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ON THE EDGE-CHROMATIC NUMBER OF A GRAPH*

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Abstract. V.G. Vizing has shown that the edge-chromatic number of any graph with maximum vertex-degree ρ is equal to either ρ or $\rho + 1$. In this paper, we describe various ways of constructing graphs whose edge-chromatic number is $\rho + 1$ and formulate a conjecture about such graphs.

If G is a graph containing no loops or multiple edges, then the *edge-chromatic number* $\chi_e(G)$ of G is defined to be the least number of colours needed to colour the edges of G in such a way that no two adjacent edges have the same colour. It follows immediately that if ρ denotes the maximum vertex-degree of G, then $\chi_e(G) \geq \rho$. A considerably stronger result was obtained in 1964 by Vizing [2], who proved that $\chi_e(G)$ can assume only the values ρ and $\rho + 1$. (For a proof of this result in English, see [1].) We shall say that G is a *graph of class 1* if $\chi_e(G) = \rho$, and that G is a *graph of class 2* if $\chi_e(G) = \rho + 1$.

The classification of graphs into class 1 and class 2 is still an unsolved problem, although it is not difficult to prove [3] that all bipartite graphs, wheels, even complete graphs (K_{2n}) and even circuit graphs are graphs of class 1. It seems that there are in fact relatively

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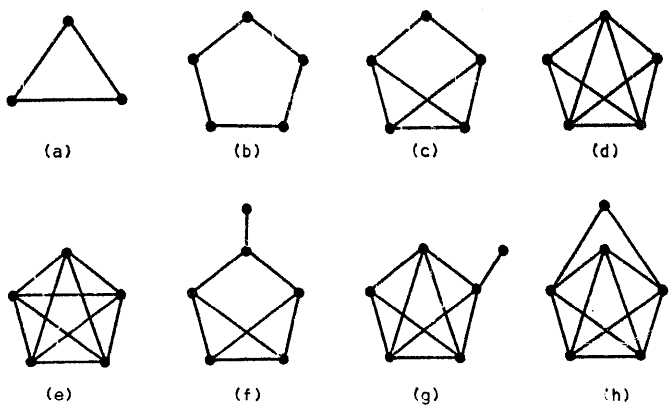


Fig. 1.

few graphs of class 2; for example, of the 143 connected graphs containing not more than six vertices, only eight are of class 2 (see Fig. 1).

In this paper, we describe some simple methods for constructing graphs of class 2 and also mention some important graphs of this type. We conclude with a few conjectures on so-called 'critical' graphs of class 2.

We begin with the following result.

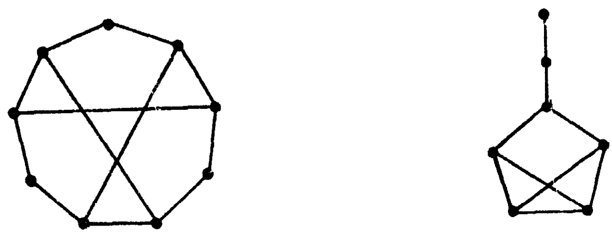
Theorem 1. *Let G be a graph with $n = 2s + 1$ vertices, m edges and with maximum degree ρ; then G is of class 2 if $m > s\rho$.*

Proof. If G is of class 1, then any proper ρ-colouring of the edges of G partitions the edges into at most ρ sets of independent edges. But the size of the largest independent set of edges in G cannot exceed $[\frac{1}{2}n]$. Hence $m \leqslant \rho[\frac{1}{2}n]$, which contradicts our hypothesis.

We observe that the hypotheses of the theorem hold when the degrees of all vertices are sufficiently close to the maximum degree.

We now show how Theorem 1 can be used to give a simple method for constructing graphs of class 2 having any specified value of ρ as maximum vertex-degree. (See Fig. 2.)

Corollary 1. *Let H be any graph with an even number of vertices which is regular of degree ρ and let G be any graph obtained from H by inserting a new vertex into any of its edges; then G is of class 2.*



(a) (b)

Fig. 2.

Proof. If H has $2s$ vertices, then G has $2s + 1$ vertices and $s\rho + 1$ edges. The result now follows from Theorem 1.

Note that the graphs (b) and (c) of Fig. 1 can both be obtained using the construction just described. We now give an alternative construction for graphs of class 2 which yields the graphs (a), (b), (d) and (e), but not (c); unlike the previous construction, it does not produce graphs of class 2 with an odd value of ρ.

Corollary 2. *Let H be any graph with an odd number of vertices which is regular of degree ρ, and let G be any graph obtained from H by deleting not more than $\frac{1}{2}\rho - 1$ edges; then G is of class 2.*

Proof. If H has $2s + 1$ vertices, then G has at least

$$\tfrac{1}{2}(2s+1)\rho - (\tfrac{1}{2}\rho - 1) = s\rho + 1$$

edges. The result follows as before.

An important special case (which includes the graphs (a), (b), (d) and (e) of Fig. 1) of Corollary 2 is obtained by splitting K_{2s+1} into s edge-disjoint Hamiltonian circuits and letting H be the union of any $\frac{1}{2}\rho$ of these circuits. For example, Fig. 3 shows the graph obtained in this way by deleting one edge from a graph for which $s = 3$ and $\rho = 4$.

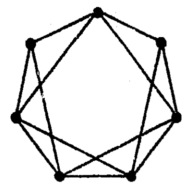


Fig. 3.

The two constructions described in Corollaries 1 and 2 may be regarded as extreme cases of the following more general theorem:

Corollary 3. *Let H be any graph with an odd number of vertices, in which one vertex has degree t and the remainder have degree ρ ($\rho \geq t$), and let G be any graph obtained from H by deleting not more than $\frac{1}{2}t-1$ edges; then G is of class 2.*

Proof. If H has $2s+1$ vertices, then G has at least

$$\tfrac{1}{2}(2s\rho + t)-(\tfrac{1}{2}t-1) = s\rho + 1$$

edges. The result follows from Theorem 1.

There are of course other methods for constructing graphs of class 2. A rather trivial way is to take any nonregular class 2 graph and adjoin a new vertex adjacent to one or more vertices whose degree is not maximum. For example, the graphs (f), (g) and (h) of Fig. 1 and the graph (b) of Fig. 2 may all be obtained in this way. The problem thus reduces to finding those class 2 graphs not obtainable in this manner. We call a graph a *class 2 vertex-critical* graph if it is of class 2 and the removal of any vertex lowers the edge-chromatic number. The first five graphs of Fig. 1 and the first graph of Fig. 2 are all vertex-critical, as are all circuits of odd length.

The determination of all vertex-critical graphs of class 2 appears to be a very difficult problem. We shall be content here with a partial result giving a method for constructing new vertex-critical graphs from old ones.

Theorem 2. *Let H be a class 2 vertex-critical graph with maximum degree ρ, m edges and* $2s + 1$ *vertices, of which k have less than the maximum degree. Let G be obtained from H by adding any set of r independent edges not in H, where*

$$r > \max \{s(\rho + 1) - m, \tfrac{1}{2}k\} .$$

Then G is a class 2 vertex-critical graph with maximum degree $\rho + 1$.

Proof. By definition, an independent set of edges is one in which no two edges have a common vertex. It follows that since $r > \frac{1}{2}k$, the addition of r independent edges increases the maximum degree by exactly 1. Hence, G has $2s + 1$ vertices, $m + r$ edges and maximum degree $\rho + 1$; and since $m + r > s(\rho + 1)$, it is of class 2 by Theorem 1. That G is vertex-critical follows immediately from the facts that H is vertex-critical and only one new colour is required for the added edges.

Repeated application of this result gives many vertex-critical graphs. In particular, since an odd circuit is vertex-critical, we have the following corollary.

Corollary 4. *If G is any graph obtained by taking a circuit of length* $2s + 1$ *and adding t sets of s independent edges* ($t \leq 2s - 2$), *then G is a class 2 vertex-critical graph.*

Some of the difficulties involved in studying class 2 vertex-critical graphs are suggested by the two graphs in Fig. 4. They appear to be very similar, but their edge-colouring properties are quite different. The first, the Petersen graph, is of class 2 whereas the second is of class 1

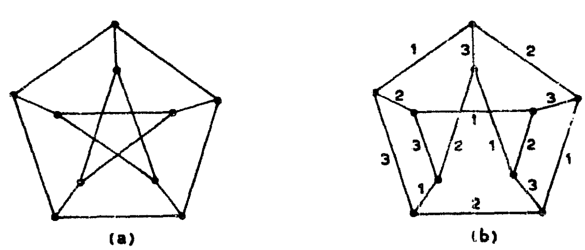


Fig. 4.

(with an edge-colouring indicated). However, the removal of any vertex from (a) still leaves a class 2 graph, that shown in Fig. 2(a). Thus, switching two edges of the Petersen graph lowers the edge-chromatic number, but deleting a vertex does not.

We conclude with a conjecture suggested by graphs (f), (g) and (h) of Fig. 1 and the vertex-critical graphs which have been indicated here.

Conjecture. *There are no class 2 vertex-critical graphs with an even number of vertices.*

Added in proof. The statement and proof of Theorem 1 are implicit in a paper of Vizing (The chromatic class of a multigraph, Cybernetics 1 (1965) 35–41), and Corollary 1 is also proved there. Our conjecture has also been stated independently by I.T. Jakobsen.

References

[1] M. Behzad and G. Chartrand, Introduction to the theory of graphs (Allyn and Bacon, Boston, Mass., 1971).
[2] V.G. Vizing, On an estimate of the chromatic class of a ρ-graph (in Russian), Diskret. Analiz. 3 (1964) 25–30.
[3] R.J. Wilson, Introduction to graph theory (Oliver and Boyd, Edinburgh, 1972).