Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 306 (2006) 2219–2221
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Boxicity of series-parallel graphs

Ankur Bohra[a], L. Sunil Chandran[b,*], J. Krishnam Raju[b]

[a]*Department of Mathematics, Indian Institute of Technology, Delhi 110016, India*
[b]*Department of Computer Science and Automation, Indian Institute of Science, Bangalore 560012, India*

Received 24 September 2005; received in revised form 23 March 2006; accepted 21 April 2006
Available online 11 July 2006

Abstract

We show that there exist series-parallel graphs with boxicity 3.

Keywords: Boxicity; Series-parallel graphs; K-tress

1. Introduction

Let $\mathscr{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe U, where V is an index set. The intersection graph $\Omega(\mathscr{F})$ of $\mathscr{F}$ has V as a vertex set, and two distinct vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$. A k-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where R_i (for $1 \leqslant i \leqslant k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph G, its boxicity is the minimum dimension k, such that there exists a family $\mathscr{F}$ of k-dimensional axis-parallel boxes with $\Omega(\mathscr{F}) = G$. We denote the boxicity of a graph G by $\mathrm{box}(G)$. The notion of boxicity was introduced by Roberts [3] and has since been studied by many authors. The complexity of finding the boxicity of a graph was shown to be NP-hard by Cozzens. This was later improved by Yannakakis and finally by Kratochvil [2] who showed that deciding whether the boxicity of a graph is at most 2 itself is NP-complete.

The three well-known graph classes, planar graphs($\mathscr{P}$), series-parallel graphs ($\mathscr{SP}$) and outer planar graphs($\mathscr{OP}$) satisfy the following proper inclusion relation: $\mathscr{OP} \subset \mathscr{SP} \subset \mathscr{P}$. It is known that $\mathrm{box}(G) \leqslant 3$ if $G \in \mathscr{P}$ [5] and $\mathrm{box}(G) \leqslant 2$ if $G \in \mathscr{OP}$ [4]. Thus it is interesting to decide whether there exist series-parallel graphs of boxicity 3. In this paper we construct a series-parallel graph with boxicity 3, thus resolving this question. Recently Chandran and Sivadasan [1] showed that for any G, $\mathrm{box}(G) \leqslant \mathrm{treewidth}(G) + 2$. They conjecture that for any k, there exists a k-tree with boxicity $k+1$. (This would show that their bound is tight but for an additive factor of 1.) The conjecture is trivial for $k = 1$. The series-parallel graph we construct in this paper is a 2-tree with boxicity 3 and thus we verify the conjecture for $k = 2$. (The reader may note that a graph is a series-parallel graph if and only if it is the subgraph of a 2-tree.)

* Corresponding author.
E-mail addresses: mau02481@ccsun50.iitd.ernet.in (A. Bohra), sunil@csa.iisc.ernet.in (L.S. Chandran), jkraju@csa.iisc.ernet.in (J.K. Raju).

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.04.014

2. The construction

The class of undirected graphs known as 2-trees is defined recursively as follows: A 2-tree on 3 vertices is a clique on 3 vertices. Given any 2-tree T_n on n vertices ($n \geqslant 3$) we construct a 2-tree on $n+1$ vertices by applying a *split operation* on an edge (a,b) of T_n. A split operation on (a,b) is the addition of a new vertex c and two new edges (a,c) and (b,c) to T_n. We say that vertex c is obtained by splitting (a,b). When describing our constructions, we use the assignment statement $c = split(a,b)$ to indicate that a split operation is performed on the edge (a,b) to obtain the new vertex c.

$I=(V,E)$ is an interval graph if and only if there exists a function Π that maps each vertex $u \in V$ to a closed interval of the form $[l(u), r(u)]$ on the real line such that $(u,v) \in E(I) \iff \Pi(u) \cap \Pi(v) \neq \emptyset$. We will call Π, an interval representation of I. In a similar way, a rectangle representation of $G=(V,E)$ is a function θ that maps each vertex $v \in V(G)$ to a 2-dimensional axis parallel box $R_1 \times R_2$, where R_i, for $1 \leqslant i \leqslant 2$, is a closed interval of the form $[a_i, b_i]$ on the real line, such that $(u,v) \in E(G) \iff \theta(u) \cap \theta(v) \neq \emptyset$. Let Π_i be the function that maps $u \in V(G)$ to R_i. Then we write $\theta = (\Pi_1, \Pi_2)$. (Note that $\Pi_i(u)$ represents the projection of the box $\theta(u)$ on the ith axis.)

Before presenting the construction of the 2-tree with boxicity 3, we present four simpler graphs which occur as subgraphs of the final 2-tree, to facilitate the presentation of the proof. To construct each of the following graphs, we start with a single edge (a,b) and then perform a few split operations:

1. The graph L_1: $c = split(a,b)$; add a pendant vertex z to c.
2. The graph L_2: $c := split(a,b)$; $x = split(a,c)$; $y := split(b,c)$.
3. The graph L_3: For $i = 1$–5 do:
   $c_i = split(a,b)$; $x_i = split(a,c_i)$; $y_i = split(b,c_i)$.
4. The graph L_4: The graph L_4 is obtained from L_3 by splitting the edge (x_i, c_i) to obtain z_i for $1 \leqslant i \leqslant 5$.

First we collect some lemmas regarding the rectangle representations of the above graphs. The first two lemmas are trivial and we leave the proofs to the reader.

Lemma 1. *Let θ be a rectangle representation of L_1. Then $\theta(c) \nsubseteq \theta(a) \cup \theta(b)$.*

Lemma 2. *Let θ be a rectangle representation of L_2, Then $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ and $\theta(c) \cap (\theta(b) - \theta(a)) \neq \emptyset$.*

Lemma 3. *Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of a graph G. If $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ then at least one of the following two conditions holds.* (1) $\Pi_1(c) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$, (2) $\Pi_2(c) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$.

Proof. If $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ then $\theta(c) \cap \theta(a) \nsubseteq \theta(b)$, which implies $\Pi(a) \cap \Pi(c) \nsubseteq \Pi(b)$, for some $\Pi \in \{\Pi_1, \Pi_2\}$, and the lemma follows. □

Lemma 4. *Let $\{a,b,c\}$ induce a triangle with representation $\theta = (\Pi_1, \Pi_2)$. If $\Pi_i(c) \nsubseteq \Pi_i(a) \cap \Pi_i(b)$ for $i = 1, 2$, then $\theta(c)$ contains a corner point of $\theta(a) \cap \theta(b)$. (If $\theta(a) \cap \theta(b)$ is a point or a line segment the corners may be taken to overlap.)*

Proof. Clearly $\theta(c) \cap (\theta(a) \cap \theta(b)) \neq \emptyset$ and therefore for $i = 1, 2$, $\Pi_i(c) \cap \Pi_i(a) \cap \Pi_i(b) \neq \emptyset$. Combining this with the assumption $\Pi_i(c) \nsubseteq \Pi_i(a) \cap \Pi_i(b)$, we can infer that $\Pi_i(c)$ contains either the left end point or the right end point of $\Pi_i(a) \cap \Pi_i(b)$, for $i = 1, 2$. Thus we conclude that $\theta(c) = \Pi_1(c) \times \Pi_2(c)$ contains at least one corner point of $\theta(a) \cap \theta(b)$. □

Definition 1. Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of G. We say that two vertices $u, v \in V(G)$ are a *crossing pair*with respect to θ if and only if $\Pi_1(u) \subseteq \Pi_1(v)$ and $\Pi_2(v) \subseteq \Pi_2(u)$.

Lemma 5. *Let $\theta = (\Pi_1, \Pi_2)$ be any rectangle representation of L_3. Then a, b cannot be a crossing pair with respect to θ.*

Proof. Suppose a, b be a crossing pair. Then we have $\Pi_1(a) \subseteq \Pi_1(b)$ and $\Pi_2(b) \subseteq \Pi_2(a)$. Now observe that for each i, $1 \leqslant i \leqslant 5$, a, b, c_i, x_i, y_i induce a subgraph isomorphic to L_2. Hence by Lemma 2, we have $\theta(c_i) \cap (\theta(a) - \theta(b)) \neq \emptyset$ and $\theta(c_i) \cap (\theta(b) - \theta(a)) \neq \emptyset$. From $\theta(c_i) \cap (\theta(a) - \theta(b)) \neq \emptyset$ we can infer (by applying Lemma 3) that at least one of the following two conditions hold:

(a) $\Pi_1(c_i) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$,
(b) $\Pi_2(c_i) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$.

But since $\Pi_1(a) \subseteq \Pi_1(b)$ we have $\Pi_1(c_i) \cap (\Pi_1(a) - \Pi_1(b)) = \emptyset$. Thus we infer that $\Pi_2(c_i) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$. It follows that $\Pi_2(c_i) \nsubseteq \Pi_2(a) \cap \Pi_2(b)$. Similarly we can infer that $\Pi_1(c_i) \nsubseteq \Pi_1(a) \cap \Pi_1(b)$. Therefore by Lemma 4, for each i, $1 \leqslant i \leqslant 5$, $\theta(c_i)$ contains a *corner point* of $\theta(a) \cap \theta(b)$. But since there are only at most 4 corner points, by pigeon hole principle there exist i, j where $1 \leqslant i, j \leqslant 5$ and $i \neq j$ such that $\theta(c_i)$ and $\theta(c_j)$ contain the same corner point, i.e. $\theta(c_i) \cap \theta(c_j) \neq \emptyset$, a contradiction since $(c_i, c_j) \notin E(L_3)$. □

Lemma 6. *Let* $\theta = (\Pi_1, \Pi_2)$ *be a rectangle representation of* L_4. *Then there exists* $c \in \{c_i : 1 \leqslant i \leqslant 5\}$ *such that either* a, c *or* b, c *is a crossing pair.*

Proof. We claim that there exists a $c \in \{c_i : 1 \leqslant i \leqslant 5\}$ such that $\Pi_1(c) \subseteq \Pi_1(a) \cap \Pi_1(b)$ or $\Pi_2(c) \subseteq \Pi_2(a) \cap \Pi_2(b)$. Suppose not. Then by Lemma 4, for each i, $1 \leqslant i \leqslant 5$, $\theta(c_i)$ contains a *corner point* of $\theta(a) \cap \theta(b)$. This leads to a contradiction since there are only at most four corner points for $\theta(a) \cap \theta(b)$ and since $\theta(c_i)$, $1 \leqslant i \leqslant 5$ are pairwise disjoint by the Definition of L_4. Therefore without loss of generality we can assume that $\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b)$. Now $\{a, b, c_1, x_1, y_1\}$ induce a graph isomorphic to L_2 in L_4. Therefore by Lemma 2, $\theta(c_1) \cap (\theta(a) - \theta(b)) \neq \emptyset$ and $\theta(c_1) \cap (\theta(b) - \theta(a)) \neq \emptyset$. By Lemma 3, $\theta(c_1) \cap (\theta(a) - \theta(b)) \neq \emptyset$ implies that at least one of the two conditions (a) $\Pi_1(c_1) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$ (b) $\Pi_2(c_1) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$ holds. But since $\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b)$, we have $\Pi_1(c_1) \cap (\Pi_1(a) - \Pi_1(b)) = \emptyset$. Thus we infer that $\Pi_2(c_1) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$. Similarly from $\theta(c_1) \cap (\theta(b) - \theta(a)) \neq \emptyset$ we can infer that $\Pi_2(c_1) \cap (\Pi_2(b) - \Pi_2(a)) \neq \emptyset$. Using these two inequalities (namely, $\Pi_2(c_1) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$ and $\Pi_2(c_1) \cap (\Pi_2(b) - \Pi_2(a)) \neq \emptyset$) and recalling that $\Pi_2(a)$ and $\Pi_2(b)$ are intervals, it is easy to conclude that $\Pi_2(a) \cap \Pi_2(b) \subseteq \Pi_2(c_1)$. Now observe that the graph induced by $\{a, b, c_1, z_1\}$ in L_4 is isomorphic to L_1. Hence by Lemma 1, $\theta(c_1) \nsubseteq \theta(a) \cup \theta(b)$. Thus, recalling that $\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b)$, we must have $\Pi_2(c_1) \nsubseteq \Pi_2(a) \cup \Pi_2(b)$. This along with $\Pi_2(a) \cap \Pi_2(b) \subseteq \Pi_2(c_1)$ allows us to infer that $\Pi_2(a) \subseteq \Pi_2(c_1)$ or $\Pi_2(b) \subseteq \Pi_2(c_1)$. It follows that either a, c_1 is a *crossing pair* or b, c_1 is a *crossing pair*. □

Now we construct the final 2-tree G, and prove that its boxicity equals 3.

1. Let (a, b) be a single edge. For $i = 1$–5 do: $c_i = split(a, b)$.
2. For each c_i where $1 \leqslant i \leqslant 5$ do: For $j = 1$–5 do: $d_{ij} = split(a, c_i)$ and $e_{ij} = split(b, c_i)$.
3. For all i, j where $1 \leqslant i, j \leqslant 5$ do: $p_{ij} = split(a, d_{ij})$; $q_{ij} = split(c_i, d_{ij})$; $r_{ij} = split(b, e_{ij})$; $s_{ij} = split(c_i, e_{ij})$.

First we show that $\mathrm{box}(G) > 2$. Suppose not. Then there exists a rectangle representation for G. Since $\{a, b\} \cup \{c_i, d_{i1}, e_{i1}, q_{i1} : 1 \leqslant i \leqslant 5\}$ induce a graph isomorphic to L_4, by Lemma 6, there exists a $c \in \{c_i : 1 \leqslant i \leqslant 5\}$ such that either a, c or b, c is a *crossing pair*. Without loss of generality let a, c_1 be a *crossing pair*. But $\{a, c_1\} \cup \{d_{1j}, p_{1j}, q_{1j} : 1 \leqslant j \leqslant 5\}$, induce a graph isomorphic to L_3. Thus by Lemma 5, a, c_1 cannot be a *crossing pair*, which is a contradiction. Thus we infer that $\mathrm{box}(G) > 2$. Since any series-parallel graph is planar we have $\mathrm{box}(G) \leqslant 3$ [5] and the result follows.

References

[1] L.S. Chandran, N. Sivadasan, Treewidth and boxicity, submitted for publication, Available at ⟨http://arxiv.org/abs/math.CO/0505544⟩, 2005.
[2] J. Kratochvil, A special planar satisfiability problem and a consequence of its NP-completeness, Discrete Appl. Math. 52 (1994) 233–252.
[3] F.S. Roberts, On the boxicity and cubicity of a graph, Recent Progresses in Combinatorics, Academic Press, New York, 1969.
[4] E.R. Scheinerman, Intersection classes and multiple intersection parameters, Ph. D. Thesis, Princeton University, Princeton, 1984.
[5] C. Thomassen, Interval representations of planar graphs, J. Combin. Theory Ser. B 40 (1986) 9–20.