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Moments, moderate and large deviations for a branching process in a random environment

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Abstract

Let (Z_n) be a supercritical branching process in a random environment ξ, and W be the limit of the normalized population size $Z_n/\mathbb{E}[Z_n|\xi]$. We show large and moderate deviation principles for the sequence $\log Z_n$ (with appropriate normalization). For the proof, we calculate the critical value for the existence of harmonic moments of W, and show an equivalence for all the moments of Z_n. Central limit theorems on $W - W_n$ and $\log Z_n$ are also established.

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1. Introduction and main results

As an important extension of the Galton–Watson process, the model of branching process in a random environment was introduced first by Smith and Wilkinson [22] for the independent environment case, and then by Athreya and Karlin [4] for the stationary and ergodic environment case. See also [3,23,24] for some basic results on the subject. The study of asymptotic properties of a branching process in a random environment has recently received attention; see for example, [1,2,15,5,6,8,7], among others. Here, for a supercritical branching process (Z_n) in a random

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environment, we shall mainly show asymptotic properties of the moments of Z_n, and prove moderate and large deviation principles for $(\log Z_n)$. In particular, our result on the annealed harmonic moments completes that of Hambly [12] on the quenched harmonic moments, and extends the corresponding theorem of Ney and Vidyashanker [21] for the Galton–Watson process; our moderate and large deviation principles complete the results of Kozlov [15], Bansaye and Berestycki [5], Bansaye and Böinghoff [6] and Böinghoff and Kersting [8] on large deviations.

Let us give a description of the model. Let $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in some space Θ, whose realization determines a sequence of probability generating functions

$$f_n(s) = f_{\xi_n}(s) = \sum_{i=0}^{\infty} p_i(\xi_n)s^i, \quad s \in [0,1],\ p_i(\xi_n) \geq 0,\ \sum_{i=0}^{\infty} p_i(\xi_n) = 1. \tag{1.1}$$

A branching process $(Z_n)_{n\geq 0}$ in the random environment ξ can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad n \geq 0, \tag{1.2}$$

where given the environment ξ, $X_{n,i}(i = 1, 2, \ldots)$ are independent of each other and independent of Z_n, and have the same distribution determined by f_n.

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given. As usual, $\mathbb{P}_\xi$ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$ in the sense that for all measurable and positive function g, we have

$$\int g d\mathbb{P} = \int \int g(\xi, y) d\mathbb{P}_\xi(y) d\tau(\xi),$$

where τ is the law of the environment ξ. The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

For $\xi = (\xi_0, \xi_1, \ldots)$ and $n \geq 0$, define

$$m_n(p) = m_n(p, \xi) = \sum_{i=0}^{\infty} i^p p_i(\xi_n) \quad \text{for } p > 0, \tag{1.3}$$

$$m_n = m_n(1), \qquad \Pi_0 = 1 \ \text{ and } \ \Pi_n = m_0 \cdots m_{n-1} \ \text{ for } n \geq 1. \tag{1.4}$$

Then $m_n(p) = \mathbb{E}_\xi X_{n,i}^p$ and $\Pi_n = \mathbb{E}_\xi Z_n$. It is well known that the normalized population size

$$W_n = \frac{Z_n}{\Pi_n}$$

is a nonnegative martingale under $\mathbb{P}_\xi$ (for each ξ) with respect to the filtration $\mathcal{F}_n = \sigma(\xi, X_{k,i}, 0 \leq k \leq n-1, i = 1, 2, \ldots)$, so that the limit

$$W = \lim_{n\to\infty} W_n$$

exists almost sure (a.s.) with $\mathbb{E}W \leq 1$. We shall always assume that

$$\mathbb{E}\log m_0 \in (0,\infty) \quad \text{and} \quad \mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty. \tag{1.5}$$

The first condition means that the process is supercritical; the second implies that W is non-degenerate. Hence (see e.g. [4])

$$\mathbb{P}_\xi(W>0) = \mathbb{P}_\xi(Z_n \to \infty) = \lim_{n\to\infty} \mathbb{P}_\xi(Z_n > 0) \quad a.s.$$

For simplicity, *we write often* p_i *for* $p_i(\xi_0)$ *and assume always*

$$p_0 = 0 \quad a.s.$$

Therefore $W > 0$ and $Z_n \to \infty$ a.s.

It is known that $\frac{\log Z_n}{n} \to \mathbb{E}\log m_0$ a.s. on $\{Z_n \to \infty\}$ (see e.g. [23]). We are interested in the asymptotic properties of the corresponding deviation probabilities. Notice that

$$\log Z_n = \log \Pi_n + \log W_n. \tag{1.6}$$

Since $W_n \to W > 0$ a.s., certain asymptotic properties of $\log Z_n$ would be determined by those of $\log \Pi_n$. We shall show that $\log Z_n$ and $\log \Pi_n$ satisfy the same limit theorems under suitable moment conditions.

At first, we present a large deviation principle. Let $\Lambda(t) = \log \mathbb{E} m_0^t$. Assume that m_0 is not a constant a.s. and that $\Lambda(t) < \infty$ for all $t \in \mathbb{R}$. Let

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{tx - \Lambda(t)\}$$

be the Fenchel–Legendre transform of Λ. It is well known [10, Lemma 2.2.5] that $\Lambda^*(\mathbb{E}\log m_0) = 0$, $\Lambda^*(x)$ is strictly increasing for $x \geq \mathbb{E}\log m_0$ and strictly decreasing for $x \leq \mathbb{E}\log m_0$; moreover,

$$\Lambda^*(x) = \begin{cases} tx - \Lambda(t) & \text{if } x = \Lambda'(t) \text{ for some } t \in \mathbb{R}, \\ \infty & \text{if } x \geq \Lambda'(\infty) \text{ or } x \leq \Lambda'(-\infty). \end{cases}$$

In fact, Λ^* is the rate function with which $\log \Pi_n$ satisfies a large deviation principle. We introduce the following assumption.

(H) *There exist constants* $\delta > 0$ *and* $A > A_1 > 1$ *such that a.s.*

$$A_1 \leq m_0 \quad \text{and} \quad m_0(1+\delta) \leq A^{1+\delta}, \tag{1.7}$$

(recall that m_0 and $m_0(1+\delta)$ were defined in (1.3) and (1.4)). Notice that the second condition implies that $m_0 \leq A$ a.s.

The theorem below shows that $\log Z_n$ and $\log \Pi_n$ satisfy the same large deviation principle.

Theorem 1.1 (*Large Deviation Principle*)**.** *Assume* (H). *If* $\mathbb{E}Z_1^s < \infty$ *for all* $s > 1$ *and* $p_1 = 0$ *a.s., then for any measurable subset* B *of* $\mathbb{R}$,

$$-\inf_{x\in B^o}\Lambda^*(x) \leq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\in B\right) \leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\in B\right) \leq -\inf_{x\in\bar{B}}\Lambda^*(x),$$

where B^o *denotes the interior of* B, *and* $\bar{B}$ *its closure.*

From Theorem 1.1, we obtain immediately the following corollary.

Corollary 1.2. *Assume* (H). *If* $\mathbb{E}Z_1^s < \infty$ *for all* $s > 1$ *and* $p_1 = 0$ *a.s., then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \le x\right) = -\Lambda^*(x) \quad \text{for } x < \mathbb{E}\log m_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \ge x\right) = -\Lambda^*(x) \quad \text{for } x > \mathbb{E}\log m_0.$$

Remark. This result was shown by Bansaye and Berestycki [5] when (H) holds with $\delta = 1$. If $\mathbb{P}(p_1 > 0) > 0$, the rate function for the lower deviation is no longer $\Lambda^*(x)$: in this case, Bansaye and Berestycki [5] proved that under certain hypothesis,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \le x\right) = -\chi(x) \quad \text{for } x < \mathbb{E}\log m_0,$$

where $\chi(x) = \inf_{t\in[0,1]}\{-t \log \mathbb{E}p_1 + (1-t)\Lambda^*(\frac{x}{1-t})\}$. Obviously, $\chi(x) \le \Lambda^*(x)$.

For the upper deviation and for branching processes with special offspring distributions, more precise results can be found in [15,8,6].

Notice that the Laplace transform of $\log Z_n$ is

$$\mathbb{E}e^{t\log Z_n} = \mathbb{E}Z_n^t.$$

Therefore, Theorem 1.1 is a consequence of the Gärtner–Ellis theorem (see e.g. [10]) and Theorem 1.3 below.

Theorem 1.3 (*Moments of* Z_n). *Let* $t \in \mathbb{R}$. *Suppose that one of the following conditions is satisfied:*

(i) $t \in (0, 1]$ *and* $\mathbb{E}m_0^{t-1} Z_1 \log^+ Z_1 < \infty$;
(ii) $t > 1$ *and* $\mathbb{E}Z_1^t < \infty$;
(iii) $t < 0$, $\mathbb{E}p_1 < \mathbb{E}m_0^t$, $\|p_1\|_\infty :=$ esssup $p_1 < 1$ *and (H) holds.*

Then for some constant $C(t) \in (0, \infty)$,

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n^t}{\left(\mathbb{E}m_0^t\right)^n} = C(t).$$

For $t < 0$, Theorem 1.3 is an extension of a result of Ney and Vidyashanker [21] on the Galton–Watson process. Theorem 1.3 can also be used to study the convergence rate in a central limit theorem for $W - W_n$ (see Theorem 1.7).

A key step in the proof of Theorem 1.3 is the study of the harmonic moments (moments of negative orders) of W, which is of interest of its own. The following result is our main result on this subject.

Theorem 1.4 (*Harmonic Moments of* W). *Let* $a > 0$. *Assume* (H) *and* $\|p_1\|_\infty < 1$. *Then*

$$\mathbb{E}W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}p_1 m_0^a < 1.$$

Theorem 1.4 reveals that under certain conditions, the number a_0 satisfying $\mathbb{E}p_1 m_0^{a_0} = 1$ is the critical value for the existence of the harmonic moments $\mathbb{E}W^{-a} (a > 0)$. More precisely, we have the following corollary.

Corollary 1.5. *Assume* (H) *and* $\|p_1\|_\infty < 1$. *If* $\mathbb{E} p_1 m_0^{a_0} = 1$, *then* $\mathbb{E} W^{-a} < \infty$ *if* $0 < a < a_0$ *and* $\mathbb{E} W^{-a} = \infty$ *if* $a \geq a_0$.

Remark. Hambly [12] proved that under an assumption similar to (H), the number $\alpha_0 := -\frac{\mathbb{E}\log p_1}{\mathbb{E}\log m_0}$ is the critical value for the a.s. existence of the quenched moments $\mathbb{E}_\xi W^{-a} (a > 0)$: namely, $\mathbb{E}_\xi W^{-a} < \infty$ a.s. if $a < \alpha_0$ and $\mathbb{E}_\xi W^{-a} = \infty$ a.s. if $a > \alpha_0$. Here, we obtain the critical value for the existence of the annealed moments instead of the quenched ones. Notice that by Jensen's inequality and the equation $\mathbb{E} p_1 m_0^{a_0} = 1$, we see the natural relation that $a_0 \leq \alpha_0$.

Now we consider moderate deviations. Let (a_n) be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty \quad \text{as } n \to \infty. \tag{1.8}$$

Similar to the case of large deviation principle, $\log Z_n$ and $\log \Pi_n$ satisfy the same moderate deviation principle.

Theorem 1.6 (*Moderate Deviation Principle*). *Assume (H) and* $\sigma^2 = \mathrm{var}(\log m_0) \in (0, \infty)$. *Then for any measurable subset* B *of* $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2\sigma^2} &\leq \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B\right) \\ &\leq \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B\right) \leq -\inf_{x\in\bar{B}} \frac{x^2}{2\sigma^2}, \end{aligned}$$

where B^o *denotes the interior of* B, *and* $\bar{B}$ *its closure.*

Here and throughout the paper, $\mathrm{var}(\log m_0)$ denotes the variance of $\log m_0$.

As in the case of large deviation principle, the proof of Theorem 1.6 is based on the Gärtner–Ellis theorem.

As another application of Theorem 1.3, we shall also establish a central limit theorem for $W - W_n$ with exponential convergence rate. Let

$$\delta_\infty^2(\xi) = \sum_{n=0}^{\infty} \frac{1}{\Pi_n}\left(\frac{m_n(2)}{m_n^2} - 1\right) \tag{1.9}$$

(recall that $m_n(2) = \sum_{i=1}^\infty i^2 p_i(\xi_n)$ by (1.3)). Then δ_∞^2 is the variance of W under $\mathbb{P}_\xi$ (see e.g. [14]) if the series converges. As usual, we write $T^n\xi = (\xi_n, \xi_{n+1}, \ldots)$ if $\xi = (\xi_0, \xi_1, \ldots)$ and $n \geq 0$.

Theorem 1.7 (*Central Limit Theorem on* $W - W_n$). *Assume* (H) *and* $\|p_1\|_\infty < 1$. *If* $\mathbb{E} p_1 < \mathbb{E} m_0^{-\epsilon/2}$, essinf $\frac{m_0(2)}{m_0^2} > 1$ *and* $\mathbb{E} Z_1^{2+\epsilon} < \infty$ *for some* $\epsilon \in (0, 1]$, *then for some constant* $C > 0$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\Pi_n(W - W_n)}{\sqrt{Z_n}\,\delta_\infty(T^n\xi)} \leq x\right) - \Phi(x)\right| \leq C\left(\mathbb{E} m_0^{-\epsilon/2}\right)^n. \tag{1.10}$$

Notice that the condition $\|p_1\|_\infty < 1$ is automatically satisfied when $\epsilon > 0$ is small enough.

Theorem 1.7 shows that $W - W_n$ (with appropriate normalization) satisfies a central limit theorem with an exponential convergence rate; it improves a recent result of Wang et al. [25]. For

Galton–Watson process, Theorem 1.7 improves the convergence rate of Heyde and Brown [13], and coincides with that of Ney and Vidyashanker [21].

Finally, as $\log \Pi_n$ satisfies a central limit theorem, it is natural that the same would hold for $\log Z_n$. In fact we have the following theorem.

Theorem 1.8 (*Central Limit Theorem on* $\log Z_n$)**.** *Assume that* $\sigma^2 = \operatorname{var}(\log m_0) \in (0, \infty)$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) = \Phi(x), \tag{1.11}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$ *is the standard normal distribution function.*

The rest of the paper is organized as follows. In Section 2, we consider the harmonic moments of W and prove Theorem 1.4. Section 3 is devoted to the study of the moments of Z_n of all orders (positive or negative) and the large deviations of $\log Z_n$, where Theorems 1.1 and 1.3 are proved with additional information. In Section 4, we consider the moderate deviations of $\log Z_n$ and prove Theorem 1.6. In Section 5, we deal with central limit theorems and prove Theorems 1.7 and 1.8. We end the paper by a short Appendix showing a general result on large deviations.

2. Harmonic moments of W

In this section, we shall study the harmonic moments of W, i.e. $\mathbb{E}W^{-s} (s > 0)$, which are closely related to the corresponding moments of W_n. The following lemma reveals their relations.

Lemma 2.1. *Assume (1.5). Then for any convex function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\lim_{n\to\infty} \mathbb{E}_\xi \varphi(W_n) = \sup_n \mathbb{E}_\xi \varphi(W_n) = \mathbb{E}_\xi \varphi(W) \quad a.s.,$$

and

$$\lim_{n\to\infty} \mathbb{E}\varphi(W_n) = \sup_n \mathbb{E}\varphi(W_n) = \mathbb{E}\varphi(W).$$

In particular, for all $s > 0$,

$$\lim_{n\to\infty} \mathbb{E}_\xi W_n^{-s} = \sup_n \mathbb{E}_\xi W_n^{-s} = \mathbb{E}_\xi W^{-s} \quad a.s.,$$

and

$$\lim_{n\to\infty} \mathbb{E} W_n^{-s} = \sup_n \mathbb{E} W_n^{-s} = \mathbb{E} W^{-s}.$$

Proof. Recall that by (1.5), $W_n \to W$ in L^1. Therefore, $W_n = \mathbb{E}(W|\mathcal{F}_n)$ a.s. By the conditional Jensen's inequality,

$$\mathbb{E}(\varphi(W)|\mathcal{F}_n) \ge \varphi(\mathbb{E}(W|\mathcal{F}_n)) = \varphi(W_n) \quad a.s.,$$

so $\mathbb{E}\varphi(W) \ge \sup_n \mathbb{E}\varphi(W_n)$. The other side comes from Fatou's lemma. The equality

$$\lim_{n\to\infty} \mathbb{E}\varphi(W_n) = \sup_n \mathbb{E}\varphi(W_n)$$

is obvious by the monotonicity of $\mathbb{E}\varphi(W_n)$. For the quenched moments, it suffices to repeat the proof above with $\mathbb{E}_\xi$ in the place of $\mathbb{E}$. □

Recall that we can estimate the harmonic moments of a positive random variable through its Laplace transform.

Lemma 2.2 (*[16, Lemma 4.4]*). *Let X be a positive random variable. For $0 < a < \infty$, consider the following statements:*

(i) $\mathbb{E}X^{-a} < \infty$; (ii) $\mathbb{E}e^{-tX} = O(t^{-a})(t \to \infty)$;
(iii) $\mathbb{P}(X \le x) = O(x^a)(x \to 0)$; (iv) $\forall b \in (0, a), \mathbb{E}X^{-b} < \infty$.

Then the following implications hold: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

Set

$$\phi_\xi(t) = \mathbb{E}_\xi e^{-tW} \quad \textit{and} \quad \phi(t) = \mathbb{E}\phi_\xi(t) = \mathbb{E}e^{-tW} \quad (t \ge 0).$$

Lemma 2.3. *Assume* (H). *Then there exist constants $\beta \in (0, 1)$ and $K \ge 1$ such that*

$$\phi_\xi(t) \le \beta \quad a.s. \ \forall t \ge \frac{1}{K}.$$

Proof. Let $p = 1 + \delta$. By a similar argument to the one used in the proof of [18, Proposition 1.3], we have $\forall k \ge 0$,

$$\mathbb{E}_\xi |W_{k+1} - W_k|^p \le \begin{cases} 2^p \Pi_k^{1-p} \overline{m}_k(p) & \text{if } 1 < p \le 2, \\ (B_p)^p \Pi_k^{-p/2} \mathbb{E}_\xi W_k^{p/2} \overline{m}_k(p) & \text{if } p > 2, \end{cases} \tag{2.1}$$

where $B_p = 2\sqrt{\lceil p/2 \rceil}$ with $\lceil p/2 \rceil = \min\{k \in \mathbb{N} : k \ge p/2\}$, and $\overline{m}_k(p) = \sum_{i=0}^{\infty} |\frac{i}{m_k} - 1|^p p_i(\xi_k)$.

The assumption (H) implies that $\|\overline{m}_0(p)\|_\infty = \|\mathbb{E}_\xi |\frac{Z_1}{m_0} - 1|^p\|_\infty < \infty$ and that $\Pi_k \ge A_1^k$ a.s. Using the inequality (2.1) and an induction argument on $[p]$ (see [18, Proposition 1.3]), we obtain

$$\mathbb{E}_\xi W^{1+\delta} = \sup_n \mathbb{E}_\xi W_n^p \le C \quad a.s.$$

for some constant C. In fact we shall only use the result for $\delta \le 1$. Assume that $\delta \in (0, 1]$, otherwise we consider $\min\{\delta, 1\}$ instead of δ. Notice that the function $\frac{e^{-x} - 1 + x}{x^{1+\delta}}$ is positive and bounded on $(0, \infty)$. So there exists a constant $C \ge 1$ such that

$$e^{-x} \le 1 - x + \frac{C}{1+\delta} x^{1+\delta} \quad \forall x > 0. \tag{2.2}$$

Take $K := \left(C \|\mathbb{E}_\xi W^{1+\delta}\|_\infty\right)^{1/\delta} \in [1, \infty)$. By (2.2), we obtain

$$\begin{aligned}\phi_\xi(t) = \mathbb{E}_\xi e^{-tW} &\le 1 - t + \frac{C}{1+\delta} t^{1+\delta} \mathbb{E}_\xi W^{1+\delta} \\ &\le 1 - t + \frac{K^\delta}{1+\delta} t^{1+\delta} \quad a.s.\end{aligned}$$

Let $g(t) = 1 - t + \frac{K^\delta}{1+\delta} t^{1+\delta}$. Obviously,

$$\min_{t>0} g(t) = g\left(\frac{1}{K}\right) = 1 - \frac{\delta}{K(1+\delta)} =: \beta \in (0, 1)$$

(it can be seen that $\beta \geq \frac{1}{2}$). Since $\phi_\xi(t)$ is decreasing, we have for $t \geq \frac{1}{K}$,

$$\phi_\xi(t) \leq \phi_\xi\left(\frac{1}{K}\right) \leq g\left(\frac{1}{K}\right) = \beta \quad a.s. \quad \square$$

Denote

$$\underline{m} = \operatorname{essinf} Z_1 = \inf\{j > 0 : \mathbb{P}(Z_1 = j) > 0\}. \tag{2.3}$$

Notice that $\mathbb{P}(Z_1 = j) = 0$ if and only if $\mathbb{P}(p_j(\xi_0) > 0) = 0$, so an alternative definition of $\underline{m}$ is

$$\underline{m} = \inf\{j > 0 : \mathbb{P}(p_j(\xi_0) > 0) > 0\}. \tag{2.4}$$

The following theorem gives an uniform bound for the quenched harmonic moments of W.

Theorem 2.1. *Assume* (H).

(i) *If* $\|p_1\|_\infty < 1$, *then for some constants* $a > 0$ *and* $C > 0$, *we have a.s.,*

$$\phi_\xi(t) \leq Ct^{-a} \quad (\forall t > 0), \qquad \mathbb{P}_\xi(W \leq x) \leq Cx^a \quad (\forall x > 0) \quad \text{and} \quad \mathbb{E}_\xi W^{-a} \leq C.$$

(ii) *If* $p_1 = 0$ *a.s., then a.s.*

$$\phi_\xi(t) \leq C_2 \exp(-C_1 t^\gamma) \quad (\forall t > 0), \qquad \mathbb{P}_\xi(W \leq x) \leq C_2 \exp(-C_1 x^{\frac{\gamma}{\gamma-1}}) \quad (\forall x > 0),$$

and $\mathbb{E}_\xi W^{-s} \leq C_s$ $(\forall s > 0)$, *where* $\gamma = \frac{\log \underline{m}}{\log A} \in (0, 1)$, C_1, C_2 *and* C_s *are positive constants independent of* ξ.

Proof. We only prove the results about $\phi_\xi(t)$, from which the results about $\mathbb{P}_\xi(W \leq x)$ and $\mathbb{E}_\xi W^{-s}$ can be deduced by Lemma 2.2 for (i), and by Tauberian theorems of exponential type (see [20]) for (ii).

(i) It is clear that $\phi_\xi(t)$ satisfies the functional equation

$$\phi_\xi(t) = f_0\left(\phi_{T\xi}\left(\frac{t}{m_0}\right)\right) \tag{2.5}$$

(recall that $T^n\xi = (\xi_n, \xi_{n+1}, \ldots)$ if $\xi = (\xi_0, \xi_1, \ldots)$ and $n \geq 0$). Hence a.s.,

$$\begin{aligned}
\phi_\xi(t) &\leq p_1(\xi_0)\phi_{T\xi}\left(\frac{t}{m_0}\right) + (1 - p_1(\xi_0))\phi^2_{T\xi}\left(\frac{t}{m_0}\right) \\
&\leq \phi_{T\xi}\left(\frac{t}{m_0}\right)\left(p_1(\xi_0) + (1 - p_1(\xi_0))\phi_{T\xi}\left(\frac{t}{m_0}\right)\right) \\
&\leq \phi_{T\xi}\left(\frac{t}{m_0}\right).
\end{aligned}$$

Similarly, we have a.s.,

$$\phi_{T\xi}\left(\frac{t}{m_0}\right) \leq \phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right)\left(p_1(\xi_1) + (1 - p_1(\xi_1))\phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right)\right) \leq \phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right).$$

Consequently, we get a.s.,

$$\begin{aligned}
\phi_\xi(t) \leq\ & \phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right)\left(p_1(\xi_1) + (1 - p_1(\xi_1))\phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right)\right) \\
& \times \left(p_1(\xi_0) + (1 - p_1(\xi_0))\phi_{T^2\xi}\left(\frac{t}{\Pi_2}\right)\right).
\end{aligned}$$

By iteration, we obtain that $\forall n \geq 1$, a.s.

$$\phi_\xi(t) \leq \phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right)\prod_{j=0}^{n-1}\left(p_1(\xi_j) + (1 - p_1(\xi_j))\phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right)\right). \tag{2.6}$$

By Lemma 2.3, a.s., $\phi_{T^n\xi}(\frac{t}{\Pi_n}) \leq \beta$ if $t \geq \frac{A^n}{K}$ and $n \geq 0$, since $\Pi_n \leq A^n$. Let $\bar{p}_1 := \|p_1\|_\infty$. As $p_1(\xi_0) \leq \bar{p}_1$ a.s., it follows that a.s.,

$$\phi_\xi(t) \leq \beta\alpha^n \quad \text{for } t \geq \frac{A^n}{K} \text{ and } n \geq 0,$$

where $\alpha = \bar{p}_1 + (1 - \bar{p}_1)\beta \in (0, 1)$. For $t \geq \frac{1}{K}$, take $n_0 = n_0(t) = [\frac{\log(Kt)}{\log A}] \geq 0$. Clearly, $t \geq \frac{A^{n_0}}{K}$ and $\frac{\log(Kt)}{\log A} - 1 \leq n_0 \leq \frac{\log(Kt)}{\log A}$. Thus for $t \geq \frac{1}{K}$, a.s.

$$\phi_\xi(t) \leq \beta\alpha^{n_0} \leq \beta\alpha^{-1}(Kt)^{\frac{\log\alpha}{\log A}} = C_0 t^{-a},$$

where $C_0 = \beta\alpha^{-1}K^{\frac{\log\alpha}{\log A}} > 0$ and $a = -\frac{\log\alpha}{\log A} > 0$; therefore, we can choose a constant $C > 0$ such that a.s., $\phi_\xi(t) \leq Ct^{-a}$ ($\forall t > 0$). Thus the first part of the theorem is proved.

(ii) By Eq. (2.5),

$$\phi_\xi(t) = f_0\left(\phi_{T\xi}\left(\frac{t}{m_0}\right)\right) \leq \left(\phi_{T\xi}\left(\frac{t}{m_0}\right)\right)^{\underline{m}} \quad a.s.$$

By iteration, using Lemma 2.3 we have

$$\phi_\xi(t) \leq \left(\phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right)\right)^{\underline{m}^n} \leq \beta^{\underline{m}^n} \quad a.s. \text{ for } t \geq \frac{A^n}{K}.$$

Like the proof of the first part, take $n_0 = n_0(t) = [\frac{\log(Kt)}{\log A}] \geq 0$. Then for $t \geq \frac{1}{K}$,

$$\phi_\xi(t) \leq \beta^{\underline{m}^{n_0}} \leq \exp\left(\underline{m}^{-1}(\log\beta)(Kt)^{\frac{\log\underline{m}}{\log A}}\right) \leq \exp(-C_1 t^\gamma) \quad a.s.,$$

where $C_1 = -\underline{m}^{-1}K^{\frac{\log\underline{m}}{\log A}}\log\beta > 0$ and $\gamma = \frac{\log\underline{m}}{\log A} \in (0, 1)$. It follows that we can choose $C_2 > 0$ such that a.s., $\phi_\xi(t) \leq C_2\exp(-C_1 t^\gamma)$, $\forall t > 0$. This completes the proof. □

We now study the annealed moments of W.

Theorem 2.2. *Assume* (H).

(i) *Then there exist constants $a > 0$ and $C > 0$ such that*

$$\phi(t) \leq Ct^{-a} \quad (\forall t > 0), \qquad \mathbb{P}(W \leq x) \leq Cx^a \quad (\forall x > 0) \quad \textit{and}$$
$$\mathbb{E}W^{-s} < \infty \ (\forall s \in (0, a)). \tag{2.7}$$

If additionally $\|p_1\|_\infty < 1$, then for each $a > 0$ with $\mathbb{E}p_1 m_0^a < 1$, (2.7) holds for some constant $C > 0$.

(ii) *If $p_1 = 0$ a.s., then*

$$\phi(t) \leq C_2\exp(-C_1 t^\gamma) \quad (\forall t > 0), \qquad \mathbb{P}(W \leq x) \leq C_2\exp(-C_1 x^{\frac{\gamma}{\gamma-1}}) \quad (\forall x > 0),$$

and $\mathbb{E}W^{-s} < \infty$ ($\forall s > 0$), where $\gamma = \frac{\log\underline{m}}{\log A} \in (0, 1)$, and C_1, C_2 are positive constants.

Notice that when $\|p_1\|_\infty < 1$, the conclusion that (2.7) holds for some $a > 0$ is also a direct consequence of Theorem 2.1(i). But Theorem 2.2(i) gives more precise information.

To prove Theorem 2.2, we need the following lemma.

Lemma 2.4 ([17], Lemma 3.2)**.** *Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function and let A be a positive random variable such that for some $0 < p < 1$, $t_0 \geq 0$ and all $t > t_0$,*

$$\phi(t) \leq p\mathbb{E}\phi(At).$$

If $pEA^{-a} < 1$ for some $0 < a < \infty$, then $\phi(t) = O(t^{-a})$ $(t \to \infty)$.

Proof of Theorem 2.2. Part (ii) is from Theorem 2.1(ii) by taking expectation $\mathbb{E}$. For part (i), we first consider the special case where $p_1 \leq \bar{p}_1$ a.s. for some constant $\bar{p}_1 < 1$. By Theorem 2.1(i), we have $\phi_\xi(t) \leq C_1 t^{-a_1}$ $a.s.$ $(\forall t > 0)$ for some positive constants C_1 and a_1. So for all $0 < \epsilon < 1$, there exists a constant $t_\epsilon > 0$ such that $\phi_\xi(t) \leq \epsilon$ a.s. for $t \geq t_\epsilon$. Thus by (2.5),

$$\phi_\xi(t) \leq (p_1 + (1 - p_1)\epsilon)\phi_{T\xi}\left(\frac{t}{m_0}\right) \quad a.s. \text{ if } t \geq At_\epsilon. \tag{2.8}$$

Notice that ξ_0 is independent of $T\xi$. Taking expectation in (2.8), we see that for $t \geq At_\epsilon$,

$$\begin{aligned}\phi(t) &\leq \mathbb{E}\left[(p_1 + (1 - p_1)\epsilon)\phi_{T\xi}\left(\frac{t}{m_0}\right)\right] \\ &= \mathbb{E}\left[(p_1 + (1 - p_1)\epsilon)\mathbb{E}\left[\phi_{T\xi}\left(\frac{t}{m_0}\right)\middle|\, \xi_0\right]\right] \\ &= \mathbb{E}\left[(p_1 + (1 - p_1)\epsilon)\phi\left(\frac{t}{m_0}\right)\right] = p_\epsilon\mathbb{E}\phi(\tilde{A}_\epsilon t),\end{aligned}$$

where $p_\epsilon = \mathbb{E}(p_1 + (1 - p_1)\epsilon) < 1$ and $\tilde{A}_\epsilon$ is a positive random variable whose distribution is determined by

$$\mathbb{E}g(\tilde{A}_\epsilon) = \frac{1}{p_\epsilon}\mathbb{E}\left[(p_1 + (1 - p_1)\epsilon)g\left(\frac{1}{m_0}\right)\right]$$

for all bounded and measurable function g. If $p_\epsilon\mathbb{E}\tilde{A}_\epsilon^{-a} < 1$, by Lemma 2.4, we have $\phi(t) = O(t^{-a})$ $(t \to \infty)$, or equivalently, $\phi(t) \leq Ct^{-a}$ $(\forall t > 0)$ for some constant $C > 0$. Since $\mathbb{E}p_1 m_0^a < 1$, we can take $\epsilon > 0$ small enough such that

$$p_\epsilon\mathbb{E}\tilde{A}_\epsilon^{-a} = \mathbb{E}\left[(p_1 + (1 - p_1)\epsilon)m_0^a\right] < 1.$$

Therefore, we have proved that $\phi(t) = O(t^{-a})$ whenever $\|p_1\|_\infty < 1$ and $\mathbb{E}p_1 m_0^a < 1$ $(a > 0)$. Now consider the general case where $\|p_1\|_\infty$ may be 1. By Lemma 2.3, we have $\phi_\xi(t) \leq \beta$ a.s. for $t \geq t_\beta = \frac{1}{K}$. So we can repeat the proof above with β in place of ϵ, showing that if $a > 0$ small enough such that

$$\mathbb{E}[(p_1 + (1 - p_1)\beta)m_0^a] \leq A^a(\mathbb{E}p_1 + (1 - \mathbb{E}p_1)\beta) < 1,$$

then $\phi(t) = O(t^{-a})$. Now we have proved the results about $\phi(t)$. By Lemma 2.2, we obtain the results about $\mathbb{P}(W \leq x)$ and $\mathbb{E}W^{-s}$. □

We now prove our main result on the harmonic moments of W already stated in the introduction at the beginning of this paper.

Proof of Theorem 1.4. If $\mathbb{E}p_1 m_0^a < 1$, then there exists $\epsilon > 0$ such that $\mathbb{E}p_1 m_0^{a+\epsilon} < 1$. So by Theorem 2.2(i), $\mathbb{E}W^{-a} < \infty$. Conversely, assume that $a > 0$ and $\mathbb{E}W^{-a} < \infty$. Notice that

$$W = \frac{1}{m_0}\sum_{i=1}^{Z_1} W_i^{(1)} \quad a.s.,$$

where $\left(W_i^{(1)}\right)_{i\geq 1}$, when ξ is given, are conditionally independent copies of $W^{(1)}$ whose distribution is $\mathbb{P}_\xi(W^{(1)} \in \cdot) = \mathbb{P}_{T\xi}(W \in \cdot)$. Since $\mathbb{P}(Z_1 \geq 2) > 0$, we have

$$\mathbb{E}W^{-a} > \mathbb{E}m_0^a \left(W_1^{(1)}\right)^{-a} \mathbf{1}_{\{Z_1=1\}} = \mathbb{E}p_1 m_0^a \mathbb{E}W^{-a}.$$

Therefore, $\mathbb{E}p_1 m_0^a < 1$. □

3. Moments of Z_n and large deviations for $\log Z_n$

We first recall some preliminary results for the existence of moments of W.

Guivarc'h and Liu [11] gave a sufficient and necessary condition for the existence of moments of positive orders of W: for $s > 1$,

$$0 < \mathbb{E}W^s < \infty \quad \text{if and only if} \quad \mathbb{E}\left(\frac{Z_1}{m_0}\right)^s < \infty \quad \text{and} \quad \mathbb{E}m_0^{1-s} < 1. \tag{3.1}$$

In particular, if $p_0 = 0$ a.s. and $\mathbb{E}Z_1^s < \infty$ for all $s > 1$, then $0 < \mathbb{E}W^s < \infty$ for all $s > 0$.

For the existence of moments of negative orders of W, Theorem 1.4 shows that, assuming (H) and $\|p_1\|_\infty < 1$, we have for $s > 0$,

$$\mathbb{E}W^{-s} < \infty \quad \text{if and only if} \quad \mathbb{E}p_1 m_0^s < 1. \tag{3.2}$$

In particular, if $p_0 = p_1 = 0$ a.s., it is clear that $\mathbb{E}W^{-s} < \infty$, for all $s > 0$.

These results will be applied in the proof of Theorem 1.3.

Proof of Theorem 1.3. Denote the distribution of ξ_0 by τ_0. Fix $t \in \mathbb{R}$ and define a new distribution $\tilde{\tau}_0$ as

$$\tilde{\tau}_0(dx) = \frac{m(x)^t \tau_0(dx)}{\mathbb{E}m_0^t},$$

where $m(x) = \mathbb{E}[Z_1|\xi_0 = x] = \sum_{i=0}^{\infty} i p_i(x)$. Consider the new branching process in a random environment whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes\mathbb{N}}$ instead of $\tau = \tau_0^{\otimes\mathbb{N}}$. The corresponding probability and expectation are denoted by $\tilde{\mathbb{P}} = \mathbb{P}_\xi \otimes \tilde{\tau}$ and $\tilde{\mathbb{E}}$, respectively. Then

$$\frac{\mathbb{E}Z_n^t}{\left(\mathbb{E}m_0^t\right)^n} = \tilde{\mathbb{E}}W_n^t.$$

It is easy to see that under $\tilde{\mathbb{P}}$, we still have $p_0 = 0$ a.s. Moreover, if (H) holds and $\|p_1\|_\infty < 1$, then the same hold under $\tilde{\mathbb{P}}$. Notice that

$$\tilde{\mathbb{E}}\log m_0 = \frac{\mathbb{E}m_0^t \log m_0}{\mathbb{E}m_0^t} \in (0, \infty].$$

We distinguish three cases as considered in the theorem.

(i) If $t \in (0, 1]$ and $\mathbb{E} m_0^{t-1} Z_1 \log^+ Z_1 < \infty$, then

$$\tilde{\mathbb{E}} \frac{Z_1}{m_0} \log^+ Z_1 = \frac{\mathbb{E} m_0^{t-1} Z_1 \log^+ Z_1}{\mathbb{E} m_0^t} < \infty,$$

so that $W_n \to W$ in L^1 under $\tilde{\mathbb{P}}$ (cf. [4,24]). Therefore,

$$\lim_{n\to\infty} \tilde{\mathbb{E}} W_n^t = \tilde{\mathbb{E}} W^t \in (0, \infty). \tag{3.3}$$

(ii) If $t > 1$ and $\mathbb{E} Z_1^t < \infty$, then

$$\tilde{\mathbb{E}} \left(\frac{Z_1}{m_0} \right)^t = \frac{\mathbb{E} Z_1^t}{\mathbb{E} m_0^t} < \infty \quad a.s. \text{ under } \tilde{\mathbb{P}},$$

so that $W_n \to W$ in L^t under $\tilde{\mathbb{P}}$ (cf. (3.1)).
(iii) If $t < 0$, $\mathbb{E} p_1 < \mathbb{E} m_0^t$, $\|p_1\|_\infty < 1$ and (H) holds, then

$$\tilde{\mathbb{E}} p_1 m_0^{-t} = \frac{\mathbb{E} p_1}{\mathbb{E} m_0^t} < 1,$$

so that $\tilde{\mathbb{E}} W^t < \infty$ from Theorem 1.4. Using Lemma 2.1, we obtain again (3.3).
Therefore, we have proved Theorem 1.3 with $C(t) = \tilde{\mathbb{E}} W^t$. □

Using Theorem 1.3, we can easily prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that the hypothesis of Theorem 1.1 ensures that $\mathbb{E} Z_1^t < \infty$ for all $t \in \mathbb{R}$. Hence by Theorem 1.3,

$$\lim_{n\to\infty} \frac{\mathbb{E} Z_n^t}{\left(\mathbb{E} m_0^t\right)^n} = C(t) \in (0, \infty) \quad \forall t \in \mathbb{R},$$

which implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} Z_n^t = \log \mathbb{E} m_0^t = \Lambda(t) \quad \forall t \in \mathbb{R}. \tag{3.4}$$

Notice that the Laplace transform of $\log Z_n$ is $\mathbb{E} e^{t \log Z_n} = \mathbb{E} Z_n^t$. As $\Lambda(t)$ is finite and derivable everywhere, from (3.4) and the Gärtner–Ellis theorem ([10], p. 52, Exercise 2.3.20), we immediately obtain Theorem 1.1. □

Theorem 1.3 can also be used to study the large deviation probabilities $\mathbb{P}\left(\frac{\log Z_n}{n} \geq x\right)$ (resp. $\mathbb{P}(\frac{\log Z_n}{n} \leq x)$) for a finite interval of x, when $\mathbb{E} W^a$ (resp. $\mathbb{E} W^{-a}$) ($a > 0$) exists only in a finite interval of a. To this end, we shall use the following version of the Gärtner–Ellis theorem adapted to the study of tail probabilities.

Lemma 3.1 (*[19, Theorem 6.1]*). *Let (μ_n) be a family of probability distribution on $\mathbb{R}$ and let (a_n) be a sequence of positive numbers satisfying $a_n \to \infty$. Assume that for some $t_0 \in [0, \infty]$ and for every $t \in [0, t_0)$, as $n \to \infty$,*

$$l_n(t) := \frac{1}{a_n} \log \int e^{a_n t x} \mu_n(dx) \to l(t) < \infty.$$

For $x \in \mathbb{R}$, *set*

$$l^*(x) = \sup\{tx - l(t); t \in [0, t_0)\}.$$

If l *is continuously differentiable on* $(0, t_0)$, *then for all* $x \in (l'(0+), l'(t_0-))$ *(where* $l'(x\pm) = \lim_{y\to x\pm} l'(y)$*),*

$$\lim_{n\to\infty} \frac{1}{a_n} \log \mu_n([x, \infty)) = -l^*(x).$$

From Theorem 1.3 and Lemma 3.1, we immediately obtain the following theorem.

Theorem 3.1. *Let* $a \in \mathbb{R}$.

(i) *Let* $a > 0$. *If* $a \in (0, 1]$ *and* $\mathbb{E}m_0^{a-1} Z_1 \log^+ Z_1 < \infty$, *or* $a > 1$ *and* $\mathbb{E}Z_1^a < \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \ge x\right) = -\Lambda^*(x), \quad \forall x \in (\mathbb{E}\log m_0, \Lambda'(a)). \tag{3.5}$$

(ii) *Let* $a < 0$. *Assume* (H) *and* $\|p_1\|_\infty < 1$. *If* $\mathbb{E}p_1 < \mathbb{E}m_0^a$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \le x\right) = -\Lambda^*(x), \quad \forall x \in (\Lambda'(a), \mathbb{E}\log m_0). \tag{3.6}$$

If $\mathbb{E}Z_1^a < \infty$ for all $a > 1$ (resp. $p_1 = 0$ $a.s.$), then Theorem 3.1 suggests that the limit in (3.5) (resp. (3.6)) would hold for any $x > \mathbb{E}\log m_0$ (resp. $x < \mathbb{E}\log m_0$). This leads to the following theorem which is more precise than Corollary 1.2. It was proved by Bansaye and Berestycki [5] when (H) holds with $\delta = 1$.

Theorem 3.2. (i) *If* $\mathbb{E}Z_1^s < \infty$ *for all* $s > 1$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \ge x\right) = -\Lambda^*(x) \quad \textit{for } x > \mathbb{E}\log m_0.$$

(ii) *Assume* (H) *and* $p_1 = 0$ *a.s., then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \le x\right) = -\Lambda^*(x) \quad \textit{for } x < \mathbb{E}\log m_0,$$

If $\Lambda'(\infty) = \infty$ and $\Lambda'(-\infty) = 0$, then Theorem 3.2 can be directly deduced from Theorem 3.1. But it is possible that $\Lambda'(\infty) < \infty$ or $\Lambda'(-\infty) > 0$. So we will give a direct proof of Theorem 3.2, following [5].

According to the large deviation principle for i.i.d. random variables, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log \Pi_n}{n} \le x\right) = -\Lambda^*(x) \quad \text{for } x \le \mathbb{E}\log m_0, \tag{3.7}$$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log \Pi_n}{n} \ge x\right) = -\Lambda^*(x) \quad \text{for } x \ge \mathbb{E}\log m_0. \tag{3.8}$$

Lemma 3.2 below gives the lower bound for both the lower and upper deviations.

Lemma 3.2 (*[5, Proposition 1]*). *Assume (1.5). Then*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\le x\right)\ge -\Lambda^*(x)\quad \text{for } x\le \mathbb{E}\log m_0, \tag{3.9}$$

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\ge x\right)\ge -\Lambda^*(x)\quad \text{for } x\ge \mathbb{E}\log m_0. \tag{3.10}$$

We remark that in Lemma 3.2, the original moment condition in [5, Proposition 1], namely, $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^s<\infty$ for some $s>1$, is weaken to $\mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1<\infty$.

The following lemma gives the upper bound for both the lower and upper deviations.

Lemma 3.3. (i) *If* $\mathbb{E}W^{-s}<\infty$ *for all* $s>1$, *then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\le x\right)\le -\Lambda^*(x)\quad \text{for } x< \mathbb{E}\log m_0. \tag{3.11}$$

(ii) *If* $\mathbb{E}W^{s}<\infty$ *for all* $s>0$, *then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\ge x\right)\le -\Lambda^*(x)\quad \text{for } x> \mathbb{E}\log m_0. \tag{3.12}$$

The inequality (3.12) was proved by Bansaye and Berestycki [5]. For readers' convenience, we shall prove simultaneously (3.11) and (3.12).

Proof of Lemma 3.3. By the decomposition (1.6), for $x\in\mathbb{R}$, $\epsilon>0$ and $s>0$, we have

$$\mathbb{P}\left(\frac{\log Z_n}{n}\le x\right)\le \mathbb{P}\left(\frac{\log \Pi_n}{n}\le x+\epsilon\right)+\mathbb{P}\left(\frac{\log W_n}{n}\le -\epsilon\right).$$

By Markov's inequality and Lemma 2.1,

$$\mathbb{P}\left(\frac{\log W_n}{n}\le -\epsilon\right)\le \frac{\mathbb{E}W_n^{-s}}{e^{s\epsilon n}}\le \frac{\mathbb{E}W^{-s}}{e^{s\epsilon n}}.$$

Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\le x\right)&\le \max\left\{\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log \Pi_n}{n}\le x+\epsilon\right),-s\epsilon\right\}\\ &=\max\{-\Lambda^*(x+\epsilon),-s\epsilon\}.\end{aligned}$$

Letting $s\to\infty$ and $\epsilon\to 0$, we obtain (3.11). For (3.12), we use a similar argument. For $\epsilon>0$ and $s>1$,

$$\begin{aligned}\mathbb{P}\left(\frac{\log Z_n}{n}\ge x\right)&\le \mathbb{P}\left(\frac{\log \Pi_n}{n}\ge x-\epsilon\right)+\mathbb{P}\left(\frac{\log W_n}{n}\ge \epsilon\right)\\ &\le \mathbb{P}\left(\frac{\log \Pi_n}{n}\ge x-\epsilon\right)+\frac{\mathbb{E}W^s}{e^{s\epsilon n}}.\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log Z_n}{n}\ge x\right)&\le \max\left\{\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{\log \Pi_n}{n}\ge x-\epsilon\right),-s\epsilon\right\}\\ &=\max\{-\Lambda^*(x-\epsilon),-s\epsilon\}.\end{aligned}$$

Again letting $s\to\infty$ and $\epsilon\to 0$, we obtain (3.12). □

Proof of Theorem 3.2. It is just a combination of Lemmas 3.2 and 3.3. □

Notice that Theorem 3.2 implies Corollary 1.2. By Lemma 4.4, we see that Corollary 1.2 is in fact equivalent to Theorem 1.1. So the direct proof of Theorem 3.2 leads to an alternative proof of Theorem 1.1.

4. Moderate deviations for $\log Z_n$

Now we turn to the proof of moderate deviation principle (Theorem 1.6). Similar to the proof of large deviation principle (Theorem 1.1), we can study the convergence rate of $\frac{\log Z_n}{n}$ by considering those of $\frac{\log \Pi_n}{n}$. Recall that (a_n) is a sequence of positive numbers satisfying (1.8). Let

$$S_n := \log \Pi_n - n\mathbb{E}\log m_0 \quad \text{and} \quad \bar{\Lambda}_n(t) = \log \mathbb{E}\exp\left(\frac{tS_n}{a_n}\right).$$

By the classic moderate deviation results for i.i.d. random variables (see [10], Theorem 3.7.1 and its proof), it is known that, if $f(t) = \mathbb{E}m_0^t < \infty$ in a neighborhood of the origin, then

$$\lim_{n\to\infty} \frac{n}{a_n^2}\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right) = \frac{1}{2}\sigma^2 t^2, \tag{4.1}$$

and for any measurable subset B of $\mathbb{R}$,

$$-\inf_{x\in B^o}\frac{x^2}{2\sigma^2} \le \liminf_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n}\in B\right)$$
$$\le \limsup_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n}\in B\right) \le -\inf_{x\in \bar{B}}\frac{x^2}{2\sigma^2}. \tag{4.2}$$

Lemma 4.1. *Let $t \in \mathbb{R}$.*

(i) *If* (H) *holds and $\|p_1\|_\infty < 1$, then for all $t < 0$,*

$$\lim_{n\to\infty}\frac{\mathbb{E}Z_n^{\frac{a_n}{n}t}}{\mathbb{E}\Pi_n^{\frac{a_n}{n}t}} = 1. \tag{4.3}$$

(ii) *If* (H) *holds, then there is a constant $c > 0$ such that for all $t > 0$,*

$$c \le \liminf_{n\to\infty}\frac{\mathbb{E}Z_n^{\frac{a_n}{n}t}}{\mathbb{E}\Pi_n^{\frac{a_n}{n}t}} \le \limsup_{n\to\infty}\frac{\mathbb{E}Z_n^{\frac{a_n}{n}t}}{\mathbb{E}\Pi_n^{\frac{a_n}{n}t}} \le 1. \tag{4.4}$$

Proof. (i) Let $t_n = \frac{a_n}{n}t$. For $t < 0$, we have $t_n < 0$. By Jensen's inequality,

$$\mathbb{E}_\xi W_n^{t_n} \ge (\mathbb{E}_\xi W_n)^{t_n} = 1 \quad a.s.$$

Thus

$$\mathbb{E}Z_n^{t_n} = \mathbb{E}\Pi_n^{t_n}\mathbb{E}_\xi W_n^{t_n} \ge \mathbb{E}\Pi_n^{t_n}, \tag{4.5}$$

which leads to

$$\liminf_{n\to\infty} \frac{\mathbb{E}Z_n^{t_n}}{\mathbb{E}\Pi_n^{t_n}} \geq 1.$$

On the other hand, if (H) holds and $\|p_1\|_\infty < 1$, then by Theorem 2.1, we have $\mathbb{E}_\xi W^{-s} \leq C_s$ a.s. for some constants $s > 0$ and $C_s > 0$. Noticing that $-t_n/s \in (0,1)$ for n large enough and that by Lemma 2.1, $\mathbb{E}_\xi W_n^{-s} \leq \mathbb{E}_\xi W^{-s}$ a.s., again by Jensen's inequality, we have

$$\mathbb{E}_\xi W_n^{t_n} = \mathbb{E}_\xi (W_n^{-s})^{-t_n/s} \leq (\mathbb{E}_\xi W_n^{-s})^{-t_n/s} \leq (\mathbb{E}_\xi W^{-s})^{-t_n/s} \leq C_s^{-t_n/s},$$

so that

$$\mathbb{E}Z_n^{t_n} \leq C_s^{-t_n/s} \mathbb{E}\Pi_n^{t_n}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\mathbb{E}Z_n^{t_n}}{\mathbb{E}\Pi_n^{t_n}} \leq 1.$$

(ii) For $t > 0$, we have $t_n = \frac{a_n}{n} t \in (0,1)$ for n large enough, so by Jensen's inequality,

$$\mathbb{E}_\xi W_n^{t_n} \leq (\mathbb{E}_\xi W_n)^{t_n} = 1 \quad a.s.$$

Thus

$$\limsup_{n\to\infty} \frac{\mathbb{E}Z_n^{t_n}}{\mathbb{E}\Pi_n^{t_n}} \leq 1.$$

On the other hand, from the proof Lemma 2.3, we know that the assumption (H) ensures that $\mathbb{E}_\xi W^s \leq C_s$ a.s. for $1 < s \leq 1+\delta$ and some constant $C_s > 0$. By Hölder's inequality,

$$\begin{aligned} 1 = \mathbb{E}_\xi W_n &\leq \mathbb{E}_\xi W_n^{t_n/p} W_n^{1-t_n/p} \\ &\leq \left(\mathbb{E}_\xi W_n^{t_n}\right)^{1/p} \left(\mathbb{E}_\xi W_n^{(1-t_n/p)q}\right)^{1/q} \quad a.s., \end{aligned} \tag{4.6}$$

for $p, q > 1$, $1/p + 1/q = 1$. Take $p = p(n) = \frac{s-t_n}{s-1}$ and $q = q(n) = \frac{s-t_n}{1-t_n}$, so that $(1-t_n/p)q = s$ and $p/q = \frac{1-t_n}{s-1}$. Notice that by Lemma 2.1, $\mathbb{E}_\xi W_n^{-s} \leq \mathbb{E}_\xi W^{-s}$ a.s. We deduce from (4.6) that

$$\mathbb{E}_\xi W_n^{t_n} \geq \left(\mathbb{E}_\xi W_n^s\right)^{-\frac{1-t_n}{s-1}} \geq \left(\mathbb{E}_\xi W^s\right)^{-\frac{1-t_n}{s-1}} \geq C_s^{-\frac{1-t_n}{s-1}}.$$

Thus

$$\mathbb{E}Z_n^{t_n} \geq C_s^{-\frac{1-t_n}{s-1}} \mathbb{E}\Pi_n^{t_n}.$$

Letting $n \to \infty$, we obtain

$$\liminf_{n\to\infty} \frac{\mathbb{E}Z_n^{t_n}}{\mathbb{E}\Pi_n^{t_n}} \geq c,$$

where $c = C_s^{-\frac{1}{s-1}} \in (0,1]$. This completes the proof. □

Theorem 4.1. *Let* $\Lambda_n(t) = \log \mathbb{E} \exp\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} t\right)$ *and* $\bar{\Lambda}_n(t) = \log \mathbb{E}\exp\left(\frac{tS_n}{a_n}\right)$. *If (H) holds, then*

$$\lim_{n\to\infty} \frac{\Lambda_n\left(\frac{a_n^2}{n}t\right)}{\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right)} = 1, \quad \forall t \neq 0 \tag{4.7}$$

and

$$\lim_{n\to\infty} \frac{\log \mathbb{E} Z_n^{\frac{a_n}{n}t}}{\log \mathbb{E} \Pi_n^{\frac{a_n}{n}t}} = 1, \quad \forall t \neq 0. \tag{4.8}$$

Proof. We only need to prove (4.7), which implies (4.8). For $t > 0$, (4.7) is a direct consequence of Lemma 4.1(ii). For $t < 0$, if additionally $\|p_1\|_\infty < 1$, then (4.7) is also a direct consequence of Lemma 4.1(i); we shall prove that the condition $\|p_1\|_\infty < 1$ is not needed for (4.7) to hold. Assume (H) and let $t < 0$. Notice that (4.5) implies that

$$\liminf_{n\to\infty} \frac{\Lambda_n\left(\frac{a_n^2}{n}t\right)}{\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right)} \geq 1.$$

It remains to show that

$$\limsup_{n\to\infty} \frac{\Lambda_n\left(\frac{a_n^2}{n}t\right)}{\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right)} \leq 1. \tag{4.9}$$

By Hölder's inequality,

$$\begin{aligned}
\exp\left(\Lambda_n\left(\frac{a_n^2}{n}t\right)\right) &= \mathbb{E}\exp\left(\frac{a_n}{n}t(\log Z_n - n\mathbb{E}\log m_0)\right) \\
&= \mathbb{E} e^{\frac{a_n}{n}tS_n} W_n^{\frac{a_n}{n}t} \\
&\leq \left(\mathbb{E} e^{\frac{a_n}{n}ptS_n}\right)^{1/p} \left(\mathbb{E} W_n^{\frac{a_n}{n}tq}\right)^{1/q} \\
&\leq \exp\left(\frac{1}{p}\bar{\Lambda}_n\left(\frac{a_n^2}{n}pt\right)\right)\left(\mathbb{E} W_n^{\frac{a_n}{n}tq}\right)^{1/q},
\end{aligned}$$

where $p, q > 1$ are constants satisfying $1/p + 1/q = 1$. By Theorem 2.2, there exists $s > 0$ such that $\mathbb{E}W^{-s} < \infty$. Noticing that $t_n q > -s$ for n large, we have

$$\mathbb{E} W_n^{t_n q} \leq 1 + \mathbb{E} W_n^{-s} \leq 1 + \mathbb{E} W^{-s}.$$

Hence for n large enough,

$$\Lambda_n\left(\frac{a_n^2}{n}t\right) \leq \frac{1}{p}\bar{\Lambda}_n\left(\frac{a_n^2}{n}pt\right) + \frac{1}{q}\log(1 + \mathbb{E}W^{-s}).$$

Therefore, considering (4.1), we have

$$\limsup_{n\to\infty} \frac{\Lambda_n\left(\frac{a_n^2}{n}t\right)}{\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right)} \le \frac{1}{p}\frac{\frac{1}{2}\sigma^2 p^2 t^2}{\frac{1}{2}\sigma^2 t^2} = p.$$

Letting $p \to 1$, (4.9) is proved. □

Proof of Theorem 1.6. From (4.7) and (4.1), we have

$$\lim_{n\to\infty} \frac{n}{a_n^2}\Lambda_n\left(\frac{a_n^2}{n}t\right) = \lim_{n\to\infty} \frac{n}{a_n^2}\bar{\Lambda}_n\left(\frac{a_n^2}{n}t\right) = \frac{1}{2}\sigma^2 t^2.$$

Applying the Gärtner–Ellis theorem ([10], p. 52, Exercise 2.3.20), we obtain Theorem 1.6. □

The following theorem about the tail probabilities is a direct consequence of Theorem 1.6.

Theorem 4.2. *Assume* (H) *and* $\sigma^2 = \operatorname{var}(\log m_0) \in (0, \infty)$. *Then for all* $x > 0$,

$$\lim_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) = -\frac{x^2}{2\sigma^2}, \tag{4.10}$$

$$\lim_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) = -\frac{x^2}{2\sigma^2}. \tag{4.11}$$

It is also possible to give a direct proof of Theorem 4.2. We shall give such a proof in the following, as it will give additional one-side results on the tail probabilities under weaker assumptions.

Lemma 4.2. *If* $f(t) = \mathbb{E}m_0^t < \infty$ *in a neighborhood of the origin, then for all* $x > 0$,

$$\liminf_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) \ge -\frac{x^2}{2\sigma^2}, \tag{4.12}$$

$$\limsup_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) \le -\frac{x^2}{2\sigma^2}. \tag{4.13}$$

Proof. Let $x > 0$. By (4.2), the moderate deviation principle for $\log \Pi_n$, we have

$$\lim_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) = -\frac{x^2}{2\sigma^2} \tag{4.14}$$

and

$$\lim_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) = -\frac{x^2}{2\sigma^2}. \tag{4.15}$$

For every $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) &\ge \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \le -x-\epsilon\right) - \mathbb{P}(W_n \ge e^{a_n\epsilon}) \\ &=: u_n - v_n = u_n(1 - v_n/u_n).\end{aligned}$$

By (4.14), we have $\forall \delta' > 0$, for n large enough,

$$u_n \geq \exp\left(-\frac{a_n^2}{n}\left(\frac{(x+\epsilon)^2}{2\sigma^2}+\delta'\right)\right).$$

Furthermore, by Markov's inequality,

$$v_n = \mathbb{P}(W_n \geq e^{a_n\epsilon}) \leq e^{-a_n\epsilon}.$$

Hence,

$$0 \leq \frac{v_n}{u_n} \leq \exp\left(-a_n\epsilon + \frac{a_n^2}{n}\left(\frac{(x+\epsilon)^2}{2\sigma^2}+\delta'\right)\right) \to 0 \quad \text{as } n \to \infty,$$

since

$$\lim_{n\to\infty} \frac{-a_n\epsilon + \frac{a_n^2}{n}\left(\frac{(x+\epsilon)^2}{2\sigma^2}+\delta'\right)}{a_n} = -\epsilon < 0.$$

Therefore,

$$\liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \leq -x\right) \geq \liminf_{n\to\infty} \frac{n}{a_n^2} \log u_n = -\frac{(x+\epsilon)^2}{2\sigma^2}.$$

Letting $\epsilon \to 0$, we obtain (4.12). For (4.13), the proof is similar. For every $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \geq x\right) \\
&\quad \leq \mathbb{P}(W_n \geq e^{a_n\epsilon}) + \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \geq x - \epsilon\right) \\
&\quad =: v_n + \tilde{u}_n = \tilde{u}_n(1 + v_n/\tilde{u}_n).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{v_n}{\tilde{u}_n} = 0$, we have

$$\limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \geq x\right) \leq \limsup_{n\to\infty} \frac{n}{a_n^2} \log \tilde{u}_n = -\frac{(x-\epsilon)^2}{2\sigma^2}.$$

Letting $\epsilon \to 0$, we get (4.13). □

To prove Theorem 4.2, we need to estimate the decay rate of the probabilities $\mathbb{P}(W_n \leq e^{-a_n\epsilon})$ for $\epsilon > 0$.

Lemma 4.3. *If $\mathbb{E}W^{-s} < \infty$ for some $s > 0$, then for any positive sequence (a_n) satisfying $a_n \to \infty$, we have for all $\epsilon > 0$,*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(W_n \leq e^{-a_n\epsilon}) \leq -s\epsilon. \tag{4.16}$$

Proof. By Markov's inequality and Lemma 2.1,

$$\mathbb{P}(W_n \leq e^{-a_n\epsilon}) \leq \frac{\mathbb{E}W_n^{-s}}{e^{sa_n\epsilon}} \leq \frac{\mathbb{E}W^{-s}}{e^{sa_n\epsilon}}.$$

Thus

$$\frac{1}{a_n} \log \mathbb{P}(W_n \leq e^{-a_n\epsilon}) \leq \frac{1}{a_n} \log \mathbb{E}W^{-s} - s\epsilon.$$

Taking the limit superior in the above inequality gives (4.16). □

Another Proof of Theorem 4.2. Lemma 4.2 gives one side of the desired results, so we only need to prove the other side. By Theorem 2.2, there exists $s > 0$ such that $\mathbb{E}W^{-s} < \infty$, so (4.16) holds for this s. For $x > 0$, we have for every $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}&\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right)\\ &\le \mathbb{P}\left(W_n \le e^{-a_n\epsilon}\right) + \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \le -x+\epsilon\right)\\ &=: v_n + u_n.\end{aligned}$$

By (4.14) and (4.16), $\lim_{n\to\infty}\frac{v_n}{u_n} = 0$, thus,

$$\limsup_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) \le \limsup_{n\to\infty}\frac{n}{a_n^2}\log u_n = -\frac{(x-\epsilon)^2}{2\sigma^2}.$$

Letting $\epsilon \to 0$, we obtain

$$\limsup_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \le -x\right) \le -\frac{x^2}{2\sigma^2}. \tag{4.17}$$

(4.12) and (4.17) yield (4.10). To prove (4.11), on account of (4.13), it remains to show that

$$\liminf_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) \ge -\frac{x^2}{2\sigma^2}. \tag{4.18}$$

Similarly, for every $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) &\ge \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} \ge x+\epsilon\right) - \mathbb{P}(W_n \le e^{-a_n\epsilon})\\ &=: \tilde{u}_n - v_n.\end{aligned}$$

Again by (4.14) and (4.16), $\lim_{n\to\infty}\frac{v_n}{\tilde{u}_n} = 0$, thus,

$$\liminf_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \ge x\right) \le \liminf_{n\to\infty}\frac{n}{a_n^2}\log\tilde{u}_n = -\frac{(x+\epsilon)^2}{2\sigma^2}.$$

Letting $\epsilon \to 0$, we obtain (4.18). □

We remark that, by Lemma 4.4 below, Theorem 4.2 is in fact equivalent to Theorem 1.6. So the direct proof of Theorem 4.2 leads to another proof of Theorem 1.6.

Lemma 4.4. *Let I be a continuous function on $\mathbb{R}$ satisfying*

(a) $I(b) = \inf_{x\in\mathbb{R}} I(x) = 0$ *for some* $b \in \mathbb{R}$*;*
(b) I *is strictly increasing on* $[b,\infty)$ *and strictly decreasing on* $(-\infty, b]$.

Let (μ_n) be a family of probability distribution on $\mathbb{R}$ and let (a_n) be a sequence of positive numbers satisfying $a_n \to \infty$. Then the following statements (i) *and* (ii) *are equivalent.*

(i) *For $x < b$,*

$$\lim_{n\to\infty}\frac{1}{a_n}\log\mu_n((-\infty, x]) = -I(x);$$

for $x > b$,

$$\lim_{n\to\infty} \frac{1}{a_n} \log \mu_n([x, +\infty)) = -I(x).$$

(ii) (μ_n) *satisfies a large deviation principle: for any measurable subset* B *of* $\mathbb{R}$,

$$-\inf_{x\in B^o} I(x) \le \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \tag{4.19}$$

$$\le \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \le -\inf_{x\in \bar{B}} I(x), \tag{4.20}$$

where B^o *denotes the interior of* B *and* $\bar{B}$ *its closure.*

This is a general result on large deviations. It shows that the large deviation principle holds if and only if the corresponding limit exists for tail events, when the rate function is continuous and strictly monotone. This result would be known; as we have not found a reference, we shall give a proof in an Appendix by the end of the paper.

5. Central limit theorems for $W - W_n$ and $\log Z_n$

In this section, we shall prove the results about central limit theorems.

We first prove the central limit theorem on $W - W_n$ with exponential convergence rate, using the results about the harmonic moments of Z_n (i.e. Theorem 1.3 with $t < 0$).

Proof of Theorem 1.7. Notice that

$$\Pi_n(W - W_n) = \sum_{i=1}^{Z_n} \left(W_i^{(n)} - 1 \right),$$

where under $\mathbb{P}_\xi$, the random variables $W_i^{(n)} (i = 1, 2, \ldots)$ are independent of each other and independent of Z_n, and have common conditional distribution $\mathbb{P}_\xi(W_i^{(n)} \in \cdot) = \mathbb{P}_{T^n\xi}(W \in \cdot)$. Notice that if $a_0 := \operatorname{essinf} \frac{m_0(2)}{m_0^2} > 1$, then $\delta_\infty^2 \ge a_0 - 1 > 0$. Therefore, the condition $\mathbb{E}Z_1^{2+\epsilon} < \infty$ implies that $\mathbb{E}\left|\frac{W-1}{\delta_\infty}\right|^{2+\epsilon} < \infty$. By the Berry–Esseen theorem (see [9, Theorem 9.1.3]), for all $x \in \mathbb{R}$,

$$\left| \mathbb{P}_\xi \left(\frac{\Pi_n(W - W_n)}{\sqrt{Z_n}\delta_\infty(T^n\xi)} \le x \right) - \Phi(x) \right| \le C_1 \mathbb{E}_{T^n\xi} \left| \frac{W - 1}{\delta_\infty} \right|^{2+\epsilon} \mathbb{E}_\xi Z_n^{-\epsilon/2}, \tag{5.1}$$

where C_1 is the Berry–Esseen constant. Taking expectation in (5.1), we obtain for all $x \in \mathbb{R}$,

$$\left| \mathbb{P} \left(\frac{\Pi_n(W - W_n)}{\sqrt{Z_n}\delta_\infty(T^n\xi)} \le x \right) - \Phi(x) \right| \le C_1 \mathbb{E} \left| \frac{W - 1}{\delta_\infty} \right|^{2+\epsilon} \mathbb{E}Z_n^{-\epsilon/2}. \tag{5.2}$$

Since $\mathbb{E}p_1 < \mathfrak{m}_0^{\epsilon/2}$, $\|p_1\|_\infty < 1$ and (H) holds, the condition (iii) of Theorem 1.3 is satisfied, so that by Theorem 1.3, there exists a constant $C_\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n^{-\epsilon/2}}{\left(\mathbb{E}m_0^{-\epsilon/2}\right)^n} = C_\epsilon.$$

Combining this with (5.2), we obtain (1.10). $\square$

We then prove the central limit theorem on $\log Z_n$, using the central limit theorem on $\log \Pi_n$.

Proof of Theorem 1.8. Let $x \in \mathbb{R}$. By the standard central limit theorem for i.i.d. random variables,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) = \Phi(x). \tag{5.3}$$

By (1.6), we have for every $\epsilon > 0$,

$$\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) \le \mathbb{P}\left(\frac{\log W_n}{\sqrt{n}} < -\epsilon\sigma\right) + \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x+\epsilon\right). \tag{5.4}$$

Since $\lim_{n\to\infty} \frac{\log W_n}{\sqrt{n}} = 0$ a.s., we have

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log W_n}{\sqrt{n}} < -\epsilon\sigma\right) = 0. \tag{5.5}$$

Taking the limit superior in (5.4), and applying (5.3) and (5.5), we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) \le \Phi(x+\epsilon).$$

Letting $\epsilon \to 0$, we get the upper bound. For the lower bound, observe that

$$\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) \ge \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x-\epsilon\right) - \mathbb{P}\left(\frac{\log W_n}{\sqrt{n}} > \epsilon\sigma\right). \tag{5.6}$$

Similarly,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log W_n}{\sqrt{n}} > \epsilon\sigma\right) = 0.$$

Taking the limit inferior in (5.6) and letting $\epsilon \to 0$, we get

$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \le x\right) \ge \Phi(x).$$

So (1.11) is proved. □

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Appendix. Proof of Lemma 4.4

Proof of Lemma 4.4. It is clear that (ii) implies (i) since I is continuous. We need to prove (i) implies (ii). First, we show (4.19). For $x \in B^o$, consider the case where $x \ge b$. Then B^o contains an interval $[x+\epsilon_1, x+\epsilon_2)$ for some $0 < \epsilon_1 < \epsilon_2$. Consequently, by (i), $\forall \epsilon > 0$, there exists $n_\epsilon > 0$ such that $\forall n \geq n_\epsilon$,

$$\begin{aligned}\mu_n(B) &\geq \mu_n([x+\epsilon_1, x+\epsilon_2)) \\ &= \mu_n([x+\epsilon_1, \infty)) - \mu_n([x+\epsilon_2, \infty)) \\ &\geq e^{-a_n(I(x+\epsilon_1)+\epsilon)} - e^{-a_n(I(x+\epsilon_2)-\epsilon)}.\end{aligned}$$

Since I is strictly increasing on $[b, \infty)$, we can take $\epsilon > 0$ small enough such that $I(x+\epsilon_1)+\epsilon < I(x+\epsilon_2)-\epsilon$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \geq -I(x+\epsilon_1) - \epsilon.$$

Letting $\epsilon, \epsilon_1 \to 0$, we get

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \geq -I(x). \tag{A.1}$$

If $x < b$, we obtain (A.1) by a similar argument. So (A.1) holds for all $x \in B^o$, which yields (4.19).

Now we show (4.20). If $b \in \bar{B}$, then (4.20) is obvious since $\mu_n(B) \leq 1$ and the right side of (4.20) is 0. Assume that $b \notin \bar{B}$. Let $B_1 = B \bigcap (-\infty, b]$ and $B_2 = B \bigcap (b, \infty)$ so that $B = B_1 \bigcup B_2$. Then

$$B_1 \subset (-\infty, b_1] \quad (\text{if } B_1 \neq \emptyset) \quad \text{and} \quad B_2 \subset [b_2, \infty) \quad (\text{if } B_2 \neq \emptyset),$$

where $b_1 := \sup B_1$ and $b_2 := \inf B_2$. Assume that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. As $b \notin \bar{B}$, we have $b_1 < b < b_2$. By (i), $\forall \epsilon > 0$, there exists $n_\epsilon > 0$ such that $\forall n \geq n_\epsilon$,

$$\begin{aligned}\mu_n(B) &\leq \mu_n([-\infty, b_1]) + \mu_n([b_2, \infty)) \\ &\leq e^{-a_n(I(b_1)-\epsilon)} + e^{-a_n(I(b_2)-\epsilon)} \\ &\leq 2e^{-a_n(I_0-\epsilon)},\end{aligned}$$

where $I_0 := \min\{I(b_1), I(b_2)\} = \inf_{x\in\bar{B}} I(x)$. Therefore,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \leq -I_0 + \epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \leq -I_0 = -\inf_{x\in\bar{B}} I(x).$$

If $B_1 = \emptyset$ or $B_2 = \emptyset$, we obtain (4.20) by a similar argument. □

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