Metric subgraphs of the chamfer metrics and the Melter–Tomescu path generated metrics

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Abstract

Chamfer metrics are determined by local distances which are chosen to ensure that each geodesic lies within one of the cones determined by the mask and contains only edges in the directions of the bounding rays of the cone. It is shown that the chamfer distances calculated within a set are the same as those calculated in the whole space if and only if the set is convex in each of the local distance directions. The result does not hold when the local distances allow more general geodesics. The results for chamfer metrics are related to corresponding results for the metrics generated by the two-, three- and four-direction graphs studied by Melter and Tomescu.

1. Introduction

The aim of the paper is to obtain necessary and sufficient conditions on a set of points in the digital plane for chamfer distances between points of the set calculated along paths within the set to be the same as those calculated along paths in the whole space. Distances calculated within a set of points in the 4-connection graph are the same as those calculated in the whole graph if and only if the set contains the horizontal and vertical segments joining pairs of points in the set. However, the natural extension of this result to sets in the 8-connection graph is false. In [5], Harary et al. have obtained a necessary and sufficient condition for distances calculated within a set of points in the 8-connection graph to be the same as that calculated in the whole of the graph. The frequency with which the chamfer metrics are used makes it advisable to check necessary and sufficient conditions under which calculations performed in restricted regions of the digital plane do indeed give the chamfer distances between points in the regions.

The conditions for chamfer metrics follow from conditions for metrics in a class studied by Melter and Tomescu [8]. These are metrics associated with graphs whose edges are constrained to lie in two, three or four of the directions parallel to the axes.
and the diagonals. In the Melter–Tomescu metrics, distances between pairs of points can be calculated using algorithms to trace minimal paths, which also lead to analytic expressions for the metrics [2]. An algorithm for tracing minimal paths in the 4-connected graph was given by Rosenfeld and Pfaltz [14]. General path tracing procedures for \( r \)-neighbour metrics are discussed in [3].

Chamfer metrics are defined in terms of local distances between neighbouring points. For each of these metrics, between any two points there is a shortest path consisting of two straight line segments [10]. In the case of the \( 3 \times 3 \) chamfer metric, each of the shortest paths is a shortest path for one of the Melter–Tomescu metrics. For a \( (2k + 1) \times (2k + 1) \) chamfer metric each of the shortest paths is a shortest path for a linear transform of one of the Melter–Tomescu metrics. The particular metric and linear transform depends on the angle between the end points of the path.

An algorithm for tracing shortest paths can be executed within a subgraph so long as distances calculated in the subgraph are the same as distances calculated in the whole graph. Such graphs are said to be metric subgraphs. The metric subgraphs of the two-gradient Melter–Tomescu metrics are characterized in Section 3. Those of the chamfer metrics and of the three and four-gradient Melter–Tomescu metrics are characterized in Sections 4 and 5, respectively.

2. The Melter–Tomescu path generated metrics

A graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is specified by its set of vertices \( \mathcal{V} \) and its set of edges \( \mathcal{E} \). A subgraph \( \mathcal{G}' = (\mathcal{V}', \mathcal{E}') \) of \( \mathcal{G} \) is a graph in which \( \mathcal{V}' \) is a subset of \( \mathcal{V} \) and \( \mathcal{E}' \) is a subset of \( \mathcal{E} \). An induced subgraph is specified by a set of vertices \( \mathcal{V}' \). The set of edges of the induced subgraph are those edges of \( \mathcal{G} \) which join two vertices in \( \mathcal{V}' \). The subgraphs used in the proofs in [5] need to be taken to be induced subgraphs, though the graphs illustrated in Figs. 2(a) and (b) of that paper are not induced subgraphs of the 8-connection graph.

A path of length \( n \) in a graph \( \mathcal{G} \) is a sequence of vertices \( v_0, v_1, \ldots, v_n \) in \( \mathcal{G} \), consecutive vertices of which are neighbours, i.e. are joined by an edge. A graph is connected if each pair of vertices is joined by a path. The distance \( d_\mathcal{G}(u, v) \) between two vertices \( u \) and \( v \) of a connected graph \( \mathcal{G} \) is the minimum of the lengths of the paths joining them. This distance function is symmetric, and positive definite. It also satisfies the triangle inequality, and so it is a metric on \( \mathcal{V} \).

A path is said to be a shortest path if its length is the distance between its end points. A geodesic is a path such that each subpath is a shortest path between its end points. For all the metrics considered in this paper each shortest path is a geodesic [11].

The edges in the graph can be assigned positive numbers, the local distances. Then the global distance between two vertices is the minimum of the sums of the local distances of the paths joining them. The chamfer metrics, which are determined by non-constant local distances, are described in Section 4.
The Melter–Tomescu path generated metrics on the digital plane $\mathbb{Z}^2$ are the metrics of certain graphs specified by their sets of edges. Four types of edges, called types $a$, $b$, $c$, $d$, were considered by Melter and Tomescu. Since more types of edges will have to be considered in this paper, edges will be specified here by their gradients. An edge joining vertices $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathbb{Z}^2$ will be said to be of type $\infty$ if $|y_1 - y_2| = 1$ with $x_1 = x_2$, and of type $g$ if $(y_1 - y_2)/(x_1 - x_2) = g$ with $y_1 - y_2$ and $x_1 - x_2$ relatively prime.

The Melter–Tomescu graphs have vertex set $\mathbb{Z}^2$ and edges selected from types $0$, $\infty$ and $\pm 1$. The graphs are all connected and so give rise to metrics on $\mathbb{Z}^2$. Metric spaces obtained from sets of edges on the same row in the list below are isometric to each other. The first five are two-gradient graphs, the next four are three-gradient graphs, while the last is a four-gradient graph. The other six selections of the four types of edges do not lead to connected graphs.

$$g(0, \infty);$$
$$g(0, 1), g(0, -1), g(\infty, 1), g(\infty, -1);$$
$$g(0, \infty, 1), g(0, \infty, -1);$$
$$g(0, 1, -1), g(\infty, 1, -1);$$
$$g(0, \infty, 1, -1).$$

For each of these graphs, the associated metric is invariant under translations. Thus it is sufficient to give the distance of a vertex $v = (x, y)$ from the origin. The notation $D(x, y)$ will be used for the distance $d((x, y), (0, 0))$ associated with one of the graphs $(\mathbb{Z}^2, g)$.

The graphs $(\mathbb{Z}^2, g(0, \infty))$ and $(\mathbb{Z}^2, g(0, \infty, 1, -1))$ are the standard 4-connection graph $(\mathbb{Z}^2, g_4)$ and 8-connection graph $(\mathbb{Z}^2, g_8)$ for which the distances of a vertex $(x, y)$ from the origin are $D_4(x, y) = |x| + |y|$ and $D_8(x, y) = \max \{ |x|, |y| \}$, respectively. Analytic expressions for the metrics of the other graphs have been given by Das [2]. They are related to algorithms for tracing minimal paths. Alternative expressions of these metrics are given here. They may be more convenient to use in some circumstances.

Melter and Tomescu noted in their conference paper [9] that the edge sets in the second row are linear transforms of the edge set $g(0, \infty)$. The process is described in detail in [11]. A one-to-one linear transformation $T$ on $\mathbb{Z}^2$ maps a graph $\mathcal{G} = (\mathbb{Z}^2, g)$ to the graph $T\mathcal{G} = (\mathbb{Z}^2, Tg)$, where $T\mathcal{G}$ contains an edge joining the vertices $u$ and $v$ if and only if $\mathcal{G}$ contains an edge joining the vertices $T^{-1}u$ and $T^{-1}v$. It maps a path $\alpha = (v_0, v_1, \ldots, v_n)$ in $\mathcal{G}$ to a path $T\alpha = (Tv_0, Tv_1, \ldots, Tv_n)$ in $T\mathcal{G}$ of the same length. For example, the graph $(\mathbb{Z}^2, g(0, 1))$ is the image of the graph $(\mathbb{Z}^2, g(0, \infty))$ under the linear transformation $T$ with matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
Thus the metric of this graph is given by
\[ D(x, y) = D_4(T^{-1} x, T^{-1} y) = D_4(x, y - x) = |x| + |y - x|. \]
The other Melter–Tomescu metrics can also be expressed in ways which emphasize their relationships with \( D_4 \) and \( D_8 \). For example the metric for the graph \((\mathbb{Z}^2, \mathcal{E}(0, \infty, 1))\) is
\[ D(x, y) = \begin{cases} \max\{|x|, |y|\} & \text{if } xy \geq 0, \\ |x| + |y| & \text{if } xy < 0, \end{cases} \]
and the metric for the graph \((\mathbb{Z}^2, \mathcal{E}(0, 1, -1))\) is
\[ D(x, y) = \max\{|x|, |y| + \delta(x, y)|, \]
where \( \delta(x, y) = 1 \), if \( x + y \) is odd with \( |y| < |x| \), and \( \delta(x, y) = 0 \), otherwise. The expressions given by Das for these metrics are \( \max\{|x|, |y|, |x - y|\} \) and \( \max\{2|I(x - y)/2|, 0\} + |y| \), respectively.

3. Metric subgraphs for the two-gradient Melter–Tomescu metrics

Since many applications of path generated digital metrics are to finite subsets of the digital plane, it is helpful to have criteria which ensure that metrics generated on subgraphs are restrictions of metrics generated on the whole graphs. If \( \mathcal{H} = (\mathcal{V}', \mathcal{E}') \) is a connected subgraph of a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) then \( d_{\mathcal{E}}(u, v) \geq d_{\mathcal{G}}(u, v) \) for each pair of vertices \( u, v \) in \( \mathcal{V}' \). It is said to be a metric subgraph if and only if \( d_{\mathcal{E}}(u, v) = d_{\mathcal{G}}(u, v) \) for each pair of vertices \( u, v \) in \( \mathcal{V}' \). For each special case of a graph \( \mathcal{G} \), the problem is to see if some less stringent condition guarantees that a subgraph \( \mathcal{H} \) contains with each pair of vertices a shortest path in \( \mathcal{G} \) joining them. Harary et al. [5] have studied the metric subgraphs of \((\mathbb{Z}^2, \mathcal{E}_4)\) and \((\mathbb{Z}^2, \mathcal{E}_8)\). The method of transforming path generated metrics on \( \mathbb{Z}^2 \) enables one to deduce characterizations of the metric subgraphs of one-to-one transformations of \((\mathbb{Z}^2, \mathcal{E}_4)\). In order to do so, the notions of axial and diagonal convexity used in [5] need to be extended.

A subgraph \( \mathcal{G} = (\mathcal{V}', \mathcal{E}') \) of \((\mathbb{Z}^2, \mathcal{E}_4)\) will be said to be g-convex if \((x_1, y_1)\) and \((x_2, y_2)\) in \( \mathcal{V}' \) with \((y_2 - y_1)/(x_2 - x_1) = g \) implies \( \mathcal{V}' \) contains at least one of the integer points in the segment joining \((x_1, y_1)\) and \((x_2, y_2)\) nearest to these end points: precisely, if \( \lambda \) is the least positive number such that \( \lambda(x_1 - x_2) \) and \( \lambda(y_1 - y_2) \) are integers then \( \mathcal{V}' \) contains either \((x_1 + \lambda(x_2 - x_1), y_1 + \lambda(y_2 - y_1))\) or \((x_2 - \lambda(x_2 - x_1), y_2 - \lambda(y_2 - y_1))\). A subgraph will be said to be axially convex if it is \( 0 \)-convex and \( \infty \)-convex. It will be said to be diagonally convex if it is \( 1 \)-convex and \( (-1) \)-convex.

Note that if an axially convex subgraph of \((\mathbb{Z}^2, \mathcal{E}_4)\) or \((\mathbb{Z}, \mathcal{E}_8)\) is induced, then it contains with each pair of vertices on a line parallel to one of the axes all the vertices between them and all the edges joining adjacent pairs of these vertices. Thus
Theorem 1 of [5] together with the method of transforming metrics leads to a characterization of metric subgraphs of all graphs whose edges are generated by two basic vectors. An alternative proof of the theorem is given here in a form which will be a basis for proofs of other results.

**Theorem 3.1.** A subgraph $(\mathcal{V}', \mathcal{E}')$ of $(\mathbb{Z}^2, \mathcal{E}_4)$ is a metric subgraph if and only if it is a connected induced subgraph which is axially convex. If

\[
T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

is a one-to-one linear transformation of $\mathbb{Z}^2$ then a subgraph $(\mathcal{V}'', \mathcal{E}')$ of $(\mathbb{Z}^2, T\mathcal{E}_4)$ is a metric subgraph if and only if it is a connected induced subgraph which is $(c/a)$-convex and $(d/b)$-convex.

**Proof.** The necessity of the conditions is clear. Suppose that $\mathcal{H}$ is an axially convex connected induced subgraph but not a metric subgraph of $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}_4).$ Then there exists an integer $n$ such that $d_\mathcal{H}(u, v) < n$ implies $d_\mathcal{G}(u, v) = d_\mathcal{G}(v_0, v_n).$ Since $\mathcal{H}$ is an induced subgraph $n > 2.$ Let $v_0, v_1, \ldots, v_n$ be a path of length $n$ from $v_0$ to $v_n$ in $\mathcal{H}.$ Then by assumption, $v_0, v_1, \ldots, v_{n-1}$ and $v_1, v_2, \ldots, v_n$ are geodesics in $\mathcal{G}$ as well as in $\mathcal{H}. \ $ Let $v_i = (x_i, y_i)$ for $0 \leq i \leq n.$ Rosenfeld [13] has characterized geodesics in $\mathcal{G}$ as paths for which both sequences of coordinates are monotone. Since $v_1, v_2, \ldots, v_{n-1}$ is a geodesic, the sequence of $x$-coordinates for $v_1, v_2, \ldots, v_{n-1}$ and $v_0, v_1, \ldots, v_n$ and $v_1, v_2, \ldots, v_{n-1}$ and also for $v_0, v_1, \ldots, v_n$ will all be non-decreasing or all be non-increasing, as will be the sequence of $y$-coordinates. It follows that $v_0, v_1, \ldots, v_n$ is a geodesic in $\mathcal{G}$ and that $d_\mathcal{G}(v_0, v_n) = n.$ The contradiction proves the sufficiency of the conditions for the four-connection graph. Now let $\mathcal{H}''$ be a connected induced subgraph of $(\mathbb{Z}^2, T\mathcal{E}_4)$ which is $(c/a)$-convex and $(d/b)$-convex. Then $T^{-1}\mathcal{H}''$ is a connected induced subgraph of $(\mathbb{Z}^2, \mathcal{E}_4)$ which is axially convex and so is a metric subgraph of $(\mathbb{Z}^2, \mathcal{E}_4).$ Lengths of paths and so distances between vertices are preserved under the transformation $T.$ Thus $\mathcal{H}''$ is a metric subgraph of $(\mathbb{Z}^2, T\mathcal{E}_4). \hfill \Box$

Characterizations of the metric subgraphs of some of the Melter–Tomescu metrics now follow as corollaries.

**Corollary 3.1.** Let $g_1$ be chosen from 0, $\infty$ and $g_2$ be chosen from $\pm 1.$ A connected induced subgraph of $(\mathbb{Z}^2, \mathcal{E}(g_1, g_2))$ is a metric subgraph if and only if it is $g_1$-convex and $g_2$-convex.

4. Metric subgraphs for chamfer metrics

Chamfer metrics are calculated using local distances on a graph determined by the appropriate mask. Given a positive integer $k$, the graph $(\mathbb{Z}^2, \mathcal{E}(k))$ for
a \((2k + 1) \times (2k + 1)\) chamfer metric has edges joining each vertex \((x_0, y_0)\) to every vertex \((x_1, y_1)\) such that \(|x_0 - x_1| \leq k\) and \(|y_0 - y_1| \leq k\) with \(x_0 - x_1\) and \(y_0 - y_1\) relatively prime. The mask divides the space into a set of rational pointed cones whose bounding rays are in the directions of the edges in \(\mathcal{E}(k)\). The local distances for these edges are chosen to ensure that each chamfer metric geodesic lies within one of the cones and consists only of edges in the directions of the bounding rays of the cone. Then no local distance is redundant. The global distance is \(\text{positive linear homogeneous}\), i.e. \(D(\lambda x, \lambda y) = \lambda D(x, y)\) for each point \((x, y)\) in \(\mathbb{Z}^2\) and each positive integer \(\lambda\). The metric is linear in each of the cones and the circles are convex polygons with different gradients in each of the cones. Subject to these restrictions, the geodesics, but not their lengths, are independent of the precise values of the local distances.

For example, the \(3 \times 3\) chamfer metrics are determined by local distances on the digital plane in which edges parallel to the axes are assigned a local distance \(a\) and diagonal edges are assigned a local distance \(b\). The two local distances are constrained by the inequalities \(a < b < 2a\). The distance of a point \((x, y)\) in the first quadrant from the origin is \(D(x, y) = \max\{ax + (b - a)y, (b - a)x + ay\}\). The function is linear in each of the cones bounded by an axial ray and a diagonal ray.

**Proposition 4.1.** Given a positive integer \(k\), the rational numbers \(g_i = \frac{Y_i}{X_i}\) with \(0 \leq Y_i \leq X_i \leq k\) and \(Y_i, X_i\) mutually prime can be ordered as the Farey sequence \(g_0, g_1, \ldots, g_{\phi(k)}\) of order \(k\). The integer points in the cones bounded by successive rays \(\frac{y_i}{x_i} = g_i\) and \(\frac{y_i+1}{x_i+1} = g_{i+1}\) are generated by positive integer combinations of \((X_i, Y_i)\) and \((X_{i+1}, Y_{i+1})\).

**Proof.** The ascending sequence of the numbers \(g_i\) is the Farey sequence of order \(k\). For two successive members of the sequence, \(Y_{i+1}X_i - X_{i+1}Y_i = 1\). This is a necessary and sufficient condition for \((X_i, Y_i)\) and \((X_{i+1}, Y_{i+1})\) to span \(\mathbb{Z}^2\) as a vector space over \(\mathbb{Z}\) [6]. \(\square\)

A set of points is a **Hilbert basis** for a cone if each integral point in the cone is a non-negative linear combination of the points in the set. The set of integer points in a rational polyhedral cone is generated by an integral Hilbert basis, and if the cone is pointed then there is a unique minimal integral Hilbert basis which generates all the integral points in the cone [7, 15]. For each pointed cone bounded by the successive rays in a \((2k + 1) \times (2k + 1)\) mask, the unique minimal integral Hilbert basis consists of two integral points nearest to the origin on the bounding rays.

The chamfer metrics are invariant under reflections in the diagonals as well as under translations. Thus the metric is determined by the distances \(D(x, y)\) from the origin of points \((x, y)\) with \(0 \leq y \leq x\). Each edge in a direction with gradient \(g_i\) is assigned a local distance \(a_i\). If \(g_i \leq y/x \leq g_{i+1}\) then there exist integers \(\lambda_i\) and \(\lambda_{i+1}\) such that \((x, y) = \lambda_i(X_i, Y_i) + \lambda_{i+1}(X_{i+1}, Y_{i+1})\). The constants \(a_i\) are chosen to ensure that \(D(x, y) = \lambda_ia_i + \lambda_{i+1}a_{i+1}\), and that each geodesic joining the origin to \((x, y)\) lies in the cone and contains only edges in the direction \(g_i\) and \(g_{i+1}\). This is equivalent to
requiring that each circle \( D(x, y) = r \) is a convex polygon with different gradients in the different cones. The gradient of the polygon in the cone bounded by the rays with gradients \( g_i \) and \( g_{i+1} \) is \( G_i = (Y_{i+1}a_i - Y_i a_{i+1})/(X_{i+1}a_i - X_i a_{i+1}) \). The constants \( a_i \) are chosen so that \( G_{i+1} > G_i, 0 \leq i < \Phi(k) \), \( G_0 > -\infty \), \( G_{\Phi(k)} < -1 \).

For points \((x, y)\) in the \( i \)th cone the metric can also be written \( D(x, y) = a_i(y - G_i x)/(Y_i - G_i X_i) \). For points in the first octant, \( 0 \leq y \leq x \), the metric is \( \max\{a_i(y - G_i x)/(Y_i - G_i X_i) \mid 0 \leq i \leq \Phi(k) \} \). Metrics of this type have been studied in [12].

For the 5 × 5 chamfer metrics, the three local distances \( a = a_0, b = a_2 \) and \( c = a_4 \), assigned to axial, diagonal and knight's connections, need to satisfy three inequalities. To obtain these inequalities it is most convenient to consider the gradients of the unit sphere in the cones \( 0 \leq 2x \leq -y \) and \( 0 \leq x \leq -y \leq 2x \). The points \((0, -1/a), (1/c, -2/c)\) and \((1/b, -1/b)\) lie on the unit sphere. The sequence of gradients of the segments joining successive pairs of points must be increasing and lie between 0 and 1. Thus \( a, b \) and \( c \) must satisfy the inequalities \( 0 < (c - 2a)/a < (2b - c)/(c - b) < 1 \), i.e. \( 2a < c, c - b < a \) and \( 3b < 2c \).

**Theorem 4.1.** For a chamfer metric of order \( k \) a connected induced subgraph \( \mathcal{H} = (\mathcal{V}', \mathcal{E}') \) of the graph \( \mathcal{G} = (\mathbb{Z}^2, \mathcal{E}(k)) \) is a metric subgraph if and only if it is \( g \)-convex for every gradient \( g_i \) and \( 1/g_i \), \( 0 \leq i \leq \Phi(k) \).

**Proof.** Given two points \((x_0, y_0)\) and \((x_1, y_1)\) in \( \mathcal{V}' \) with \((y_1 - y_0)/(x_1 - x_0) = g_i \), the only geodesic in \( \mathcal{G} \) joining them is the line segment between them. Thus if \( \mathcal{H} \) is a metric subgraph it is \( g_i \)-convex for each \( i \), and by symmetry for each of the directions bounding the chamfer cones. Now suppose that \( \mathcal{H} \) is a connected and induced subgraph which is convex in each of the directions bounding the chamfer cones, but is not a metric subgraph. As in the proof of Theorem 3.1, there is a path \( v_0, v_1, \ldots, v_n \) of length \( n > 2 \) which is a geodesic in \( \mathcal{H} \) but not in \( \mathcal{G} \) and for which \( v_0, v_1, \ldots, v_{n-1} \) and \( v_1, v_2, \ldots, v_n \) are geodesics in \( \mathcal{G} \) as well as in \( \mathcal{H} \). There is no loss of generality in supposing that \( v_0 = 0 \) and that \( v_n \) is in the first octant, \( 0 \leq y \leq x \), of the plane. Then the whole geodesic \( v_0, v_1, \ldots, v_{n-1} \) lies in a cone determined by a pair of gradients \( g_i, g_{i+1} \) and the whole geodesic \( v_1, v_2, \ldots, v_n \) lies in a cone determined by a pair of gradients \( g_j, g_{j+1} \). If \( i < j \) then \( i + 1 = j \). The points \( v_1, v_2, \ldots, v_{n-1} \) all lie on the ray with gradient \( g_j \). Then \( v_0 \) also lies on the same ray so that the whole path \( v_0, v_1, \ldots, v_n \) lies in the cone determined by the pair of gradients \( g_i, g_{i+1} \). The geodesics in this cone for the chamfer metric are those for the transformation of the 4-connection metric by the linear transformation

\[
T = \begin{bmatrix}
X_i & Y_i+1 \\
X_i & Y_i+1
\end{bmatrix}
\]

It follows from Theorem 3.1 that \( v_0, v_1, \ldots, v_n \) is a geodesic in this transformed metric and so in the chamfer metric. The contradiction proves the theorem. □
5. Metric subgraphs for three and four-gradient Melter–Tomescu metrics

It would be natural to expect that axial and diagonal convexity would be necessary and sufficient for a subgraph of \((\mathbb{Z}^2, d_8)\) to be a metric subgraph. However, although the conditions are sufficient they are not necessary. The induced subgraph of \((\mathbb{Z}^2, d_8)\) whose vertices are \((0, 0), (1, 1)\) and \((2, 0)\) is a metric subgraph. It is diagonally convex but not axially convex. The problem is that the metric \(d_8\) is linear in each of the four cones bounded by the diagonals. For each cone, the Hilbert basis consists of a point on each of the diagonals together with a point on an axis. For example, the Hilbert basis of the cone which is entirely in the right-half of the plane consists of the points \((1, 1), (1, 0)\) and \((1, -1)\). The points on the diagonals generate some but not all of the points on the axis within the cone. Thus some points on an axis can be joined to the origin by a geodesic which has no edges parallel to the axis, while others can be joined to the origin by a geodesic which has just one edge parallel to the axis.

A similar problem can arise with chamfer type metrics if some of the inequalities controlling the local distances are replaced by equalities. For example, a \(7 \times 7\) chamfer metric is determined by five local distances \(a_i, 0 \leq i \leq 4\), associated with the gradients 0, 1/3, 1/2, 2/3, 1. In order that the chamfer metric circles be convex polygons with different gradients in each of the cones determined by the \(7 \times 7\) mask, the local distances must satisfy the five inequalities \(a_1 > 3a_0, a_0 + a_2 > a_1, a_1 + a_3 > 3a_2, a_2 + a_4 > a_3, 2a_3 > 5a_4\). If the local distances satisfy the equation \(a_1 + a_3 = 3a_2\) and the other four inequalities, then the circles for the metric are still convex polygons. However, the gradient of the circle in the cone bounded by the rays through the points \((3, 1)\) and \((2, 3)\) is now the same as that in the cone bounded by the rays through the points \((2, 3)\) and \((3, 2)\). The metric has the same linear form in each of these two cones.

The piecewise linear form of the metric determines an alternative subdivision of the plane into cones, one of which is bounded by the rays through the points \((3, 1)\) and \((2, 3)\). These points are not a Hilbert basis for the cone — the unique minimal integral Hilbert basis for the cone consists of these two points together with the point \((2, 1)\). Thus the local distance \(a_2\) is necessary in order to determine the distance from the origin of those points in the cone which are not generated by the points \((3, 1)\) and \((3, 2)\). Nevertheless, the local distances \(a_1\) and \(a_3\) completely determine the linear form of the metric in the cone, and the points on the ray through \((2, 1)\) which are generated by the points \((3, 1)\) and \((3, 2)\) can be joined to the origin by geodesics which have no other points on that ray, so that in this case a metric subgraph need not be \((1/2)\)-convex.

Harary et al. [5] characterized the metric subgraphs of \((\mathbb{Z}^2, d_8)\) as diagonally convex subgraphs which do not contain certain patterns of edges. The significance of the forbidden patterns illustrated in Fig. 4 of [5] lies as much in what they exclude as in what they include. Here the implied double negative condition will be replaced by a positive condition. A subgraph will be required to contain with each pair of vertices on neighbouring diagonals a path joining them which is a geodesic in the whole graph.
Rosenfeld [13] has shown that a path in $(\mathbb{Z}^2, \mathcal{E}_8)$ is a geodesic if and only if the sequence of $x$ coordinates or the sequence of $y$ coordinates of the vertices in the path is strictly monotone. Thus when two vertices lie on neighbouring diagonals a geodesic joining them runs part of the way along one of the diagonals and then switches to run the rest of the way along the other diagonal. The notion of near-diagonal convexity is introduced for a characterization of metric subgraphs of $(\mathbb{Z}^2, \mathcal{E}_8)$.

A subgraph $\mathcal{G} = (V', \mathcal{E}')$ of $(\mathbb{Z}^2, \mathcal{E}_8)$ will be said to be near-diagonally convex if

(i) $(x_1, y_1)$ and $(x_2, y_2)$ in $V'$ with $x_1 < x_2$ and $x_2 - x_1 = y_2 - y_1 + 1$ implies $V'$ contains $(x_1 + 1, y_1 + 1)$ or $(x_2 - 1, y_2 - 1)$, and

(ii) $(x_1, y_1)$ and $(x_2, y_2)$ in $V'$ with $x_1 < x_2$ and $x_2 - x_1 = y_1 - y_2 + 1$ implies $V'$ contains $(x_1 + 1, y_1 - 1)$ or $(x_2 - 1, y_2 + 1)$.

Note that if a diagonally convex subgraph is induced then it contains with each pair of vertices on a diagonal all the vertices of a geodesic in $(\mathbb{Z}^2, \mathcal{E}_8)$ joining them. If a near-diagonally convex subgraph is induced then it contains with each pair of vertices on a pair of neighbouring diagonals all the vertices of a geodesic in $(\mathbb{Z}^2, \mathcal{E}_8)$ joining them.

The next theorem is a form of Theorem 2 of [5] which characterizes metric subgraphs of $(\mathbb{Z}^2, \mathcal{E}_8)$.

**Theorem 5.1.** A subgraph $\mathcal{H} = (V'', \mathcal{E}'')$ of the graph $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}_8)$ is a metric subgraph if and only if it is a connected induced subgraph which is diagonally and near-diagonally convex.

**Proof.** The Rosenfeld conditions for geodesics in $\mathcal{G}$ show that the conditions are necessary. Suppose that an induced subgraph $\mathcal{H} = (V'', \mathcal{E}'')$ is connected and diagonally and near-diagonally convex, but is not a metric subgraph. As in the proof of Theorem 3.1 there is a path $v_0, v_1, \ldots, v_n$ of length $n > 2$ which is a geodesic in $\mathcal{H}$ but not in $\mathcal{G}$ and for which $v_0, v_1, \ldots, v_{n-1}$ and $v_1, v_2, \ldots, v_n$ are geodesics in $\mathcal{G}$ as well as in $\mathcal{H}$. Let $v_i = (x_i, y_i)$ for $0 \leq i \leq n$. Suppose that $v_1$ and $v_{n-1}$ lie on a line parallel to one of the coordinate axes, say $x_1 = x_{n-1}$ with $y_1 < y_{n-1}$. Then the Rosenfeld condition guarantees that $y_0 < y_1 < \cdots < y_{n-1}$ and $y_1 < y_2 < \cdots < y_n$ from which it follows that $d_{\mathcal{G}}(v_0, v_n) = n$. The contradiction shows that $x_1 \neq x_{n-1}$ and similarly $y_1 \neq y_{n-1}$.

Now the case $x_1 < x_{n-1}$ and $y_1 < y_{n-1}$ will be discussed in detail. The other possible inequalities can be treated in similar ways. Since $v_0$ is a neighbour of $v_1$ and $v_{n-1}$ is a neighbour of $v_n$ in $\mathcal{G}$ we have $x_0 \leq x_{n-1}$, $y_0 \leq y_{n-1}$, $x_1 \leq x_n$ and $y_1 \leq y_n$. Thus the Rosenfeld conditions for geodesics in $\mathcal{G}$ ensure that

(i) $x_0 < x_1 < \cdots < x_{n-1}$ or $y_0 < y_1 < \cdots < y_{n-1}$

and also

(ii) $x_1 < x_2 < \cdots < x_n$ or $y_1 < y_2 < \cdots < y_n$.

but

(iii) neither $x_0 < x_1 < \cdots < x_n$ nor $y_0 < y_1 < \cdots < y_n$. 

Hence either

\[(iv) \quad x_0 < x_1 < \cdots < x_{n-1} \text{ and } y_1 < y_2 < \cdots < y_n,\]

or

\[(v) \quad x_1 < x_2 < \cdots < x_n \text{ and } y_0 < y_1 < \cdots < y_{n-1}.\]

Thus certainly

\[(vi) \quad x_1 < x_2 < \cdots < x_{n-1} \text{ and } y_1 < y_2 < \cdots < y_{n-1},\]

with either

\[(vii) \quad x_0 < x_1, \quad x_{n-1} \geq x_n, \quad y_0 \geq y_1 \text{ and } y_{n-1} < y_n,\]

or

\[(viii) \quad x_0 \geq x_1, \quad x_{n-1} < x_n, \quad y_0 < y_1 \text{ and } y_{n-1} \geq y_n.\]

In case (vii) \(v_0 = (x_1 - 1, y_1)\) or \((x_1 - 1, y_1 + 1)\) and \(v_n = (x_{n-1}, y_{n-1} + 1)\) or \((x_{n-1} - 1, y_{n-1} + 1)\). In case (viii), \(v_0 = (x_1, y_1 - 1)\) or \((x_1 + 1, y_1 - 1)\) and \(v_n = (x_{n-1} + 1, y_{n-1})\) or \((x_{n-1} + 1, y_{n-1} - 1)\). Hence \(v_0\) and \(v_n\) are either on the same diagonal or on neighbouring diagonals. Since \(\mathcal{H}\) is diagonally and near-diagonally convex, it follows that \(d_{\mathcal{H}}(v_0, v_n) = d_{\mathcal{H}}(v_0, v_n) < n\). The contradiction proves the sufficiency of the conditions. \(\square\)

A combination of the arguments used to prove earlier theorems leads to a characterization of metric subgraphs of \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\) and \((\mathbb{Z}^2, \mathcal{S}(0, \infty, -1))\).

**Theorem 5.2.** Let \(g\) be either 1 or \(-1\). A subgraph \(\mathcal{H} = (\mathcal{V}, \mathcal{E}')\) of the graph \(\mathcal{G} = (\mathbb{Z}^2, \mathcal{S}(0, \infty, g))\) is a metric subgraph if and only if it is a connected induced subgraph which is \(g\)-convex and axially convex.

**Proof.** Consider the case \(g = 1\). Let \(D(x, y)\) be the distance in \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\) from the origin to a point \((x, y)\). Since \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\) contains more edges than \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\), \(D(x, y) \geq D_{\mathcal{G}}(x, y)\). When \(x > 0\) and \(y > 0\), a geodesic from the origin to \((x, y)\) in \((\mathbb{Z}^2, \mathcal{S}_\mathcal{G})\) contains only edges with gradients 0, \(\infty\) and 1. In this case, \(D(x, y) \leq D_{\mathcal{G}}(x, y)\) and the path is also a geodesic in \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\). Now suppose \(x > 0\) and \(y < 0\), and let \(v_0, v_1, \ldots, v_n\) be a geodesic from the origin to \((x, y)\). Suppose that the geodesic contains an edge with gradient 1 and let the edge \((v_t, v_{t+1})\) be the first in this direction. On this edge either the \(x\)-coordinate decreases or the \(y\)-coordinate increases. In the first case the edge is preceded by a last edge \((v_t, v_{t+1})\) parallel to the \(x\)-axis with increasing \(x\)-coordinate. Then there is a shorter path from \(v_t\) to \(v_{t+1}\) parallel to the \(y\)-axis. In the second case, the edge is preceded by a last edge \((v_t, v_{t+1})\) parallel to the \(y\)-axis with decreasing \(y\)-coordinate. Then there is a shorter path from \(v_t\) to \(v_{t+1}\) parallel to the \(x\)-axis. The contradiction shows that geodesics from the origin to points in the fourth quadrant are the same in \((\mathbb{Z}^2, \mathcal{S}(0, \infty, 1))\) as in
(\mathbb{Z}^2, \mathcal{E}_4). The plane can be seen as the union of six cones centred on the origin and bounded by the axes and the diagonal \( y = x \). For each cone, the Hilbert basis is a pair of points nearest to the origin on the bounding rays. The proof now follows that of Theorem 4.1.

The problem of characterizing the metric subgraphs of the remaining Melter–Tomescu graphs \((\mathbb{Z}^2, \mathcal{E}(0, 1, -1))\) and \((\mathbb{Z}^2, \mathcal{E}(\infty, 1, -1))\) is more delicate. Since the two graphs are obtained from one another by reflection in a diagonal it is sufficient to give the details of a characterization of the metric subgraphs of the second one. In this case, the circles for the metric are not convex polygons. In each of the cones \(|y_2 - y_1| \geq |x_2 - x_1|\) the metric associated with \((\mathbb{Z}^2, \mathcal{E}(\infty, 1, -1))\) is \(d_8\). The conditions (i) and (ii) in Theorem 5.3 are modifications of the near-diagonal convexity conditions of Theorem 5.1 to cover only near-diagonals in these cones. In each of the cones, \(|y_2 - y_1| \leq |x_2 - x_1|\), the metric associated with \((\mathbb{Z}^2, \mathcal{E}(\infty, 1, -1))\) is \(d_8\) when \(|y_2 - y_1| + |x_2 - x_1|\) is even and \(d_8 + 1\) when \(|y_2 - y_1| + |x_2 - x_1|\) is odd. The metric is not positive linear homogeneous in these cones. Condition (iii) in Theorem 5.3 covers geodesics in these cones.

**Theorem 5.3.** A subgraph \(\mathcal{H} = (\mathcal{V}', \mathcal{E}')\) of the graph \(\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}(\infty, 1, -1))\) is a metric subgraph if and only if it is a diagonally convex, connected, induced subgraph which satisfies the following conditions.

(i) If \((x_1, y_1)\) and \((x_2, y_2)\) are in \(\mathcal{V}'\) with \(x_1 < x_2\) and \(x_2 - x_1 = y_2 - y_1 - 1\) then \(\mathcal{V}'\) contains \((x_1 + 1, y_1 + 1)\) or \((x_2 - 1, y_2 - 1)\).

(ii) If \((x_1, y_1)\) and \((x_2, y_2)\) are in \(\mathcal{V}'\) with \(x_1 < x_2\) and \(x_2 - x_1 = y_1 - y_2 - 1\) then \(\mathcal{V}'\) contains \((x_1 + 1, y_1 - 1)\) or \((x_2 - 1, y_2 + 1)\).

(iii) If \((x_1, y_1)\) and \((x_2, y_2)\) are in \(\mathcal{V}'\) with \(x_1 < x_2\) and \(|y_2 - y_1| < x_2 - x_1\) then \(\mathcal{V}'\) contains one of \((x_1 + 1, y_1 + 1), (x_1 + 1, y_1 - 1), (x_2 - 1, y_2 + 1), (x_2 - 1, y_2 - 1)\).

The proof of the theorem is preceded by a lemma.

**Lemma 5.1.** Suppose that a subgraph \(\mathcal{H} = (\mathcal{V}', \mathcal{E}')\) of the graph \(\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}(\infty, 1, -1))\) satisfies the conditions of Theorem 5.3. If \(v' = (x', y')\) and \(v'' = (x'', y'')\) are in \(\mathcal{V}'\) with \(x' < x''\) and \(|y'' - y'| \leq x'' - x' + 1\) then there is a path \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\) in \(\mathcal{V}'\) joining \(v'\) to \(v''\) such that \(x_0 \leq x_1 \leq \cdots \leq x_n\) with at most one equality.

**Proof.** The lemma will be proved by induction on \(x'' - x'\). First suppose that \(x'' - x' = 1\). If \(|y'' - y'| = 0\) then \(v'\) and \(v''\) are the ends of a path of length 1 in \(\mathcal{V}'\). For \(|y'' - y'| = 2\), conditions (i) and (ii), ensure that \(v'\) and \(v''\) are joined by a path of length 2 with either \(x_0 = x_1 < x_2\) or \(x_0 < x_1 = x_2\). Now suppose that there is a path of the required type joining \(v'\) and \(v''\) when \(x'' - x' < t\), and let \(v'\) and \(v''\) be vertices in \(\mathcal{V}'\) with \(x'' - x' = t\). The conditions ensure that \(\mathcal{V}'\) contains vertices \(V' = (X', Y')\) and \(V'' = (X'', Y'')\) such that either \((x', y') = (X', Y')\) and \(X'' = x'' - 1\) with \(|Y'' - Y'| \leq X'' - X' + 1\), or \((x'', y'') = (X'', Y'')\) and
\( X' = x' + 1 \) with \( |Y'' - Y'| \leq X'' - X' + 1 \). Since by the inductive assumption \( Y'' \) and \( Y'' \) can be joined in \( \mathcal{V}' \) by a path of the required type, so also can \( v' \) and \( v'' \).

**Proof of Theorem 5.3.** Melter and Tomescu [9] have shown that a path \( \alpha = (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n) \) of length \( n \) in \( \mathcal{G} = (Z^2, \mathcal{E}((\infty, 1, -1)) \) with \( x_0 \leq x_n \) is a geodesic if and only if \( y_0 < y_1 < \cdots < y_n \) or \( y_0 > y_1 > \cdots > y_n \) or \( x_0 \leq x_1 \leq \cdots \leq x_n \) with at most one equality. Let the number of edges of type \( g \) in the geodesic \( \alpha \) be \( k(\alpha, g) \). Then \( k(\alpha, \infty) + k(\alpha, 1) + k(\alpha, -1) = n \) with \( k(\alpha, \infty) \leq 1 \). Suppose first that \( \mathcal{H} \) is a metric subgraph of \( \mathcal{G} \). Then it must be connected and the set \( \mathcal{E}' \) must be induced from the set \( \mathcal{V}' \). If \( \alpha \) is a geodesic with \( y_0 < y_1 < \cdots < y_n \) then \( y_n - y_0 = n \) and \( x_n - x_0 = k(\alpha, 1) - k(\alpha, -1) \) with \( x_n - x_0 = n \) if \( k(\alpha, 1) = n \) and \( x_n - x_0 \leq n - 2 \) if \( k(\alpha, -1) \geq 1 \). If \( y_0 > y_1 > \cdots > y_n \) then \( y_0 - y_n = n \) and \( x_n - x_0 = k(\alpha, -1) - k(\alpha, 1) \) with \( x_n - x_0 = n \) if \( k(\alpha, 1) = n \) and \( x_n - x_0 \leq n - 2 \) if \( k(\alpha, 1) \geq 1 \). If \( x_0 \leq x_1 \leq \cdots \leq x_n \) with at most one equality then \( x_n - x_0 = k(\alpha, 1) + k(\alpha, -1) = n - k(\alpha, \infty) \) and \( |y_n - y_0| = |k(\alpha, 1) - k(\alpha, -1) + k(\alpha, \infty)| \). It follows that if \( x_n - x_0 = y_n - y_0 \) then \( k(\alpha, 1) = n \) and if \( x_n - x_0 = y_n - y_0 - 1 \) then \( k(\alpha, 1) = n - 1 \) and \( k(\alpha, \infty) = 1 \). Thus \( \mathcal{H} \) is diagonally convex and satisfies conditions (i) and (ii). It follows also that if \( |y_1 - y_2| < x_2 - x_1 \) then \( k(\alpha, \infty) \leq 1 \) so that \( \mathcal{H} \) also satisfies condition (iii). Hence all the conditions are necessary.

Now suppose that \( \mathcal{H} \) satisfies the conditions but is not a metric subgraph of \( \mathcal{G} = (Z^2, \mathcal{E}((\infty, 1, -1)) \). As in the proof of Theorem 3.1 there is a path \( \alpha = \{v_0, v_1, \ldots, v_n\} \) of length \( n > 2 \) which is a geodesic in \( \mathcal{H} \) but not in \( \mathcal{G} \) and for which \( v_0, v_1, \ldots, v_{n-1} \) and \( v_1, v_2, \ldots, v_n \) are geodesics in \( \mathcal{G} \) as well as in \( \mathcal{H} \). Let \( v_i = (x_i, y_i) \) for \( 0 \leq i \leq n \). The Melter–Tomescu conditions for geodesics ensure that one cannot have both \( y_0 < y_1 < \cdots < y_n \) and \( y_1 < y_2 < \cdots < y_n \) or both of these with all the inequalities reversed. Suppose that both \( x_0 \leq x_1 \leq \cdots \leq x_n \) with at most one equality and \( x_1 \leq x_2 \leq \cdots \leq x_n \) with at most one equality. By assumption one does not have \( x_0 \leq x_1 \leq \cdots \leq x_n \) with at most one equality. Hence \( x_0 = x_1 < x_2 < \cdots < x_{n-1} = x_n \) so that \( x_n - x_0 = n - 2 \) and \( |y_n - y_0| \leq n - 1 \). Now suppose that one but not both subsequences of \( y \) values is strictly monotone and one but not both of the subsequences of \( x \) values is increasing with at most one equality. For example, suppose that \( y_0 < y_1 < \cdots < y_{n-1} \) and \( y_n = y_0 = y_{n-1} = \cdots = y_1 \) and that \( x_1 \leq x_2 \leq \cdots \leq x_n \) with at most one equality but not \( x_0 \leq x_1 \leq \cdots \leq x_{n-1} \) with at most one equality. Then either \( x_0 > x_1 \) or \( x_0 = x_1 \) and \( x_i = x_{i+1} \) for some \( i \) such that \( 1 \leq i \leq n - 2 \). In the first case, \( x_n - x_0 = n - 2 \) or \( n - 3 \) while \( y_n - y_0 = n - 2 \). In the second case, \( x_n - x_0 = n - 2 \) while \( y_n - y_0 = n - 2 \). In all these cases, it follows from Lemma 5.1 that there is a path in \( \mathcal{V}' \) from \((x_0, y_0)\) to \((x_n, y_n)\) which is a geodesic in \( \mathcal{G} \). Thus \( d_{\mathcal{H}}(v_0, v_n) = d_{\mathcal{G}}(v_0, v_n) < n \). The contradiction proves the theorem. \( \square \)

**References**