Isogeometric Analysis as a New FEM Formulation - Simple Problems of Steady State Thermal Analysis

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Abstract

The subject of this article concerns Isogeometric Analysis as a new formulation within Finite Element Method. Motivation for this new approach was presented together with theoretical foundations of the method. The main subject of the paper is numerical implementation of the method in the Matlab environment. Special focus was put on the new concept of element and new geometry data interpretation in terms of analysis. Object oriented programming was utilized to produce more universal tool for future investigations. Several examples for Poisson’s partial differential equation solution were presented. The h-refinement was utilized to show method convergence. Steady state thermal analysis was performed with temperatures and heat flux as boundary conditions. Quick geometry data transfer and mesh refinement from Rhino software with Grasshopper plug-in was also prepared and described in the paper. Finally, authors’ conclusions concerning new method were presented and comparison between FEM and IGA was made on the basis of the studied examples.

Keywords: Isogeometric Analysis (IGA); Finite Element Method (FEM); B-spline; NURBS; T-spline; CAD; Computer Aided Engineering;

1. Introduction

Most of recent approaches in the field of finite element method focus on solving certain difficulties for the narrow class of problems. Isogeometric Analysis has entirely different motivation and takes into account the FEM and its problems as a whole. Authors of this method [2] took industry needs and problems into consideration and came with
the solution that aims to integrate geometry description used in CAD systems with the one applied during FEM analysis. It would appear that the topic was already covered by very widely developed market of mesh generators and geometry transfer tools. However this new approach makes geometry transfer needless and introduces full automation that answers problems with design process bottleneck in CAD-CAE industry.

The subsequent change in IGA element concept in comparison to FEM element concept demands thorough and detailed study of CAD geometry representation both from mathematical and numerical point of view. It is a necessary knowledge for a designer that enables intuitive engineering approach.

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>(\xi,\eta)</td>
<td>parametric domain direction</td>
</tr>
<tr>
<td>(\Xi,\Delta)</td>
<td>knot vector for a respective parametric direction</td>
</tr>
<tr>
<td>(p,q)</td>
<td>Polynomial order of curve/surface in (\xi,\eta) direction respectively</td>
</tr>
<tr>
<td>(N_{i,p})</td>
<td>Shape function of order (p) for the (i)-th span of parametric domain</td>
</tr>
<tr>
<td>(N_{i1,i2,p,q})</td>
<td>Shape function tensor product for 2D geometry of order (p, q) in (\xi,\eta) direction respectively for the (i_1, i_2) span</td>
</tr>
<tr>
<td>(P_k)</td>
<td>Control point (k) coordinates for one-dimensional geometry</td>
</tr>
<tr>
<td>(P_{k,l})</td>
<td>Control point (k,l) coordinates for two-dimensional geometry</td>
</tr>
<tr>
<td>(C(\xi))</td>
<td>Parametric curve B-spline or NURBS</td>
</tr>
<tr>
<td>(S(\xi,\eta))</td>
<td>Parametric surface B-spline or NURBS</td>
</tr>
<tr>
<td>(n_i)</td>
<td>number ok control points in (i=\xi) or (i=\eta) direction</td>
</tr>
<tr>
<td>(u_i)</td>
<td>point (i) coordinates in physical domain</td>
</tr>
<tr>
<td>(w_k)</td>
<td>weights for (k) control points of NURBS curve</td>
</tr>
<tr>
<td>(w_{k1,k2})</td>
<td>weights for (k_1,k_2) control points of NURBS curve</td>
</tr>
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1.1. CAD systems and geometry description

Computer Aided Design systems are now widespread design tool in every branch of industry. CAD model can be created in many different ways, with use of different file formats, geometry description, algorithms etc. Some of CAD designs are transferred at different stages of design process to the Computer Aided Engineering systems. Those latter ones use Finite Element Analysis most commonly to perform static structural, thermal, dynamic, hydrodynamic,... analysis on the provided geometry. From academic point of view the formal transfer from CAD to CAE may seem irrelevant issue but it has become a huge problem recently. The main obstacle is the proper model for FEA preparation out of data received from CAD stage of design process. It is estimated in many companies and research institutes that 80% of total FEA time for the product is related to the input geometry adjustment. This encompasses simplifications in the CAD model, partitioning, mesh generation and necessity to repeat those operations in case of resulting model unsuitability for analysis.

1.2. B-Splines and NURBS

B-Spline curves are parametric curves defined recursively from the polynomial basis functions by Cox de Boor formula [4]:

\[
N_{i,0} = \begin{cases} 
1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\
0 & \text{otherwise} 
\end{cases} \quad \text{For } p=1, 2, 3, \ldots \tag{1}
\]

\[
N_{i,p} = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1} + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1} \tag{2}
\]
Basis functions are defined in each parametric direction using formula from above and a B-spline curve for a given direction has the form of:

\[
C(\hat{x}) = \sum_{k=1}^{n} P_k N_{k,p}(\hat{x})
\]  

For the two-dimensional generalization we use tensor product and B-Spline surface has a form of:

\[
S(\xi, \eta) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} P_{k_1,k_2} N_{k_1-p+1,k_2-q+1} N_{k_1-p,k_2-q} (\xi) M_{k_1-p+1,k_2-q} (\eta)
\]  

As can be seen from above, it is necessary to define set of control points (net of control points in 2D case), vector of non-decreasing numbers \(\vec{\xi}, \vec{\eta}\) for each parametric direction and a set of evaluating points \(\hat{x}\) on the domain to find curve or surface evaluation in the physical space. This resulting curve or surface can be then adjusted using the same input parameters.

NURBS are B-Spline generalization and enable more control over local geometry shape due to non-uniform treatment of each control point influence on the curve. This is obtained by the use of weights for each control point respectively according to the formula:

\[
C(\hat{x}) = \sum_{k=1}^{n} w_k P_k N_{k,p}(\hat{x})
\]

NURBS surfaces are defined analogously.

From the implementation point of view we need to define linear combination of basis functions with control points as weights for B-Spline evaluation. We perform this operation for a set of \(\xi, \eta\) values - not just one. For the NURBS evaluation the procedure is twice B-Spline evaluation for the numerator and denominator. Note that we can assume that for the denominator control points values are all equal to one multiplied by weights. Therefore B-Spline procedure is a basis for all numerical operations concerning both B-splines and NURBS.

### 1.3. T-Splines

T-splines are relatively new concept in CAD systems [1]. In the aforementioned paragraph, it was established that B-splines and NURBS topology is defined by set or net of control points. Therefore the parametric domain is an orthogonal grid and any geometry represented by this algorithm has the same topology. This imposes many constraints on the shape design and demands trimming and adjusting between different topologies to represent certain geometry. T-Splines are NURBS generalization in terms of topology. The grid of points is not restricted to the orthogonal and therefore allows much greater range of geometry representation in a uniform way. However T-Spline grid requires more complex and therefore even less intuitive approach to discretization for FEM purposes.

### 1.4. IGA element idea

In Isogeometric Analysis element is defined by knot spans of the parametric domain - Fig. 1. Element nodes are control points and they are not interpolator except for the knots that are repeated at least \(p\) times. Number of degrees of freedom for one element is then number of control points times number of degrees of freedom for each control point. Depending on the curve degree, an element can have more nodes but it does not affect global number of degrees of freedom because more nodes are shared between adjacent elements.
Fig. 1. Figure shows idea of an element concept on two geometries of which one (right side) has minimum number of control points and second one (left side) has additional points that do not change geometry itself. The difference in knot vectors can be observed. Note that number of nodes affecting element depends on the polynomial order in the given direction. In this case: \( p = 2; q = 1 \).

1.5. CAD geometry transfer for analysis

In order to utilize CAD algorithms for B-Splines and NURBS and to introduce and study automation in the IGA implementation, very popular software tools were used: Rhino 3D and Grasshopper plug-in. The second one allows for the necessary modifications and data export between Rhino and Matlab environment. A code in visual programming language is presented in the Fig. 2.

![Fig. 2. Schematics created in Grasshopper plug-in for Rhinoceros. Simple program extracts data from the NURBS surface created in Rhino. Yellow boxes contain those data ready to import into Matlab environment.](image)

2. IGA algorithm description

The IGA algorithm numerical implementation is very strongly supported in the B-Spline/NURBS evaluating algorithms. As can be observed in Fig. 3, this algorithm is utilized at many stages and it fulfills the concept of
parametric element as both geometry of the design and the space of the solution are supported by the same functions. Furthermore the geometry from CAD is not changed in any way and does not need adjustments and simplifications in order to be suitable for the analysis. Algorithm is more complicated than the classical FEM algorithm at the initial stage due to the full CAD geometry definition incorporation. Recursiveness of the B-Spline/NURBS definition causes numerical problems but they can be easily overcome in the future by ready-made, efficient CAD algorithms application. It is worth noticing that element formulation can be introduced into code separately, which allows to cover wide field of problems.

Fig. 3. Algorithm graphical interpretation. The code was created with the help of course from [3].

3. Problem example - Poisson equation

As a benchmark example for code correctness verification Poisson equation with known analytical solution was used. A general form of equation is presented in Eq. 6 and the respective weak form in the Eq. 7. A right hand side function for this case is specified by Eq. 8. Physical interpretation was assumed to be temperature field in the domain with heat source as surface load function \( f(x,y) \).

\[
-\nabla^2 u(x,y) = f(x,y) \quad \forall x, y \in \Omega 
\]  
\[ (6) \]

\[
-\int_{\Omega} \nabla^2 u \cdot v dx = \int_{\Omega} f \cdot v dx 
\]  
\[ (7) \]

\[
f(x,y) = \frac{8-9 \cdot \sqrt{x^2 + y^2}}{x^2 + y^2} \cdot \sin \left( 2 \cdot \arctan \left( \frac{y}{x} \right) \right)
\]  
\[ (8) \]

The problem was solved on the quarter of circular ring domain with external and internal radius of 2 m and 1 m respectively. A good result was obtained without mesh refinement and for the minimal amount of elements necessary to exactly represent geometry - namely two elements for the ring quarter.
4. Heat transfer in steady state

Next example concerns temperature field in the given geometry (Fig. 4) with assumed boundary conditions on selected edges. Two subsequent refinement on the initial mesh were obtained in Rhinoceros software together with Grasshopper as a tool extracting geometry parameters. H-refinement is simply obtained by knot insertion (InsertKnots command in Rhinoceros). The results from Matlab code were compared with those obtained from ANSYS WORKBENCH software. Results show that while in simple problem of steady state thermal analysis we need minimal amount of elements to obtain good results with respect to the quantity, refinement is necessary for geometry representation. The only boundary conditions in this case are fixed temperatures on two edges.

5. Conclusions

Isogeometric Analysis proves to be effective in cases where exact geometry representation plays crucial role. Moreover, there exist numerous examples of problems in which IGA gives better results than classical FEM formulation. This may be both due to the geometry correctness and better numerical performance of B-Spline/NURBS basis function. Besides, it is worth pursuing this method simply to introduce full automation into CAD/CAE industry. However promising the new approach is, it still introduces certain problems in the field of boundary conditions interpretation and application. Furthermore, to fully utilize the advantages of the method, new systems have to be developed and tested. Fortunately, they can profit from the already existing efficient FEM algorithms as well as from CAD software geometry representation algorithms.

References