \in [1,\alpha)$ and all $u, t \geq 0$ we have

$$\left\|X_t^2 - X_u^2\right\|_{L^p(P)} = K_{p,\alpha}\, \|\phi(t-\cdot) - \phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq K_{p,\alpha} c\, |t-u|, \tag{5.13}$$

for some constant $K_{p,\alpha} > 0$ only depending on p and α. By letting $p \in (1,\alpha)$, (5.13) and the Kolmogorov–Čentsov Theorem show that $(X_t^2)_{t\geq 0}$ has a continuous modification. Moreover, by letting $p = 1$ (5.13) shows that this modification is of integrable variation on each compact interval. This completes the proof. □

Motivated by Lemma 5.2 we study in the following proposition infinitely divisible processes $(X_t)_{t\geq 0}$ of bounded variation, where $(X_t)_{t\geq 0}$ is on the form $X_t = \int_{\mathbb{R}} f(t,s)\mathrm{d}Z_s$. Assume that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation. Rosiński [29, Theorem 4] shows that $t \mapsto f(t,s)$ is of bounded variation for λ-a.a. $s \in \mathbb{R}$. Extending this we show that the total variation of $f(\cdot,s)$ must satisfy an integrability condition which is equivalent to the existence of $\int_{\mathbb{R}} \mathrm{V}_t(f(\cdot,s))\mathrm{d}Z_s$ for all $t > 0$ when $(Z_t)_{t\in\mathbb{R}}$ is symmetric and has no Brownian component.

Proposition 5.6. *Let $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ denote a measurable function such that $X_t = \int_{\mathbb{R}} f(t,s)\mathrm{d}Z_s$ is well-defined for all $t \geq 0$. If $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation then*

$$\iint \left(1 \wedge |x\, \mathrm{V}_t(f(\cdot,s))|^2\right) \nu(\mathrm{d}x)\mathrm{d}s < \infty, \quad \forall t > 0. \tag{5.14}$$

Let $(\epsilon_i)_{i\geq 1}$ denote a Rademacher sequence, i.e. $(\epsilon_i)_{i\geq 1}$ is an i.i.d. sequence such that $P(\epsilon_1 = -1) = P(\epsilon_1 = 1) = 1/2$. It is well-known that if $(\alpha_i)_{i\geq 1} \subseteq \mathbb{R}$ then $\sum_{i=1}^{\infty} \epsilon_i \alpha_i$

converges P-a.s. if and only if $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$. Let B denote a Banach space satisfying (5.8). Following [38, page 99], a B-valued random element X is called a vector-valued Rademacher series if there exists a sequence $(x_i)_{i\geq 1}$ in B such that $\sum_{i=1}^{\infty} F^2(x_i) < \infty$ for all $F \in D$ and $(F_1(X), \dots, F_n(X))$ equals $(\sum_{i=1}^{\infty} \epsilon_i F_1(x_i), \dots, \sum_{i=1}^{\infty} \epsilon_i F_n(x_i))$ in distribution for all $n \geq 1$ and all $F_1, \dots, F_n \in D$.

Proof of Proposition 5.6. By a symmetrization argument we may and do assume that $\sigma^2 = 0$ and $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Define

$$Y_t = \sum_{j=1}^{\infty} \epsilon_j C_j f(t, U_j), \quad t \geq 0, \tag{5.15}$$

where $(\epsilon_j)_{j\geq 1}$ is a Rademacher sequence, $(\tau_j)_{j\geq 1}$ are the partial sums of i.i.d. standard exponential random variables and $(U_j)_{j\geq 1}$ are i.i.d. standard normal random variables with density ρ, and $(\epsilon_j)_{j\geq 1}$, $(\tau_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ are independent. Let $\nu^{\leftarrow}\colon \mathbb{R}_+ \to \mathbb{R}_+$ denote the right-continuous inverse of the mapping $x \mapsto \nu((x,\infty))$, that is, $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x,\infty)) \leq s\}$, and let $C_j := \nu^{\leftarrow}(\tau_j \rho(U_j))$ for all $j \geq 1$. By [29, Proposition 2], the series (5.15) converges P-a.s. and $(Y_t)_{t\geq 0}$ has the same finite dimensional distributions as $(X_t)_{t\geq 0}$. Thus, $(Y_t)_{t\geq 0}$ has a càdlàg modification of locally bounded variation. Hence we may and do assume that $(X_t)_{t\geq 0}$ is given by (5.15). Moreover, we may define $(\epsilon_j)_{j\geq 1}$ on a probability space $(\Omega', \mathcal{F}', P')$, $(\tau_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ on a probability space $(\Omega'', \mathcal{F}'', P'')$ and $(X_t)_{t\geq 0}$ on the product space. Let $T = [0, t]$ denote a compact interval in $\mathbb{R}_+$ and let B denote the subspace of $\mathbb{R}^{\mathrm{T}}$ consisting of functions which are càdlàg and of bounded variation. Inspired by Marcus and Rosiński [23] let us fix $\omega'' \in \Omega''$ and consider $\underline{X} = (X_t)_{t\in T}$ as a B-valued Rademacher series under P'. From [38, Theorem 4.8] it follows that $E'[\mathrm{e}^{\alpha \|\underline{X}\|^2}] < \infty$ for all $\alpha > 0$, which in particular shows that $(X_t)_{t\in T}$ is of P'-integrable variation. By Khinchine's inequality there exists a constant $c > 0$ such that $E'[|X_t - X_u|] \geq c\, \|X_t - X_u\|_{L^2(P')}$ for all $u, t \geq 0$. Together with the triangle inequality in l^2 this shows that

$$\begin{aligned} E'\left[\sum_{i=1}^{n} \left|X_{t_i} - X_{t_{i-1}}\right|\right] &\geq c \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} C_j^2 (f(t_i, U_j) - f(t_{i-1}, U_j))^2\right)^{1/2} \\ &\geq c \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} \left|C_j (f(t_i, U_j) - f(t_{i-1}, U_j))\right|\right)^2\right)^{1/2} \\ &= c \left(\sum_{j=1}^{\infty} \left(\left|C_j\right| \sum_{i=1}^{n} \left|f(t_i, U_j) - f(t_{i-1}, U_j)\right|\right)^2\right)^{1/2}. \end{aligned} \tag{5.16}$$

Thus, by monotone convergence we conclude

$$E'[\mathrm{V}_t(X)] \geq c \left(\sum_{j=1}^{\infty} \left(C_j \,\mathrm{V}_t(f(\cdot, U_j))\right)^2\right)^{1/2}, \tag{5.17}$$

and in particular $(C_j \,\mathrm{V}_t(f(\cdot, U_j)))_{j\geq 1} \in l^2$. Thus, we have shown that the series $\sum_{j=1}^{\infty} \epsilon_j C_j \,\mathrm{V}_t(f(\cdot, U_j))$ converges P-a.s. and from Theorem 2.4 and Proposition 2.7 in [39] it follows that

$$\int_0^\infty \int \left(1 \wedge H(u,v)^2\right) \rho(v) \mathrm{d}v \mathrm{d}u < \infty, \tag{5.18}$$

where $H(u,v) = \nu^{\leftarrow}(u\rho(v))\, \mathrm{V}_t(f(\cdot, v))$. Furthermore, (5.18) equals

$$\begin{aligned} &\iint \left(1 \wedge (\nu^{\leftarrow}(u)\, \mathrm{V}_t(f(\cdot,v)))^2\right) \frac{1}{\rho(v)} \mathrm{d}u\, \rho(v) \mathrm{d}v \\ &= \iint \left(1 \wedge (u\, \mathrm{V}_t(f(\cdot,v)))^2\right) \nu(\mathrm{d}u) \mathrm{d}v, \end{aligned} \tag{5.19}$$

which shows (5.14). □

References

[1] J.L. Doob, Stochastic Processes, Wiley Classics Library, John Wiley & Sons Inc., New York, 1990, Reprint of the 1953 original, A Wiley-Interscience Publication.

[2] A. Rocha-Arteaga, K.-i. Sato, Topics in Infinitely Divisible Distributions and Lévy Processes, in: Aportaciones Matemáticas: Investigación [Mathematical Contributions: Research], vol. 17, Sociedad Matemática Mexicana, México, 2003.

[3] G. Samorodnitsky, M.S. Taqqu, Stable non-Gaussian random processes, in: Stochastic Modeling, Chapman & Hall, New York, 1994, Stochastic models with infinite variance.

[4] P. Protter, Volterra equations driven by semimartingales, Ann. Probab. 13 (2) (1985) 519–530.

[5] M. Reiß, M. Riedle, O. van Gaans, On Émery's inequality and a variation-of-constants formula, Stochastic Anal. Appl. 25 (2) (2007) 353–379.

[6] O.E. Barndorff-Nielsen, J. Schmiegel, Ambit processes: With applications to turbulence and tumour growth, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, Springer, Berlin, 2007, pp. 93–124.

[7] F. Biagini, B. Øksendal, A. Sulem, N. Wallner, An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion, Proc. Roy. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 460 (2041) (2004) 347–372, Stochastic analysis with applications to mathematical finance.

[8] F. Delbaen, W. Schachermayer, A general version of the fundamental theorem of asset pricing, Math. Ann. 300 (3) (1994) 463–520.

[9] K. Bichteler, Stochastic integration and L^p-theory of semimartingales, Ann. Probab. 9 (1) (1981) 49–89.

[10] F.B. Knight, Foundations of the Prediction Process, in: Oxford Studies in Probability, vol. 1, The Clarendon Press, Oxford University Press, Oxford Science Publications, New York, 1992.

[11] A. Cherny, When is a moving average a semimartingale? MaPhySto – Research Report2001–28, 2001. Available from http://www.maphysto.dk/cgi-bin/gp.cgi?publ=318.

[12] P. Cheridito, Gaussian moving averages, semimartingales and option pricing, Stochastic Process. Appl. 109 (1) (2004) 47–68.

[13] A. Basse, Spectral representation of Gaussian semimartingales. Thiele Centre – Research Report 2008–03, 2008b. Available from http://www.imf.au.dk/publs?id=672.

[14] T. Jeulin, M. Yor, Moyennes mobiles et semimartingales, Sémin. Probab. XXVII (1557) (1993) 53–77.

[15] A. Basse, Gaussian moving averages and semimartingales, Electron. J. Probab. 13 (39) (2008) 1140–1165.

[16] K. Sato, Lévy Processes and Infinitely Divisible Distributions, in: Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original, Revised by the author.

[17] J. Bertoin, Lévy Processes, in: Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996.

[18] P.E. Protter, Stochastic Integration and Differential Equations, Second ed., in: Applications of Mathematics (New York), vol. 21, Springer-Verlag, Berlin, 2004, Stochastic Modelling and Applied Probability.

[19] B.S. Rajput, J. Rosiński, Spectral representations of infinitely divisible processes, Probab. Theory Related Fields 82 (3) (1989) 451–487.

[20] M.B. Marcus, J. Rosiński, L^1-norms of infinitely divisible random vectors and certain stochastic integrals, Electron. Comm. Probab. 6 (2001) 15–29 (electronic).

[21] J. Rosiński, On stochastic integral representation of stable processes with sample paths in Banach spaces, J. Multivariate Anal. 20 (2) (1986) 277–302.
[22] M. Braverman, G. Samorodnitsky, Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes, Stochastic Process. Appl. 78 (1) (1998) 1–26.
[23] M.B. Marcus, J. Rosiński, Sufficient conditions for boundedness of moving average processes, in: Stochastic Inequalities and Applications, in: Progr. Probab., vol. 56, Birkhäuser, Basel, 2003, pp. 113–128.
[24] S. Kwapień, M.B. Marcus, J. Rosiński, Two results on continuity and boundedness of stochastic convolutions, Ann. Inst. H. Poincaré Probab. Statist. 42 (5) (2006) 553–566.
[25] B.B. Mandelbrot, J.W. Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.
[26] G.H. Hardy, J.E. Littlewood, Some properties of fractional integrals. I, Math. Z. 27 (1) (1928) 565–606.
[27] N. Dunford, J.T. Schwartz, Linear Operators: Part I: General Theory, in: Pure and Applied Mathematics, vol. 7, Interscience Publishers, Inc., New York, 1957.
[28] W. Rudin, Functional Analysis, Second ed., in: International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.
[29] J. Rosiński, On path properties of certain infinitely divisible processes, Stochastic Process. Appl. 33 (1) (1989) 73–87.
[30] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Second ed., in: Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991.
[31] C. Stricker, Semimartingales gaussiennes—application au problème de l'innovation, Z. Wahrsch. Verw. Gebiete 64 (3) (1983) 303–312.
[32] C. Dellacherie, P.-A. Meyer, Probabilities and Potential B: Theory of Martingales, in: North-Holland Mathematics Studies, vol. 72, North-Holland Publishing Co., Amsterdam, 1982.
[33] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, Second ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003.
[34] J. Rosiński, G. Samorodnitsky, Distributions of subadditive functionals of sample paths of infinitely divisible processes, Ann. Probab. 21 (2) (1993) 996–1014.
[35] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker Inc., New York, 1991.
[36] T. Marquardt, Fractional Lévy processes with an application to long memory moving average processes, Bernoulli 12 (6) (2006) 1099–1126.
[37] A. Benassi, S. Cohen, J. Istas, On roughness indices for fractional fields, Bernoulli 10 (2) (2004) 357–373.
[38] M. Ledoux, M. Talagrand, Probability in Banach Spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes.
[39] J. Rosiński, On series representations of infinitely divisible random vectors, Ann. Probab. 18 (1) (1990) 405–430.