Starlike trees are determined by their Laplacian spectrum

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Abstract

In this paper, we show that if G is a starlike tree, then it is determined by its Laplacian spectrum. Moreover we prove some facts about trees with the same adjacency spectrum as a starlike tree.

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1. Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph with n vertices and the adjacency matrix A(G). Let D(G) be the diagonal matrix with degrees of the corresponding vertices of G on the main diagonal and zero elsewhere. The matrix L(G) = D(G) − A(G) is called the Laplacian matrix of G. Since A(G) and L(G) are real symmetric matrices, their eigenvalues are real numbers. So we can assume that λ1 ≥ λ2 ≥ · · · ≥ λn and μ1 ≥ μ2 ≥ · · · ≥ μn are the adjacency and the Laplacian eigenvalues of G, respectively. The multiset of eigenvalues of A(G) (L(G)) is called the adjacency (Laplacian) spectrum of G. Two graphs are said to be cospectral with respect to adjacency (Laplacian) matrix

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if they have the same adjacency (Laplacian) spectrum. A graph is said to be determined (DS for short) by the adjacency (Laplacian) spectrum if there is no other non-isomorphic graph with the same spectrum of adjacency (Laplacian).

The problem of characterizing the graphs which are determined by their spectrum goes back for about half a century and originates from chemistry. In 1956, Gunthard and Primas [3] raised the question in a paper that relates the theory of graph spectra to Huckel’s theory from chemistry (see also Chapter 6 of [2]). At that time it was believed that every graph is determined by its adjacency spectrum until one year later Collatz and Sinogowitz [1] presented a pair of cospectral trees. Up to now only few families of DS graphs are known. So finding new families of DS graphs is an interesting problem. For the background and some known results about this problem, we refer the reader to [9] and the references therein.

A tree is called starlike if it has exactly one vertex of degree greater than two. The starlike with maximum degree 3 is called T-shape. We will denote by \( T(l_1, l_2, l_3) \) (assume \( l_1 \leq l_2 \leq l_3 \) without loss of generality in the sequel) the unique T-shape tree such that \( T(l_1, l_2, l_3) - v = P_{l_1} \cup P_{l_2} \cup P_{l_3} \), where \( P_{l_i} \) is the path on \( l_i \) vertices \( (i = 1, 2, 3) \), and \( v \) is the vertex of degree 3. In [5] it was proved that there are no two non-isomorphic starlike trees with the same Laplacian spectrum. Also it was shown in [8] that \( T(1, 1, n - 1) \) and some graphs related to it are determined by their adjacency spectrum as well as their Laplacian spectrum. More recently in [11] it was proved that T-shape trees are determined by their Laplacian spectrum. In this note, we show that starlike trees are determined by their Laplacian spectrum.

In [6] it was proved that two starlike trees are cospectral with the adjacency matrix if and only if they are isomorphic. Also in [10] it was proved that \( T(l_1, l_2, l_3) \) is determined by its adjacency spectrum if and only if \( (l_1, l_2, l_3) \neq (l, l, 2l - 2) \) for any integer \( l \geq 2 \). In this paper, we show that if a non-starlike tree \( G \) has the same adjacency spectrum as a starlike tree of maximum degree \( \Delta \), then the maximum degree of \( G \) is less than \( \Delta \) and \( G \) has at most two vertices of degree greater than 4. Finally, we conjecture that a tree \( T \) and a starlike tree \( T' \) are cospectral with respect to the adjacency matrix if and only if they are isomorphic.

2. Preliminaries

First we give some facts that are needed in the next section.

**Lemma 1** ([9] Interlacing). Suppose that \( A \) is a symmetric \( n \times n \) matrix with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Then eigenvalues \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \) of a principal submatrix of \( A \) of size \( m \) satisfy \( \lambda_i \geq \mu_i \geq \lambda_{n-m+i} \) for \( i = 1, \ldots, m \).

**Theorem 1** [5]. No two non-isomorphic starlike trees are cospectral with respect to the Laplacian matrix.

**Theorem 2** [6]. No two non-isomorphic starlike trees are cospectral with respect to the adjacency matrix.

**Lemma 2** [4]. Let \( T \) be a tree with \( n \) vertices and \( L(T) \) be its line graph. Then for \( i = 1, \ldots, n \), \( \mu_i(T) = \lambda_i(L(T)) + 2 \).

**Corollary 1.** If two trees \( T \) and \( T' \) are cospectral with respect to the Laplacian matrix, then \( L(G) \) and \( L(G') \) are cospectral with respect to the adjacency matrix.
We summarize some results of [7,9] in the following lemma.

**Lemma 3.** Let $G$ be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum.

(i) The number of vertices.
(ii) The number of edges.
(iii) Whether $G$ is regular.
(iv) Whether $G$ is regular with any fixed girth.
   For the adjacency matrix the following follows from the spectrum.
(v) The number of closed walk of any length.
(vi) Whether $G$ is bipartite.
   For the Laplacian matrix the following follows from the spectrum.
(vii) The number of spanning trees.
(viii) The number of components.
(ix) The sum of the squares of degrees of vertices.

3. Main results

Using the previous facts, we show that any starlike tree is determined by its Laplacian spectrum.

**Theorem 3.** Let $G$ be a starlike tree. Then $G$ is determined by its Laplacian spectrum.

**Proof.** Suppose that $G$ and $G'$ are cospectral with respect to Laplacian matrix. By (viii) of Lemma 3, $G'$ is a connected graph and by (vii) of Lemma 3, $G'$ is a tree. Now suppose that $G'$ has $n_i$ vertices of degree $i$, for $i = 1, \ldots, A'$, where $A'$ is the maximum degree of $G'$. We know that $G$ has one vertex of degree $A > 2$, $A$ vertices of degree 1 and $n - A - 1$ vertices of degree 2. By (i), (ii), (ix) of Lemma 3 we have

$$
\sum_{i=1}^{A'} n_i = n, \quad (1)
$$

$$
\sum_{i=1}^{A'} in_i = 2(n - 1), \quad (2)
$$

$$
\sum_{i=1}^{A'} i^2 n_i = A^2 + A + 4(n - A - 1).
$$

So we get

$$
\sum_{i=1}^{A'} (i^2 - 3i + 2) n_i = A^2 - 3A + 2. \quad (3)
$$

Therefore, for $l > A$ we have $n_l = 0$. If $n_A \neq 0$, then $n_A = 1$ and hence for $i = 3, \ldots, A - 1$ we have $n_i = 0$ and by (1) and (2), we have $n_1 = A, n_2 = n - A - 1$. Therefore $G'$ is a starlike tree.
and by Theorem 1, $G'$ is isomorphic to $G$. Now let $n_A = 0$. Since $G$ and $G'$ are cospectral with respect to Laplacian matrix, by Corollary 1, their line graphs $L(G)$ and $L(G')$ are cospectral with respect to adjacency matrix and so by (v) of Lemma 3, they have the same number of triangles (closed walks of length 3). It follows that

$$\sum_{i=1}^{A-1} \binom{i}{3} n_i = \binom{A}{3}.$$ 

Therefore

$$\binom{A}{3} = \sum_{i=1}^{A-1} \binom{i}{3} n_i < \frac{A}{6} \sum_{i=1}^{A-1} (i - 1)(i - 2)n_i.$$ 

So

$$A^2 - 3A + 2 < \sum_{i=1}^{A-1} (i^2 - 3i + 2)n_i.$$ 

Which is contrary to (3). □

**Theorem 4.** Let $H$ and $G$ be non-isomorphic cospectral trees with respect to the adjacency matrix and $H$ be a starlike tree of maximum degree $\Delta$. Then the maximum degree of $G$ is less than $\Delta$ and $G$ has at most two vertices of degree at least 5.

**Proof.** Let $G$ and $H$ be cospectral with respect to the adjacency matrix. Now suppose that $G$ has $n_i$ vertices of degree $i$, for $i = 1, \ldots, A'$, where $A'$ is the maximum degree of $G$. Not that $H$ has one vertex of degree $\Delta$, $\Delta$ vertices of degree 1 and $n - \Delta - 1$ vertices of degree 2. By (i), (ii) of Lemma 3 we have

$$\sum_{i=1}^{A'} n_i = n, \quad \text{(4)}$$

$$\sum_{i=1}^{A'} in_i = 2(n - 1), \quad \text{(5)}$$

It is easy to see that in any graph, the number of closed walk of length 4 equals twice the number of edges plus four times the number of induced paths of length two plus eight times the number of 4-cycles. So by (v) of Lemma 3, $G$ and $H$ have the same number of induced paths of length two. Therefore

$$\sum_{i=1}^{A'} \binom{i}{2} n_i = \binom{\Delta}{2} + (n - \Delta - 1).$$

So we have

$$\sum_{i=1}^{A'} (i^2 - 3i + 2)n_i = A^2 - 3A + 2.$$ 

Therefore for $l > A$ we have $n_l = 0$. If $n_A \neq 0$, then $n_A = 1$ and hence for $i = 3, \ldots, A - 1$ we have $n_i = 0$. So by (4) and (5), we have $n_1 = A$, $n_2 = n - A - 1$. Therefore $G$ is starlike tree.
which is contrary to Theorem 2. So \( n_A = 0 \). By deleting the vertex of degree \( A \) from \( H \) and using Lemma 1 we can see that \( H \) has \( \lambda_2 < 2 \) and so \( G \) has \( \lambda_2 < 2 \). If \( G \) has two non-adjacent vertices of degree at least 5, then \( G \) has an induce subgraph which is a union of two disjoint \( K_{1,4} \). So by Lemma 1 \( G \) has more than one eigenvalue at least 2 which is contrary to the fact that \( G \) has \( \lambda_2 < 2 \). So any two vertices of degree at least 5 are adjacent and \( G \) has at most two vertices of degree at least 5. \( \square \)

References

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