

Were the Fibonacci Series and the Golden Section Known in Ancient Egypt?

Corinna Rossi and Christopher A. Tout

Churchill College, Cambridge CB3 0DS, United Kingdom

The Fibonacci series and the Golden Section have often been used to explain the proportions of ancient Egyptian art and architecture. All such theories, however, are based on our modern mathematical system. They have never been examined in the realm of ancient Egyptian mathematics as we understand it from studying the surviving mathematical sources. This article analyses the compatibility of the Fibonacci series with ancient Egyptian mathematics and suggests how an ancient scribe could have handled it. The conclusion is that concepts such as ϕ and the convergence to ϕ have little in common with the surviving ancient Egyptian mathematical documents and that they are quite far from the ancient Egyptian mentality. © 2002 Elsevier Science (USA)

La serie di Fibonacci e la Sezione Aurea sono state spesso utilizzate per spiegare le proporzioni nell'arte e nell'architettura dell'antico Egitto. Tali teorie, tuttavia, sono basate sul nostro sistema matematico moderno, e non sono mai state esaminate nel contesto di ciò che è sopravvissuto della matematica Egizia. Questo articolo analizza la compatibilità della serie di Fibonacci con la matematica Egizia e suggerisce un modo in cui un antico scriba potrebbe averla gestita. La conclusione è che concetti come ϕ e la convergenza a ϕ hanno poco in comune con gli antichi documenti matematici, e sono assolutamente distanti dalla mentalità Egizia. © 2002 Elsevier Science (USA)

MSC 1991 subject classification 01A16.

Key Words: Egypt; Fibonacci; Golden Section; architecture.

INTRODUCTION

From the second half of the 19th century onwards, the ratio known as the Golden Section has often been used by scholars to explain the design and proportions of ancient monuments. Ancient Egyptian architecture has not escaped this trend. From it have emerged a wide range of more or less interesting results, from the first confused hints suggested by Choisy, (1889, 51–58) to the complex geometrical analyses published by Lawlor (1982, 54–55, 61–62) inspired by Schwaller de Lubicz (1957), from Ghyka's ideal schemes (1931, plates 26, 27) to Fournier des Corats' imaginary geometrical convolutions (Fournier des Corats 1957) based on the pyramid of Cheops (the Great Pyramid, a favourite target of numerological and esoteric theories) down to the more sober and acceptable theory suggested by Badawy (1965).

Many of these interpretations just play with the complexity that can be derived from any geometrical figure however simple it may be. In this way, it seems possible to uncover hidden and meaningful mathematical connections that may produce an endless chain of symbolic links. Even if many of these studies go off at a tangent and end up very far from any historical and archaeological evidence, this does not mean that the study of the

proportions of ancient monuments should be entirely dismissed. Instead, this emphasises the importance of pursuing such research with a greater respect for the ancient sources, in particular for their contemporary mathematical systems. Too often modern scholars tend to forget that ancient architects did not necessarily use our own mathematical system and that sometimes what works with our numbers would not have worked with theirs.

While some of the other theories are so far-fetched (or even absurd) that spending time to analyse them and prove them wrong may seem pointless, Badawy's study is definitely worth accurate criticism. A complete discussion of the subject, from some side aspects of his study to its repercussions on the history of ancient Egyptian architecture, is out of the scope of this short article. Here we shall analyse only a basic but extremely important point of his theory, its compatibility with the ancient Egyptian mathematical sources.

This paper has greatly benefited from discussions with Barry J. Kemp and Serafina Cuomo and from the comments of the two anonymous referees. We express our gratitude to Churchill College, Cambridge, for giving us the opportunity to work together and to Andrew Webber for suggesting it in the first instance.

THE THEORY OF ALEXANDER BADAWY

The relationship between the Fibonacci series and the Golden Section forms the basis of Badawy's theory on the proportions of ancient Egyptian architecture from the Old Kingdom to the Graeco-Roman Period (thus including monuments from ca. 2600 B.C. to the 1st century A.D.). In comparison with some extremely complicated Golden Section-based theories by other scholars, Badawy was certainly more successful. First of all he was able to analyse, by means of his system, over 50 buildings whilst other authors often contented themselves with a couple of examples and considered them enough to claim a universal pattern. Second, he suggested a relatively simple method that the ancient architects could have used to achieve in practice what in theory could be a rather complicated proportion.

Badawy believed that the ancient Egyptians designed plans and elevations of their buildings using a geometrical process based on the square, the rectangle, and especially a number of triangles, among which the most important was the so-called 8 : 5 triangle, that is, an isosceles triangles in which the base is equal to 8 units and the height to 5. The preference for this ratio was due to the fact that 8 : 5 gives 1.6 as a result, a good approximation to the Golden Section ϕ , the irrational number $\frac{1+\sqrt{5}}{2} = 1.618033989\dots$. Badawy suggested that, in order to achieve a Golden Section-based pattern in the design of their buildings, the Egyptians often gave their architectural elements dimensions corresponding to numbers of the Fibonacci series 1, 2, 3, 5, 8, 13, 21, 34, 55, ... (each term is the sum of the two previous terms), because the sequence of ratios of consecutive numbers of this sequence converges to ϕ :

$$\frac{2}{1} = 2$$

$$\frac{3}{2} = 1.5$$

$$\frac{5}{3} = 1.666666\dots$$

$$\frac{8}{5} = 1.6$$

$$\frac{13}{8} = 1.625$$

$$\frac{21}{13} = 1.615384 \dots$$

$$\frac{34}{21} = 1.619047 \dots$$

$$\frac{55}{34} = 1.617647 \dots$$

$$\frac{89}{55} = 1.618181 \dots$$

$$\frac{144}{89} = 1.617977 \dots$$

$$\frac{233}{144} = 1.618055 \dots$$

$$\frac{377}{233} = 1.618025 \dots$$

and so on. Alternatively (or contemporarily), the Egyptians would use a network of 8 : 5 triangles that would approximate the proportion even if the Fibonacci numbers were not present in the building.

Even if Badawy approached the subject in a more consistent way than many of his colleagues, the main problem with his theory is that he did not go as far as to check whether the use of Fibonacci numbers and their eventual link to the Golden Section was compatible with the surviving ancient Egyptian mathematics. This point is crucial, because it may allow us to draw a line dividing conscious calculation from coincidence. It may be noted that proving the incompatibility between the calculation of ϕ and ancient Egyptian mathematics leaves open the question of a psychological tendency towards Golden Section-based geometrical figures. Since Fechner's first studies, a series of psychological experiments have been carried out to establish whether modern Western cultures really display a preference for this proportion.¹ The results are uncertain, and anyway there is always doubt as to whether these results may be extended to an ancient culture that disappeared long ago.

CHARACTERISTICS OF EGYPTIAN MATHEMATICS AND FIBONACCI SERIES

In order to proceed, it is useful to summarise a few basic concepts of ancient Egyptian mathematics. Our sources are a number of mathematical texts written on papyri, ostraca,

¹ On psychological experiments, see for example Berlyne (1970), Eysenck and Castle (1970), Benjafeld and Adams-Webber (1976), McManus (1980), Shalit (1980), Benjafeld (1985). On a psychological trend towards the Golden Section in ancient Egyptian art and architecture see for example Davis (1989, 48) and Kemp & Rose (1991). For the presence of Fibonacci numbers in ancient architecture see also Preziosi (1968) and Fletcher (1976).

and leather dating to the second half of the Middle Kingdom and the Second Intermediate Period, that is, more or less between 1800 and 1600 B.C., the most important of which are the Rhind Mathematical Papyrus (Peet 1923; Chace *et al.* 1929; Robins & Shute 1987) (usually abbreviated as RMP), the Moscow Mathematical Papyrus (Struve 1930) (MMP), the Kahun Papyri (Griffith 1897), and the Egyptian Mathematical Leather Roll (Glanville 1927) (EMLR). The computational procedure adopted by the Middle Kingdom scribes survived well into the Graeco-Roman period, as is attested by a set of Demotic papyrus dating from the third century B.C. to the second century A.D. (Parker 1972) and beyond.²

These documents do not correspond to our idea of a mathematical textbook or treatise. Apart from rare cases, they do not contain formulae or general rules that may be applied to solve problems. They contain, instead, table texts, such as the doubling of unit fractions, and problem texts, the majority of which (but not all of them) have a practical character, such as the division of loaves among men, the calculation of the area of a field or the volume of a granary, and so on. Although in theory we might not exclude the possibility that other more theoretical texts did not survive and that by chance we have only a partial glimpse of what ancient Egyptian mathematics produced, in practice there is no evidence to support this suggestion.

Concerning the numerical notation, the Egyptians used integers and fractions, but only the so-called unit fractions, with a numerator of 1, such as $\frac{1}{2}$, $\frac{1}{15}$, $\frac{1}{42}$, or $\frac{1}{140}$, with the exception of the fraction $\frac{2}{3}$. Ratios such as $\frac{3}{5}$, for example, were expressed by means of a sum of unit fractions. However, at least from the Middle Kingdom onwards,³ the result was never $\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$ but could be, for example, $\frac{1}{2} + \frac{1}{10}$. In many cases, it happens that more than one representation exists; that is, more than one combination of unit fractions can be used to express the same quantity. As we shall see below, the Egyptians were fully aware of this and were, in fact, able to use this characteristic profitably.

It is important to note that there is no evidence of the use of the Fibonacci series 1, 2, 3, 5, 8, 13, . . . (or of any similar series, such as 1, 3, 4, 7, 11, 18, . . . or 1, 4, 5, 9, 14, 23, . . . and so on) in any Egyptian mathematical source. Architectural remains unfortunately do not help because Badawy's drawings showing the use of the Fibonacci series in architecture are neither numerous nor convincing. Nonetheless, numerical series quite close in concept to the Fibonacci series were known and used by the Egyptians.

Multiplications and divisions, for example, were performed by doubling or halving the initial number, that is, by using the geometric progression 1, 2, 4, 8, 16, 32, 64, . . . , in which each term is twice the previous one, and its reciprocal $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$. In some cases, the short succession $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}$, in which each term is half the previous one, was also used.⁴ A property of the progression 1, 2, 4, 8, 16, 32, 64, . . . is that any integer can be expressed by means of the sum of some of its terms, and this is how ancient Egyptian multiplication worked. For instance, in order to calculate 15×13 , the scribe would have doubled 15 until the next multiplier (1, 2, 4, etc.) exceeded the multiplicand (13). Then he would find that $1 + 4 + 8 = 13$, tick these numbers, add the corresponding results ($15 + 60 + 120 = 195$),

² For the use of unit fractions in Greek, Coptic and Byzantine texts, see Knorr (1982), Hasitzka (1990, 265–284, 302–312), and Thompson (1914).

³ For the Old Kingdom see Silverman (1975).

⁴ Robins & Shute (1987, 22–24). See also Knorr (1982, 136) for the use of these two series in the doubling of unit fractions.

and write the result below:

$$\begin{array}{r}
 / 1 \times 15 = 15 \\
 2 \times 15 = 30 \\
 / 4 \times 15 = 60 \\
 / 8 \times 15 = 120 \\
 \hline
 \text{Total } 13 \times 15 = 195
 \end{array}$$

Further knowledge of geometric progressions seems to be attested by RMP 79:

An inventory of a household(?)

/ 1	2,801	7	houses
/ 2	5,602	49	cats
/ 4	11,204	343	mice
Total	19,607	2,301 (<i>sic</i>)	spelt
		16,807	<i>hekat</i>
		19,607	total

This calculation has been interpreted as a nursery problem that might have run as follows: “seven houses; in each house seven cats; each cat kills seven mice; each mouse would have eaten seven grains of spelt; each grain of spelt would produce 7 *hekat* [unit of measurement of capacity]. What is the total?”⁵ On the left the scribe performed a quick multiplication of 7×2801 ,⁶ interpreted by Gillings as a proof that the scribe was aware that the geometrical progression 7, 49, 343, 2401, 16,807 . . . has the property that

$$\begin{array}{r}
 \text{the sum of the } f \text{ rst } 2 \text{ terms is} \\
 3 \\
 4 \\
 5
 \end{array}
 \begin{array}{l}
 56 = 7 \times (1 + f \text{ rst term}) = 7 \times 8 \\
 399 = 7 \times (1 + f \text{ rst two terms}) = 7 \times 57 \\
 2,800 = 7 \times (1 + f \text{ rst three terms}) = 7 \times 400 \\
 19,607 = 7 \times (1 + f \text{ rst four terms}) = 7 \times 2801
 \end{array}$$

and so on. Gillings suggested that the scribe might have discovered this property in the well-known progression 1, 2, 4, 8, 16, 32, . . . and extended it to any geometrical progression where the common ratio is the same as the first term (2 for this series, 7 for the series of RMP 79) (Gillings 1972, 167).

The knowledge and use of arithmetic progressions is attested by RMP problems 40 and 64, which deal with distribution of goods. The first asks to divide 100 loaves among five men, so that the shares of the three highest are together seven times the shares of the two lowest. In the second, 10 *hekat* of barley must be divided among 10 men so that the difference of each man over his neighbour is $\frac{1}{8}$ of a *hekat*.

In conclusion, considering how familiar the ancient Egyptian scribes were with numerical series, the knowledge of what we call today the Fibonacci series does not seem incompatible with ancient Egyptian mathematics. However, even if this is the case, it does not necessarily imply any further step in the direction of more theoretical concepts such as the tendency

⁵ Cf. Peet (1923, 121), Gillings (1972, 168–170), and Robins & Shute (1987, 56). Note the scribal error in the left column (2301 instead of 2401).

⁶ In the original text there are no ticks beside the multipliers.

towards a limit. Anyway, let us assume that the Fibonacci series was known and let us try to find out if and how a scribe could have realised that one of the properties of this series was to converge to what we call ϕ . This could have happened in at least two ways, which we shall analyse in the following paragraphs.

WHAT AN ORTHODOX SCRIBE COULD (NOT) SEE

One possibility is that a Middle Kingdom scribe, while calculating for some unknown reason the consecutive ratios of Fibonacci numbers, realised that there was a pattern in his results. As we shall see, however, this is extremely unlikely. If he was adopting the computational procedure that he usually followed to perform his calculations, he would have seen nothing.

Even if, from a mathematical point of view, unit fractions allow the expression of any rational number, the calculations performed by the ancient Egyptians were limited by a certain number of factors. The *recto* of the Rhind Mathematical Papyrus provides an interesting case study. It contains the division of the number 2 by the odd numbers 3 to 101, that is, the doubling of the unit fractions from $\frac{1}{3}$ to $\frac{1}{101}$, which would have proved useful in any process of multiplication involving fractions.

Even if a ratio can be expressed by means of several combinations of unit fractions, the ancient scribes appear to have chosen one single solution among the various possibilities that they had. In order to reconstruct the way the solutions were chosen in preference to others, Gillings and Hamblin prepared a computer program that would list all the possible solutions for each ratio and then compared the results with the scribe's choice. Gillings was thus able to suggest five criteria which seem to explain the choice of the ancient scribe:

1. Of the possible equalities, those with the smaller numbers are preferred, but none (in the doubling of unit fractions) as large as 1000.
2. An equality of only two terms is preferred to one of three terms, and one of three terms is preferred to one of four terms, but an equality of more than four terms is never used.
3. The unit fractions are always set down in decreasing order of magnitude; that is, the larger fraction (corresponding to the smaller number) comes first, but never the same fraction twice.
4. The smallness of the first number is the main consideration, but the scribe would accept a slightly larger first number if it would greatly reduce the last number.
5. Even numbers are preferred to odd numbers, even though they might be larger, and even though the number of terms might thereby increase.

Whether these were rules consciously adopted by the scribes (as Gillings (1972, 45–70) seems to believe⁷) or whether they are simply able to describe an already established situation generated by some other method (as Bruckheimer and Salomon (1977, 445–446) suggest), remains to be established.⁸ At any rate, we decided to take Gillings' approach and results as a model to apply to our problem with the ratios of Fibonacci numbers. Whatever the solution to the dispute between Gillings and Bruckheimer and Salomon may be, by adopting Gillings' method we are likely to obtain a result which is, if not absolutely correct, at least in line with what seemed acceptable to the scribe of the Rhind Mathematical Papyrus.

⁷ All the criteria are taken from p. 49.

⁸ See Vogel's more neutral approach (Vogel 1959, 42).

The convergence to ϕ is apparent when using our numerical system, where we use only one way to express our results. What happens in a system where there can be more than one solution? And is there anything which could have prevented the Egyptians from noticing the convergence? In an attempt to answer these questions, we prepared a computer program which lists all the possible combinations of up to four unit fractions which express each ratio.

In order to isolate the solutions which would have appeared natural to the Egyptians, we followed the first three precepts, that is, calculated all the possible 2-, 3-, and 4-term combinations of unit fractions expressing the ratios of consecutive numbers of the Fibonacci Series for denominators smaller than 1000.⁹ We stopped after 43 ratios, but a scribe would have probably stopped earlier. In RMP Problem 66, the scribe calculated that, if 3200 *ro*¹⁰ of fat are issued for a year, the daily amount would have been $8 + \frac{2}{3} + \frac{1}{10} + \frac{1}{2190}$ *ro*, which implies that scribes did not hesitate to use extremely small fractions whenever it seemed useful. In the case of the Fibonacci series, however, proceeding with very complicated calculations does not seem justified, especially because, as we shall see, the results do not appear particularly encouraging.

The complete results of the computer program are summarised in Table I. It may be seen immediately that there are no allowed representations of the ratios $\frac{144}{233}$ and $\frac{610}{987}$, and that after that, apart from two exceptions, there are no 2-, 3-, or 4-term combinations of unit fractions that can express the following ratios. This allows us to conclude immediately that, in this way, all that a scribe could have done was to find, if possible, an interesting sequence for the first 8 or 10 ratios,¹¹ which is not at all close to the concept of limit. If we perform a further selection on the results by applying the last two criteria, we obtain the sequence contained in Table II. In this case, there is no evident trend in the series of results. If the scribe was simply calculating these ratios without any special purpose, he could have seen nothing.

If he was looking, for some reason, for a tendency in his results, if he decided therefore to ignore some of his usual methods, and if he had at his disposal a list of results such as we now have, he could have isolated a number of series of sequences beginning with $\frac{1}{2} + \frac{1}{9}$, or $\frac{1}{2} + \frac{1}{10}$, or $\frac{1}{2} + \frac{1}{11}$, and so on. Even so, no initial sequences of unit fractions generate complete sets of results within the 4-term combinations. The most complete sequence of results is that beginning with $\frac{1}{2} + \frac{1}{10}$, shown in Table III, but as long as the scribe avoided 5-term combinations or larger denominators, this series too would have been incomplete. Evidently, this approach does not achieve any result.

WHAT AN INGENIOUS SCRIBE MIGHT HAVE DONE

There is another way in which a scribe might have constructed a sequence of unit fractions representing the ratios of the Fibonacci numbers, that might have either been inspired by the discontinuous sequence $\frac{1}{2} + \frac{1}{10}$ shown above (see underlined combinations in Table III), or

⁹ We calculated the ratio of successive terms, that is, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{8}{13}$, and so on. It is a special property of the Fibonacci numbers that the reciprocals of these ratios are equal to $1 +$ the ratio between the two previous numbers, for example: $\frac{8}{5} = 1 + \frac{3}{5}$. This means that, once we have calculated the ratios $\frac{3}{5}$, $\frac{5}{8}$, $\frac{8}{13}$, . . . , we can at any time obtain their reciprocals $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, . . . by simply adding 1 to the already calculated values.

¹⁰ The *ro* was the smallest unit of measure for grain, and corresponded to $\frac{1}{320}$ of the *hekat*.

¹¹ There was no need to express the first two ratios, $\frac{1}{2}$ and $\frac{2}{3}$, in a different form.

TABLE I
Number of 2-, 3-, and 4-Term Combinations of Unit Fractions
Expressing the Ratios of Consecutive Fibonacci Numbers

Ratio	Number of 2-term combinations	Number of 3-term combinations	Number of 4-term combinations
3/5	1	7	105
5/8	1	5	76
8/13	—	3	35
13/21	—	4	61
21/34	—	1	15
34/55	—	1	31
55/89	—	—	1
89/144	—	2	41
144/233	—	—	—
233/377	—	—	3
377/610	—	—	8
610/987	—	—	—
987/1597	—	—	—
1597/2584	—	—	2
2584/4181	—	—	—
4181/6765	—	—	—
6765/10,946	—	—	—
10,946/17,711	—	—	—
17,711/28,657	—	—	—
28,657/46,386	—	—	1
Next 21 ratios	—	—	—

TABLE II
“Simplest” Combination of Unit Fractions Expressing Ratios of Consecutive Fibonacci Numbers
According to the RMP

Ratio	“Simplest” combination of unit fractions	Comments
3/5	$1/2 + 1/10$	
5/8	$1/2 + 1/8$	
8/13	$1/2 + 1/10 + 1/65$	Unless the scribe preferred the 4-term combination of even fractions $1/2 + 1/12 + 1/52 + 1/78$.
13/21	$1/2 + 1/12 + 1/28$	There are no 2-term combinations.
21/34	$1/2 + 1/12 + 1/34 + 1/204$	The scribe would have probably discarded the only 3-term combination $1/2 + 1/9 + 1/153$.
34/55	$1/2 + 1/10 + 1/55$	Unless the scribe preferred the 4-term combination of even fractions $1/2 + 1/10 + 1/80 + 1/176$.
55/89	$1/2 + 1/9 + 1/178 + 1/801$	There are no 2- or 3-term combinations.
89/144	$1/2 + 1/16 + 1/18$	There are no 2-term combinations.
144/233	—	There are no 2-, 3-, or 4-term combinations.
233/377	$1/12 + 1/10 + 1/65 + 1/377$	Only 4-term combinations which contain odd numbers.
377/610	$1/2 + 1/10 + 1/60 + 1/732$	There are no 2- or 3-term combinations.
610/987	—	There are no 2-, 3-, or 4-term combinations.

TABLE III
Most Complete Sequence of Combinations of Unit Fractions Expressing
the Ratios of Consecutive Fibonacci Numbers

Ratio	Sequence $1/2 + 1/10$
3/5	$\frac{1}{2} + \frac{1}{10}$
5/8	$\frac{1}{2} + \frac{1}{10} + \frac{1}{40}$
8/13	$\frac{1}{2} + \frac{1}{10} + \frac{1}{65}$
13/21	$\frac{1}{2} + \frac{1}{10} + \frac{1}{56} + \frac{1}{840}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{57} + \frac{1}{665}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{60} + \frac{1}{420}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{63} + \frac{1}{315}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{273}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{70} + \frac{1}{210}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{75} + \frac{1}{175}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{77} + \frac{1}{165}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{84} + \frac{1}{140}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{90} + \frac{1}{126}$
21/34	$\frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{68} + \frac{1}{340}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{85} + \frac{1}{170}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{90} + \frac{1}{153}$
34/55	$\frac{1}{2} + \frac{1}{10} + \frac{1}{55}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{60} + \frac{1}{660}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{66} + \frac{1}{330}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{80} + \frac{1}{176}$
55/89	—
89/144	$\frac{1}{2} + \frac{1}{10} + \frac{1}{60} + \frac{1}{720}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{72} + \frac{1}{240}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{80} + \frac{1}{180}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{90} + \frac{1}{144}$
144/233	—
233/377	$\frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{377}$
377/610	$\frac{1}{2} + \frac{1}{10} + \frac{1}{60} + \frac{1}{732}$ or $\frac{1}{2} + \frac{1}{10} + \frac{1}{61} + \frac{1}{610}$
610/987	—
Next 29 ratios	—

found independently by simply playing with numbers. If we consider the Fibonacci series

$$1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \quad 144 \quad 233 \quad \dots$$

we can construct a series of fractions in the following way.

—The ratio between the first two terms is $\frac{1}{2}$.

—The ratio between the third and fourth terms is $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$, that is, the previous ratio plus a unit fraction whose denominator is given by the multiplication of 2 and 5, respectively the second and fourth terms of the series.

—The ratio between the fifth and the sixth terms is $\frac{8}{13} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65}$, that is, the previous ratio plus a unit fraction whose denominator is given by the multiplication of 5 and 13, respectively the fourth and sixth terms of the series, and so on. In this way, it is possible to construct a sequence by adding a unit fraction whose denominator is the product of discontinuous pairs of terms:

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

$$\frac{8}{13} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65}$$

$$\frac{21}{34} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442}$$

$$\frac{55}{89} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{3026}$$

$$\frac{144}{233} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{3026} + \frac{1}{20,737}$$

and so on.

The intermediate values (such as $\frac{2}{3}$, $\frac{5}{8}$, and so on) can be calculated by adding to the previous ratio a unit fraction whose denominator is given by the multiplication of the two terms of the ratio (for instance, $\frac{2}{3}$ is given by $\frac{1}{2} + \frac{1}{6}$, where $6 = 2 \times 3$, and so on). Thus the complete sequence is

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$$

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

$$\frac{5}{8} = \frac{1}{2} + \frac{1}{10} + \frac{1}{40}$$

$$\frac{8}{13} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65}$$

$$\frac{13}{21} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{273}$$

$$\frac{21}{34} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442}$$

$$\frac{34}{55} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{1870}$$

$$\frac{55}{89} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{3026}$$

$$\frac{89}{144} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{3026} + \frac{1}{12,816}$$

$$\frac{144}{233} = \frac{1}{2} + \frac{1}{10} + \frac{1}{65} + \frac{1}{442} + \frac{1}{3026} + \frac{1}{20,737}$$

and so on. Very quickly the denominators involved become very large, but the trend is clearly visible from the first ratios, just as in our modern numerical system.

If the Egyptians knew the Fibonacci series, the creation of the sequence of unit fractions shown above falls within the range of their capabilities. However, all we have done so far is to show that it is not impossible that an ancient scribe, playing around with numbers without the constraints of the usual practical rules or for a purpose that must for the moment remain obscure, found out that he could express the ratios of consecutive terms of the Fibonacci series as shown above and therefore could have noticed their convergence. This does not imply that he actually did it, nor that, even if he did it, he took any interest in it.

We must also ask whether he would have actually noted the convergence. What he might have found is the way to construct an infinite sequence of fractions, where it is always possible to add a tiny quantity. We say that the sequence converges to a limit, because we have the concept of the limit and we know the irrational number ϕ . In order to claim that the ancient Egyptians calculated the convergence of Fibonacci numbers to ϕ , we ought to prove not only that they knew this particular irrational number, but also that they accepted the idea of convergence to something quite far from unity.

As for the first point, it might be suggested that the Egyptians had a geometrical concept of ϕ , just as the Greeks had a geometrical concept of π , and that they tried to approximate it using an infinite sequence of fractions. However, the first evidence of a geometrical concept of the Golden Section is to be found in Euclid's *Elements*, dating to the third century BC (Fowler 1982), about 15 centuries after our Middle Kingdom scribes compiled their documents. No ancient Egyptian mathematical source contains any element which may be interpreted as pointing to an earlier knowledge of ϕ . As for the concept of limit, it may be worth observing that, on the contrary, the ancient Egyptians seem to have displayed a marked tendency towards the completion of a unity. In the mathematical papyri, for instance, a common problem is the completion to 1 of a certain quantity (Peet 1923, 53–60; Gillings 1972, Chap. 8; Robins & Shute 1987, 19–21).

Finally, it may be observed that the surviving ancient Egyptian architectural working drawings seem to have been produced by the same practical mentality that generated the surviving mathematical sources.¹² Even if theoretical reasoning, such as the calculation of the convergence of the consecutive ratios of the Fibonacci numbers, were ever carried out by an ancient Egyptian scribe, it does not seem that it could have had a great practical impact. As a consequence, the chances that something like that could be used in building practice are very small indeed. The discussion eventually revolves around the difference between

¹² Compare, for example, the sketch plans on ostraca of a peripteral chapel (Glanville 1930; Van Siclen 1986) and of a four-pillared chamber (Engelbach 1927; Reeves 1986) with RMP problems 56–60.

theory and practice, provided that in ancient Egypt such a difference existed. Those who still believe that other mathematical sources of a different nature existed but did not survive might suggest that, because of its abstract nature, a theoretical problem would not find space in a practical document such as the Rhind Mathematical Papyrus. Seemingly, some scholars who firmly believe that the ancient Egyptians hid in their buildings complicated mathematical relationships that were supposed to remain secret will not, unfortunately, be discouraged by the lack of evidence. These assumptions, however, can be discussed at length but cannot be tested by facts.

In conclusion, on the basis of the available mathematical sources, we believe that there is no evidence to assume that, even if the Egyptians knew the Fibonacci numbers, the convergence of the succession of ratios would have been noticed and, even if it were, that it would have had any major impact on their mathematics and therefore on their architecture. In general, future studies on the proportions in ancient Egyptian architecture are likely to be more effective if they are founded on the extant ancient Egyptian mathematical sources. Adopting the correct language is a necessary, even if not sufficient, condition for attempting a reconstruction of the ancient architectural theory and practice.

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