Groups with normality conditions for non-abelian subgroups

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Abstract

Two relevant theorems of B.H. Neumann characterize groups with finite conjugacy classes of subgroups and groups in which every subgroup has finite index in its normal closure as central-by-finite groups and finite-by-abelian groups, respectively. These results have later been extended to the case of groups with similar restrictions on abelian subgroups. Moreover, Romalis and Sesekin have studied groups in which all non-abelian subgroups are normal, and in this paper we consider groups with normality conditions of Neumann’s type for non-abelian subgroups.

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1. Introduction

In a famous article of 1955, B.H. Neumann [9] proved that a group $G$ has finite conjugacy classes of subgroups if and only if the centre $Z(G)$ has finite index. In that paper, Neumann also considered groups in which every subgroup has finite index in its normal closure, and char-

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acterized such groups as those with finite commutator subgroup. The same conclusions hold if these restrictions are imposed only on abelian subgroups (see [5] and [14]). The aim of this paper is to study groups with similar normality conditions for non-abelian subgroups. Of course, the first step in this context is to consider the so-called metahamiltonian groups, i.e. groups whose non-abelian subgroups are normal. Groups with this property were introduced and investigated by G.M. Romalis and N.F. Sesekin [11–13], who proved in particular that any locally soluble metahamiltonian group has finite commutator subgroup. This result has recently been extended to the case of groups with finitely many normalizers of non-abelian subgroups (see [4]).

It follows from Neumann’s first theorem that if a group has finite conjugacy classes of subgroups, then the orders of such classes are bounded; moreover, Neumann’s second theorem yields that if $G$ is a group in which every subgroup has finite index in its normal closure, then there exists a positive integer $k$ such that $|X^G : X| \leq k$ for each subgroup $X$ of $G$. We shall prove that corresponding results hold for locally finite groups with the same normality conditions for non-abelian subgroups. Although this is no longer true in the case of non-periodic groups, in the first part of the paper it will be shown that the theorem of Romalis and Sesekin can be generalized to groups with boundedly finite conjugacy classes of non-abelian subgroups and also to groups in which the indices of non-abelian subgroups in their normal closures are boundedly finite. Removing the bound assumption, it seems to be difficult to study the structure of groups with the above normality conditions on non-abelian subgroups. However, we will give here a full description of locally nilpotent groups with these properties; in particular, it will turn out that such groups are nilpotent, and they either are finite-by-abelian or have nilpotency class at most 3.

Most of our notation is standard and can be found in [10].

2. Bounded normality conditions

Recall that the $FC$-centre of a group $G$ is the subgroup consisting of all elements of $G$ having only finitely many conjugates, and a group $G$ is called an $FC$-group if $G$ coincides with its $FC$-centre. Thus a group $G$ has the property $FC$ if and only if the centralizer $C_G(x)$ has finite index in $G$ for each element $x$; in particular, abelian-by-finite $FC$-groups are central-by-finite. We will use the monograph [15] as a general reference on $FC$-groups.

A subgroup $X$ of a group $G$ is said to be almost normal if it has finitely many conjugates, or equivalently if the normalizer $N_G(X)$ has finite index in $G$, while $X$ is called nearly normal if the index $|X^G : X|$ is finite. Thus a group has the property $FC$ if and only if all its cyclic subgroups are almost normal, and the property $FC$ is also equivalent to the requirement that every cyclic subgroup is nearly normal. On the other hand, it is easy to see that almost normality and near normality are in general incomparable conditions for a subgroup. However, it follows from Neumann’s results quoted in the introduction and from the celebrated theorem of Schur that if in a group $G$ each subgroup is almost normal, then all subgroups of $G$ are also nearly normal. Our first lemma is a slight improvement of this statement.

Lemma 2.1. Let $G$ be a group in which every subgroup is either almost normal or nearly normal. Then all subgroups of $G$ are nearly normal, and hence the commutator subgroup $G'$ of $G$ is finite.

Proof. If $x$ is any element of $G$ such that the index $|\langle x \rangle^G : \langle x \rangle|$ is finite, then $\langle x \rangle^G / \langle x \rangle_G$ is finite and so $\langle x \rangle$ is almost normal in $G$. Thus all cyclic subgroups of $G$ are almost normal and $G$ is an $FC$-group. Let $X$ be any almost normal subgroup of $G$, and let $E$ be a finitely generated subgroup of $G$ such that $G = N_G(X)E$. As $G$ is an $FC$-group, the subgroup $[E, G]$ is finite, and
hence $X$ has finite index in the normal subgroup $X[E,G]$ of $G$. In particular, $|X^G : X|$ is finite and the subgroup $X$ is nearly normal in $G$. Therefore all subgroups of $G$ are nearly normal, and $G'$ is finite by Neumann’s theorem. \ \Box

**Lemma 2.2.** Let the group $G = \langle a \rangle \wr \langle b \rangle$ be the standard wreath product of a cyclic group $\langle a \rangle$ of prime order $p$ by an infinite cyclic group $\langle b \rangle$. Then $G$ contains a non-abelian subgroup which neither is almost normal nor nearly normal.

**Proof.** The non-abelian subgroup $\langle a, b^p \rangle$ of $G$ has infinite index and

$$N_G(\langle a, b^p \rangle) = \langle a, b^p \rangle,$$

so that $\langle a, b^p \rangle$ is not almost normal. Moreover, as the normal closure of $\langle a, b^p \rangle$ has finite index in $G$, the subgroup $\langle a, b^p \rangle$ is not nearly normal. \ \Box

Recall that a group is minimax if it has a series of finite length whose factors either satisfy the minimal or the maximal condition on subgroups. A relevant theorem of P.H. Kropholler [6] proves that a finitely generated soluble group $G$ is minimax if and only if there is no prime number $p$ for which $G$ has a section isomorphic to the standard wreath product of a group of order $p$ by an infinite cyclic group. Thus it follows from Lemma 2.2 that if $G$ is any group whose non-abelian subgroups either are almost normal or nearly normal, then each finitely generated soluble subgroup of $G$ is minimax.

**Lemma 2.3.** Let $G$ be a group in which every non-abelian subgroup is either almost normal or nearly normal. Then there exists a finitely generated subgroup $E$ of $G$ such that $E' = G'$. Moreover, either $G'$ is finitely generated or $G'$ is abelian and $E$ is minimax.

**Proof.** Suppose first that $G'$ is not abelian, so that it contains a finitely generated non-abelian subgroup $H$. Since $H$ is either almost normal or nearly normal in $G$, its normal closure $H^G$ is finitely generated. Moreover, each subgroup of the factor group $G/H^G$ is either almost normal or nearly normal, and hence $G'/H^G$ is finite by Lemma 2.1. Therefore $G'$ is finitely generated, and of course in this case there exists a finitely generated subgroup $E$ of $G$ with $E' = G'$.

Assume now that $G'$ is abelian. If $K$ is a finitely generated non-abelian subgroup of $G$, it can be proved as above that $K^G$ is finitely generated and $G'/K^G/K^G$ is finite, so that $X = G'/K^G$ is likewise finitely generated. In particular, $X$ is a minimax group, so that $G'$ is minimax and hence it contains a finitely generated subgroup $A$ such that $G'/A$ is periodic. Clearly, $A$ is contained in a subgroup of $G'$ which is generated by finitely many commutators, and so the subgroup $K$ can be chosen in such a way that $A \leq K'$. Since

$$A \leq X' \leq G' \leq X$$

and the factor group $G'/X'$ is finite, there exists a finitely generated subgroup $L/X'$ of $X/X'$ such that $L' = G'$. Therefore $E = XL$ is a finitely generated (and so minimax) subgroup of $G$ and $E' = G'$. The lemma is proved. \ \Box

**Corollary 2.4.** Let $G$ be a group in which every non-abelian subgroup is either almost normal or nearly normal. If $G$ locally satisfies the maximal condition on subgroups, then its commutator subgroup $G'$ is finitely generated.
Note that the commutator subgroup of a metabelian group whose non-abelian subgroups are almost normal (or nearly normal) need not be finitely generated. In fact, let $A$ be the additive group of rational numbers whose denominators are powers of 2 and let $x$ be the automorphism of $A$ defined by $ax = 2a$ for each $a \in A$. Then $G = \langle x \rangle \rtimes A$ is a metabelian group in which every non-abelian subgroup has finite index (and so is both almost normal and nearly normal), but $G' = A$ is not finitely generated.

Combining Corollary 2.4 with the results quoted in the introduction, we obtain the following two consequences for locally finite groups.

**Corollary 2.5.** For a locally finite group $G$ the following statements are equivalent:

(i) Every subgroup of $G$ has finite index in its normal closure;

(ii) Every abelian subgroup of $G$ has finite index in its normal closure;

(iii) Every non-abelian subgroup of $G$ has finite index in its normal closure;

(iv) The commutator subgroup $G'$ of $G$ is finite.

**Corollary 2.6.** Let $G$ be a locally finite group with finite conjugacy classes of non-abelian subgroups. Then the commutator subgroup $G'$ of $G$ is finite.

**Lemma 2.7.** Let $G$ be a group with finite conjugacy classes of non-abelian subgroups, and let $X$ be a non-abelian subgroup of $G$. Then there exists a subgroup $K$ of $G$ of finite index such that $K' \leq X \leq K$.

**Proof.** The normalizer $N_G(X)$ has finite index in $G$, and each subgroup of the group $N_G(X)/X$ has finitely many conjugates. Thus the centre $K/X = Z(N_G(X)/X)$ has finite index in $N_G(X)/X$, and hence $K$ is a subgroup of finite index of $G$ with $K' \leq X \leq K$. 

Let $G$ be a group and let $\mathcal{X}$ be a set of subgroups of $G$. We say that $\mathcal{X}$ satisfies the weak minimal condition if there exists no infinite descending chain

$$X_1 > X_2 > \cdots > X_n > X_{n+1} > \cdots$$

of $\mathcal{X}$-subgroups such that the index $|X_n : X_{n+1}|$ is infinite for each $n$. R. Baer [2] and D.I. Zaïčev [16] independently proved that a soluble group is minimax if and only if the set of all its subgroups satisfies the weak minimal condition. It turns out that groups whose non-abelian subgroups are almost normal or nearly normal satisfy the weak minimal condition for some collections of commutator subgroups. If $G$ is any group, we put

$$\mathcal{C}(G) = \{X' \mid X \in \mathfrak{F}^*(G)\},$$

where $\mathfrak{F}^*(G)$ is the set of all subgroups of finite index of $G$.

**Lemma 2.8.** Let $G$ be a group in which every non-abelian subgroup is either almost normal or nearly normal. Then the set $\mathcal{C}(G)$ satisfies the weak minimal condition.

**Proof.** Assume first that $G$ is soluble-by-finite. Then all finitely generated subgroups of $G$ are minimax, and so $G'$ is minimax by Lemma 2.3. Therefore in this case the set of all subgroups
of $G'$ satisfies the weak minimal condition. Suppose now that $G$ is not soluble-by-finite, and let $X$ and $Y$ be subgroups of finite index of $G$ such that $Y' \subseteq X'$. Replacing $Y$ by its core, it can be assumed without loss of generality that $Y$ is a normal subgroup of $G$, so that $Y'$ is likewise normal in $G$. Clearly, $Y'$ is not abelian and hence every subgroup of $G/Y'$ either is almost normal or nearly normal. Thus $G'/Y'$ is finite by Lemma 2.1 and so in particular $Y'$ has finite index in $X'$. It follows that the set $C(G)$ satisfies the weak minimal condition. □

Our next result characterizes groups $G$ with finite conjugacy classes of non-abelian subgroups when the set $C(G)$ satisfies the minimal condition; in particular, it completes the description of locally finite groups with this property.

**Theorem 2.9.** Let $G$ be a group for which the set $C(G)$ satisfies the minimal condition. Then $G$ has finite conjugacy classes of non-abelian subgroups if and only if there exist a subgroup $H$ of $G$ of finite index and a finitely generated subgroup $E$ such that $H'$ is contained in every non-abelian subgroup of $G$ and $E' = G'$. Moreover, in this case there is a bound for the orders of the conjugacy classes of non-abelian subgroups of $G$.

**Proof.** Suppose first that each non-abelian subgroup of $G$ has finitely many conjugates, and among all subgroups of finite index of $G$ choose one $H$ whose commutator subgroup $H'$ is a minimal element of $C(G)$. Assume that $H'$ is not contained in some non-abelian subgroup $X$ of $G$. By Lemma 2.7 there exists a subgroup $K$ of $G$ such that $|G : K|$ is finite and $K' \subseteq X \subseteq K$; then $H \cap K$ is also a subgroup of finite index of $G$ and

$$(H \cap K)' \subseteq H' \cap X < H'.$$

This contradiction shows that $H'$ lies in all non-abelian subgroups of $G$. Moreover, it follows from Lemma 2.3 that $G$ contains a finitely generated subgroup $E$ such that $E' = G'$.

Conversely, suppose that $G$ contains a subgroup $H$ of finite index whose commutator subgroup lies in all non-abelian subgroups of $G$ and $G' = E'$ for some finitely generated subgroup $E$. Replacing $H$ by its core and $G$ by the factor group $G/(H_G)'$, it can be assumed without loss of generality that $H$ is an abelian normal subgroup of $G$. Then $G$ is abelian-by-finite and hence $G' = E'$ is polycyclic-by-finite. Assume for a contradiction that $G'$ is infinite, and let $M$ be a subgroup of finite index of $G$ such that $M'$ is minimal among all infinite elements of $C(G)$. Put $A = H \cap M$, and let $L$ be a finitely generated subgroup of $M$ such that $M = AL$. Clearly, the centralizer $C_A(L)$ is contained in $Z(M)$ and so $M/C_A(L)$ must be infinite by Schur’s theorem. In particular, $A/C_A(L)$ is infinite and so there exists $x \in L$ such that $A/C_A(x)$ is likewise infinite. As $A/C_A(x)$ is isomorphic to $[A, x]$, it is finitely generated and hence the index $|A : A^kC_A(x)|$ is finite for each positive integer $k$. Then the subgroup $X = A^kC_A(x)$ of $M$ has finite index in $G$ and

$$X' = [A^k, x] = [A, x]^k$$

is infinite, so that $[A, x]^k = M'$. It follows that $[A, x]^k = [A, x]$ for each $k > 0$, a contradiction since $[A, x]$ is a finitely generated infinite abelian group. Therefore $G'$ is finite. Put $|G'| = m$ and $|G : H'| = n$. If $X$ is any non-abelian subgroup of $G$, then $H \cap X$ is a normal subgroup of $HX$ and $|X : H \cap X| \leq n$. Thus $|XG' : H \cap X| \leq mn$, so that $X$ has finitely many conjugates in $HX$.
and $|HX : N_{HX}(X)| \leq 2^{mn}$. It follows that also the conjugacy class of $X$ in $G$ is finite, and $|G : N_{G}(X)| \leq n 2^{mn}$. The statement is proved.

We prove now the main result of this section, which describes groups with bounded normality conditions for non-abelian subgroups and generalizes the theorem of Romalis and Sesekin. Recall first that in any $n$-generator group the number of subgroups of index $k$ is boundedly finite in terms of $n$ and $k$ (see [1, Lemma 4]). As a consequence of this property, we have the following result that we state here as a lemma.

**Lemma 2.10.** Let $G$ be a group, and let $X$ be a finitely generated nearly normal subgroup of $G$. Then $X$ is almost normal in $G$. Moreover, if $X$ is $n$-generated and $|X^G : X| = k$, then $|G : N_{G}(X)| \leq k(k!)^{kn}$.

Recall also that a group $G$ is **locally graded** if every finitely generated non-trivial subgroup of $G$ contains a proper subgroup of finite index. The class of locally graded groups is quite large, and in particular it contains all locally (soluble-by-finite) groups. Of course, Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are metahamiltonian, but the assumption that the group is locally graded is enough to avoid Tarski groups and other similar pathologies.

**Theorem 2.11.** Let $G$ be a locally graded group. If there exists a positive integer $k$ such that for each non-abelian subgroup $X$ of $G$ either $|G : N_{G}(X)| \leq k$ or $|X^G : X| \leq k$, then the commutator subgroup $G'$ of $G$ is finite.

**Proof.** Let $H$ be any 2-generator non-abelian subgroup of $G$. It follows from Lemma 2.10 that there exists a positive integer $n$, depending only on $k$, such that $|G : N_{G}(H)| \leq n$. Thus $G^n \leq N_{G}(H)$, where $m = n!$. Since every non-abelian subgroup of $G$ is generated by its 2-generator non-abelian subgroups, we have that $G^m \leq N_{G}(X)$ for each non-abelian subgroup $X$ of $G$. Clearly, $G^m$ is a metahamiltonian group, so that $(G^m)'$ is finite and hence $G/G^m$ is likewise locally graded. It follows that $G/G^m$ is locally finite (see for instance [3, Lemma 4.9]). By Lemma 2.3 there exists a finitely generated subgroup $E$ of $G$ such that $G' = E'$. Replacing $G$ by $E$, we may suppose without loss of generality that $G$ is finitely generated, so that $G$ is finite-by-abelian-by-finite and hence even abelian-by-finite. Assume that the statement is false, and choose a counterexample $G$ containing a torsion-free abelian normal subgroup $A$ of finite index with smallest rank $r$. As $G$ is finitely generated and $G'$ is infinite, there exists an element $x$ of $G$ with infinitely many conjugates. Clearly, $[A^m, x] \neq [1]$ and so there is an infinite set $\pi$ of prime numbers such that $[A^{mp}, x] \neq [1]$ for every $p \in \pi$. On the other hand, $A^m$ normalizes all non-abelian subgroups of $G$, so that each $\langle x, A^{mp} \rangle$ is normal in $\langle x, A^m \rangle$. Thus

$$K = \bigcap_{p \in \pi} \langle x, A^{mp} \rangle$$

is likewise a normal subgroup of $\langle x, A^m \rangle$, and hence it is almost normal in $G$. It follows that $\langle x \rangle$ is properly contained in $K$, so that $\langle x \rangle \cap A^m \neq [1]$ and $\langle x, A^m \rangle/\langle x \rangle \cap A^m$ contains a torsion-free abelian subgroup of finite index having rank less than $r$. Therefore $\langle x, A^m \rangle/\langle x \rangle \cap A^m$ has finite commutator subgroup, and hence $x$ has finitely many conjugates in $\langle x, A^m \rangle$ and so also in $G$. This contradiction completes the proof of the theorem. □
Corollary 2.12. Let $G$ be a locally graded group with boundedly finite conjugacy classes of non-abelian subgroups. Then the commutator subgroup $G'$ of $G$ is finite.

Corollary 2.13. Let $G$ be a locally graded group. Then there exists a positive integer $k$ such that $|X^G : X| \leq k$ for each non-abelian subgroup $X$ of $G$ if and only if the commutator subgroup $G'$ of $G$ is finite.

3. Torsion-free locally nilpotent groups

The first result of this section shows in particular that if $G$ is a locally nilpotent group with finite conjugacy classes of non-abelian subgroups, then every non-abelian subgroup of $G$ is nearly normal.

Lemma 3.1. Let $G$ be a locally nilpotent group in which every non-abelian subgroup is either almost normal or nearly normal. Then all non-abelian subgroups of $G$ are nearly normal.

Proof. Let $g$ be any element of the commutator subgroup $G'$ of $G$. Consider a finitely generated non-abelian subgroup $E$ of $G$, and let $H$ be a finitely generated subgroup of $G$ such that $E \leq H$ and $g \in H'$. Suppose first that the index $|E^H : E|$ is finite. The factor group $H/E^H$ has finite commutator subgroup by Lemma 2.1, and so in particular $g^m \in E$ for some positive integer $m$. Assume now that $E$ is almost normal in $H$, and put $K = N_H(E)$. Another application of Lemma 2.1 yields that $K/E$ has finite commutator subgroup. On the other hand, $H$ is a finitely generated nilpotent group and the index $|H : K|$ is finite, so that $|H' : K'|$ is also finite, and hence also in this case we obtain that $g^m \in E$ for some $m > 0$.

Let $X$ be any non-abelian subgroup of $G$ which is almost normal, and let $X^{X^1}, \ldots, X^{X^t}$ be the conjugates of $X$ in $G$. It follows from the first part of the proof that for each $i \leq t$ there is a positive integer $m_i$ such that $g^{m_i} \in X^{X^i}$, and hence $g^k$ belongs to the core $X_G$ of $X$ for some $k > 0$. On the other hand, $G'$ is finitely generated by Corollary 2.4 and hence $G'X_G/X_G$ is finite. Thus $X$ has finite index in $XG'$ and so it is nearly normal in $G$. □

The next lemma is an elementary property of locally nilpotent groups, which is certainly well known.

Lemma 3.2. Let $G$ be a locally nilpotent group whose commutator subgroup $G'$ is finitely generated. Then $G$ is nilpotent.

Proof. As $G'$ is a finitely generated nilpotent group, also the factor group $G/C_G(G')$ is finitely generated (see [10, Part 1, Theorem 3.27]) and hence there exists a finitely generated subgroup $E$ of $G$ such that $G' = E'$ and $G = EC_G(G')$. It follows that $\gamma_i(G) = \gamma_i(E)$ for each positive integer $i$ and so $G$ is nilpotent. □

Corollary 3.3. Let $G$ be a locally nilpotent group whose non-abelian subgroups are nearly normal. Then $G$ is nilpotent.

Proof. The subgroup $G'$ is finitely generated by Corollary 2.4, and so $G$ is nilpotent by Lemma 3.2. □
Let $X$ be a subgroup of a group $G$. Recall that the isolator of $X$ in $G$ is the subset $I_G(X)$ consisting of all $g \in G$ such that $g^n$ belongs to $X$ for some positive integer $n$, and the subgroup $X$ is said to be isolated in $G$ if $I_G(X) = X$. In general, the isolator of a subgroup need not be a subgroup; however, in locally nilpotent groups isolators of subgroups are likewise subgroups. Note also that a normal subgroup $N$ of a group $G$ is isolated if and only if the factor group $G/N$ is torsion-free.

**Lemma 3.4.** Let $G$ be a locally nilpotent group whose non-abelian subgroups are nearly normal. Then the commutator subgroup of $G$ is contained in every isolated non-abelian subgroup.

**Proof.** Let $X$ be any isolated non-abelian subgroup of $G$. As the index $|X^G : X|$ is finite, we have that $X^G = X$ is normal in $G$. The factor group $G/X$ is torsion-free and all its subgroups are nearly normal, so that $G/X$ is abelian and $G'$ is contained in $X$. □

**Lemma 3.5.** Let $G$ be a torsion-free locally nilpotent group whose non-abelian subgroups are nearly normal. Then the commutator subgroup $G'$ of $G$ is a 2-generator abelian group and $G$ is nilpotent with class at most 3.

**Proof.** The group $G$ is nilpotent by Corollary 3.3. Let $A$ be a maximal abelian normal subgroup of $G$. Then $C_G(A) = A$, so that $A$ is isolated in $G$ (see [7, 2.3.8]); it follows from Lemma 3.4 that $G'$ lies in $A$ and hence $G'$ is abelian. Suppose that $G$ is not abelian, so that there exist elements $x, y$ of $G$ such that $xy \neq yx$ and $[x, y]$ commutes with both $x$ and $y$. Then every abelian subgroup of $\langle x, y \rangle$ is 2-generated, so that also all finitely generated abelian subgroups of the isolator $I_G(\langle x, y \rangle)$ are 2-generated. As $G'$ is contained in $I_G(\langle x, y \rangle)$ by Lemma 3.4, it follows that $G'$ is 2-generated. Thus $G'$ is contained in the second term of the upper central series of $G$, and so $G$ has nilpotency class at most 3. □

The following lemma applies in particular to groups with infinite cyclic commutator subgroup.

**Lemma 3.6.** Let $G$ be a group such that all non-trivial commutators have infinite order and the commutator subgroup $G'$ is finite-by-cyclic. Then all non-abelian subgroups of $G$ are nearly normal. Moreover, if the factor group $G/(G')^n$ is central-by-finite for each positive integer $n$, then all non-abelian subgroups of $G$ are almost normal.

**Proof.** Let $X$ be any non-abelian subgroup of $G$, and let $x$ and $y$ be elements of $X$ such that $xy \neq yx$. Then $[x, y]$ has infinite order and hence $X \cap G'$ is infinite. It follows that $X \cap G'$ has finite index in $G'$ and so $(G')^k$ is contained in $X$ for some $k > 0$. Thus $X$ has finite index in its normal closure since $G/(G')^k$ has finite commutator subgroup. Moreover, if $G/(G')^k$ is central-by-finite, the subgroup $X$ has obviously finitely many conjugates in $G$. □

Our next theorem characterizes torsion-free nilpotent groups of class 2 with Neumann’s type normality conditions for non-abelian subgroups.

**Theorem 3.7.** Let $G$ be a torsion-free nilpotent group of class 2.

(a) Every non-abelian subgroup of $G$ is nearly normal if and only if the commutator subgroup $G'$ of $G$ is cyclic.
(b) Every non-abelian subgroup of $G$ is almost normal if and only if $G'$ is cyclic and the factor group $G/(G')^n$ is central-by-finite for each positive integer $n$.

Proof. (a) Suppose that every non-abelian subgroup of $G$ is nearly normal, and let $x$ and $y$ be elements of $G$ such that $xy \neq yx$. Then the subgroup $Z(\langle x, y \rangle)$ is cyclic and hence the isolator $I_G(\langle x, y \rangle)$ has locally cyclic centre. On the other hand, it follows from Lemma 3.4 that $G'$ is contained in $I_G(\langle x, y \rangle)$ and so $G'$ is cyclic by Corollary 2.4. The converse statement follows directly from Lemma 3.6.

(b) Suppose first that every non-abelian subgroup of $G$ is almost normal. Then all non-abelian subgroups of $G$ are nearly normal by Lemma 3.1 and so $G'$ is cyclic by (a). Let $x$ and $y$ be elements of $G$ such that $c = [x, y] \neq 1$, and let $n$ be any positive integer. Then $[x^n, y^n] = c^{n^2} \neq 1$ and the normalizer $N_G(\langle x^n, y^n \rangle)$ has finite index in $G$. Moreover, the factor group

$$N_G(\langle x^n, y^n \rangle)/\langle x^n, y^n \rangle$$

has finite conjugacy classes of subgroups and hence it is central-by-finite. Put

$$Z/\langle x^n, y^n \rangle = Z(N_G(\langle x^n, y^n \rangle)/\langle x^n, y^n \rangle);$$

then $Z$ has finite index in $G$ and

$$Z' \leq Z(\langle x^n, y^n \rangle) = \langle c^{n^2} \rangle \leq (G')^n.$$ It follows that $G/(G')^n$ is an abelian-by-finite group with finite commutator subgroup, and so it is central-by-finite. Conversely, if $G'$ is cyclic and $G/(G')^n$ is central-by-finite for each positive integer $n$, it follows from Lemma 3.6 that all non-abelian subgroups of $G$ are almost normal. □

It follows from the above theorem that for torsion-free nilpotent groups of class 2 the finiteness of conjugacy classes of non-abelian subgroups is not equivalent to the property that all non-abelian subgroups are nearly normal. In fact, let

$$G = \langle x_n \mid n \in \mathbb{N} \rangle$$

be the torsion-free nilpotent group of class 2 with the additional relations

$$[x_m, x_n] = c \quad \text{if} \quad m \neq n,$$

where $c$ is a fixed non-trivial element of $G$. Then $G' = \langle c \rangle$ is cyclic and so every non-abelian subgroup of $G$ is nearly normal. On the other hand, for each $n > 1$ the factor group $G/(c^n)$ is not central-by-finite and hence Theorem 3.7(b) yields that $G$ contains a non-abelian subgroup with infinitely many conjugates. Note also that, if $m, n, k$ are positive integers and $m \neq n$, the non-abelian subgroup $\langle x_m, x_n^k \rangle$ has index $k$ in its normal closure. Moreover, if $H = \langle x, y \rangle$ is any 2-generator torsion-free nilpotent group of class 2, it follows from Theorem 3.7(b) that $H$ has finite conjugacy classes of non-abelian subgroups. On the other hand, for each positive integer $k$ the normalizer of the non-abelian subgroup $\langle x, y^k \rangle$ has index $k$ in $H$, and hence there is no bound for the orders of conjugacy classes of non-abelian subgroups of $H$. 
Lemma 3.8. Let $G$ be a group such that $G/Z(G)$ is polycyclic. Then every nearly normal subgroup of $G$ is almost normal.

Proof. Let $X$ be any nearly normal subgroup of $G$, and put $|X^G : X| = k$. Then $(X^G)^{k!}$ is contained in $X_G$ and in particular $X/X_G$ is periodic. As $X/X_G$ is polycyclic, it follows that $X/X_G$ is finite, so that $X^G/X_G$ is likewise finite. Therefore $X$ is almost normal in $G$. □

Lemma 3.9. Let $G$ be a torsion-free nilpotent group of class 3 whose non-abelian subgroups are nearly normal. Then $X \cap G'$ is non-cyclic for each non-abelian subgroup $X$ of $G$.

Proof. Let $X$ be any non-abelian subgroup of $G$. Then $X$ has finite index in its normal closure and all subgroups of $G/X^G$ are nearly normal, so that $G' X^G/X^G$ is finite, and so the index $|G' : X \cap G'|$ is likewise finite. On the other hand, $G'$ is not cyclic because $G$ has class 3, and hence also $X \cap G'$ is non-cyclic. □

The last result of this section describes the structure of torsion-free nilpotent groups of class 3 whose non-abelian subgroups are either almost normal or nearly normal, and so it completes the study of torsion-free locally nilpotent groups with these properties.

Theorem 3.10. Let $G$ be a torsion-free nilpotent group of class 3. The following statements are equivalent:

(i) Every non-abelian subgroup of $G$ is almost normal.
(ii) Every non-abelian subgroup of $G$ is nearly normal.
(iii) The centre $Z(G)$ is locally cyclic and $G = Z(G) K$, where $K = \langle a \rangle \rtimes H$ is a 3-generator subgroup of rank 4 and $H$ abelian.

Proof. It follows from Lemma 3.1 that (ii) is a consequence of (i). Suppose now that all non-abelian subgroups of $G$ are nearly normal. Let $x$ and $y$ be arbitrary elements of the centralizer $C = C_G(G')$, and assume that $xy \neq yx$. Then $Z(\langle x, y \rangle)$ is cyclic and in particular $\langle x, y \rangle \cap G'$ is a cyclic subgroup, which is impossible by Lemma 3.9. This contradiction shows that $C$ is abelian. It follows from Lemma 3.5 that both $G' \cap Z(G)$ and $G'/G' \cap Z(G)$ are infinite cyclic, so that $G/C$ is isomorphic to a subgroup of the group

$$\text{Hom}(G'/G' \cap Z(G), G'/G' \cap Z(G))$$

and hence it is infinite cyclic. Therefore there exists a non-trivial element $a$ of $G$ such that $G = \langle a \rangle \rtimes C$. Clearly, $G' = [C, a]$ and so the mapping $x \mapsto [x, a]$ is an epimorphism from $C$ onto $G'$ with kernel $C_C(a) = Z(G)$. Thus $C/Z(G)$ is a free abelian group of rank 2 and hence

$$C = Z(G) \times \langle u \rangle \times \langle v \rangle,$$

where $u$ and $v$ are suitable non-trivial elements of $C$ with

$$G' = \langle [u, a] \rangle \times \langle [v, a] \rangle.$$
Put $K = \langle a, u, v \rangle$, so that $G = Z(G)K$. As $G'Z(G)/Z(G)$ is cyclic, it can be assumed without loss of generality that $G'$ is contained in $Z(G) \times \langle u \rangle$. It follows that $[u, a]$ belongs to $Z(G)$, so that

$$\langle u, a \rangle = \langle a \rangle \times \langle u, [u, a] \rangle$$

has rank 3 and hence also its isolator $I_G(\langle u, a \rangle)$ has rank 3. On the other hand, $G' \leq I_G(\langle u, a \rangle)$ by Lemma 3.4 and $G' \langle u \rangle \cap \langle a \rangle = \{1\}$, so that $G' \langle u \rangle$ has rank 2. Thus $G' \langle u \rangle = \langle u \rangle \times \langle Z(G) \cap G' \langle u \rangle \rangle$ for some $b \in Z(G)$ and $[u, a] = bk$ with $k > 0$. Since $K/K \cap G' \langle u \rangle$ is a torsion-free 2-generator abelian group, it follows that $K$ has rank 4. Let $z$ be any element of $Z(G)$. Then $[uz, a] = [u, a] = bk$ and $\langle uz, a \rangle \cap G'$ cannot be cyclic by Lemma 3.9. Moreover,

$$\langle uz, a \rangle \cap G' = \langle uz, a \rangle \cap (Z(G) \langle u \rangle) \cap G' = \langle uz, b^k \rangle \cap G' = \langle \langle uz \rangle \cap G' \rangle b^k,$$

so that $\langle uz \rangle \cap G'$ cannot be trivial and hence $G' \langle u \rangle \cap \langle z \rangle \neq \{1\}$. Therefore $K \cap \langle z \rangle \neq \{1\}$ since $G' \langle u \rangle$ lies in $K$, and so $Z(G)/K \cap Z(G)$ is periodic. As

$$K \cap Z(G) = Z(K) = \langle b \rangle$$

is cyclic, it follows that $Z(G)$ is locally cyclic.

Suppose finally that $Z(G)$ is locally cyclic and $G = Z(G)K$, where $K$ is a 3-generator subgroup of rank 4, and let $X$ be any non-abelian subgroup of $G$. Clearly, $Z(G) \cap K \neq \{1\}$, so that

$$G/K \simeq Z(G)/Z(G) \cap K$$

is periodic and hence $Y = X \cap K$ is not abelian. In particular, $Y$ has rank at least 3. Assume that the index $|YK' : Y|$ is infinite. Then $YK'$ has rank 4 and so it has finite index in $K$; it follows that $Y$ itself has finite index in $K$, since $K$ is a finitely generated nilpotent group, and this contradiction shows that the index $|YK' : Y|$ is finite. Therefore $X$ has finite index in $XG'$ and so also in $X^G$, so that $X$ is almost normal in $G$ by Lemma 3.8. □

4. Mixed locally nilpotent groups

The first lemma of this section deals with the behaviour of elements of finite order in certain groups with normality conditions for non-abelian subgroups.

**Lemma 4.1.** Let $G$ be a group in which every non-abelian subgroup is either almost normal or nearly normal. If $G$ locally satisfies the maximal condition on subgroups and $G'$ is infinite, then every normal locally finite subgroup of $G$ is contained in $Z(G)$.

**Proof.** Assume for a contradiction that $N$ is a normal locally finite subgroup of $G$ which is not contained in $Z(G)$, and let $g$ be an element of $G$ such that $gx \neq xg$ for some $x \in N$. Then the subgroup $\langle x, g \rangle$ either is almost normal or nearly normal in $G$, so that $\langle x, g \rangle^G$ is finitely generated and hence it satisfies the maximal condition on subgroups. It follows that $E = \langle x \rangle^G$ is
a finite normal subgroup of $G$. Since $\langle E, g \rangle$ is either almost normal or nearly normal in $G$, the cyclic subgroup $\langle gE \rangle$ lies in the $FC$-centre of $G/E$, and so $g$ belongs to the $FC$-centre $F$ of $G$. Therefore $G \setminus F$ is contained in $C_G(N)$, and hence $G = F$ is an $FC$-group. On the other hand, the infinite subgroup $G'$ is finitely generated by Corollary 2.4 and this contradiction proves the lemma. □

Recall that a group $G$ has finite torsion-free rank if it has a series of finite length whose factors either are periodic or infinite cyclic. The number $r_0(G)$ of infinite cyclic factors in such a series is an invariant, called the torsion-free rank of $G$.

**Lemma 4.2.** Let $G$ be a nilpotent group whose non-abelian subgroups are nearly normal, and let $T$ be the subgroup consisting of all elements of finite order of $G$. If the commutator subgroup of $G/T$ is not cyclic, then $G'$ is torsion-free.

**Proof.** As the commutator subgroup of $G/T$ is not cyclic, it follows from Theorem 3.7(b) that $G/T$ has class 3, so that Theorem 3.10 yields that

$$G/T = \langle aT \rangle \ltimes H/T,$$

where $H/T$ is an abelian normal subgroup. Let $M$ be a maximal torsion-free normal subgroup of $G$. Then $G/M$ has no non-trivial torsion-free normal subgroups, and in particular its centre is periodic. Moreover, $G'M/M$ is finite by Corollary 2.4, so that $G/M$ is periodic over its centre and hence $G/M$ itself is a periodic group. Thus $G/T$ is periodic over $MT/T \cong M$ and so also $M$ has class 3. The commutator subgroup $M'$ of $M$ is a free abelian group of rank 2 by Lemma 3.5, and hence $N = M' \cap Z(M)$ is an infinite cyclic normal subgroup of $G$. Let $X$ be any non-abelian subgroup of $G$ such that $X'$ is periodic, and let $x$ and $y$ be elements of $X$ such that $xy \neq yx$. As $T$ is contained in $Z(G)$ by Lemma 4.1, it follows that $\langle x, y \rangle$ has torsion-free rank 2. Put $\bar{G} = G/N$, so that in particular $\bar{G}'$ is infinite and $\bar{x}\bar{y} \neq \bar{y}\bar{x}$. A second application of Lemma 4.1 yields that also $\langle \bar{x}, \bar{y} \rangle$ has torsion-free rank 2. Thus $(x, y) \cap N = \{1\}$. On the other hand, all subgroups of $G/\langle x, y \rangle^G$ are nearly normal and so $G'(x, y)^G/\langle x, y \rangle^G$ is finite; as the index $|\langle x, y \rangle^G : \langle x, y \rangle|$ is finite, it follows that $\langle x, y \rangle \cap G'$ has finite index in $G'$, which is impossible. This contradiction shows that every non-abelian subgroup of $G$ has non-periodic commutator subgroup, and so in particular $H$ is abelian. Clearly, $G = \langle a \rangle \ltimes H$ and hence

$$G' = [H, a] = \{[h, a] \mid h \in H\}.$$ 

Let $h$ be any element of $H$ such that $[h, a]$ has finite order; then $\langle h, a \rangle'$ is finite, so that $\langle h, a \rangle$ is abelian and $[h, a] = 1$. Therefore $G'$ is torsion-free. □

**Theorem 4.3.** Let $G$ be a nilpotent group and let $T$ be the subgroup consisting of all elements of finite order of $G$. If the commutator subgroup of $G/T$ is not cyclic, then the following statements are equivalent:

(i) Every non-abelian subgroup of $G$ is almost normal.

(ii) Every non-abelian subgroup of $G$ is nearly normal.

(iii) $G$ can be embedded into the direct product of a periodic abelian group and a torsion-free nilpotent group of class 3 whose non-abelian subgroups are almost normal.
Proof. It follows from Lemma 3.1 that (ii) is a consequence of (i). Suppose now that all non-abelian subgroups of $G$ are nearly normal. The commutator subgroup $G'$ of $G$ is torsion-free by Lemma 4.2, and we may consider a maximal torsion-free subgroup $M$ of $G$ containing $G'$; then $M$ is normal in $G$ and $G/M$ is a periodic abelian group. If $T$ is the subgroup consisting of all elements of finite order of $G$, the factor group $G/T$ is a torsion-free nilpotent group whose non-abelian subgroups are nearly normal. Moreover, it follows from Theorem 3.7(b) that $G/T$ has class 3 and all its non-abelian subgroups are almost normal. As $M \cap T = \{1\}$, the group $G$ can be embedded into the direct product of $G/M$ and $G/T$.

Suppose finally that $G$ can be embedded into a direct product $W = A \times B$, where $A$ is a periodic abelian group and $B$ is a torsion-free nilpotent group of class 3 whose non-abelian subgroups are almost normal. It follows from Theorem 3.10 that $B/Z(B)$ is finitely generated, so that also $W/Z(W)$ is finitely generated and hence by Lemma 3.8 it is enough to show that each non-abelian subgroup $X$ of $W$ is nearly normal. As $X$ is contained in $X A \cap X B = (A \cap X B)(B \cap X A)$,

the subgroup $Y = B \cap X A$ cannot be abelian; thus $Y$ has finite index in $Y^B$ and all subgroups of $B/Y^B$ are nearly normal, so that $B/Y^B$ has finite commutator subgroup and hence also the index $|B' : B' \cap Y|$ is finite. Let $w$ be any element of $W' \cap Y = B' \cap Y$, and write $w = xa$, where $x \in X$ and $a \in A$, so that in particular $a$ has finite order $n$. Then $w^n = x^n$ belongs to $W' \cap X$, and so $W' \cap Y$ is periodic over $W' \cap X$. It follows that the index $|W' : W' \cap X|$ is finite, so that the subgroup $X$ has finite index in $XW'$ and hence it is nearly normal in $W$.  

Corollary 4.4. Let $G$ be a locally nilpotent group whose non-abelian subgroups are nearly normal. Then either $G'$ is finite or $G$ is nilpotent with class at most 3.

Proof. Suppose that $G'$ is infinite. It follows from Lemma 4.1 that the subgroup $T$ consisting of all elements of finite order of $G$ lies in $Z(G)$. If the commutator subgroup of $G/T$ is not cyclic, then $G$ has class 3 by Theorem 4.3. Finally, if $G'T/T$ is cyclic, it is contained in $Z(G/T)$, and hence $G$ has nilpotency class at most 3.  

As the commutator subgroup of a nilpotent group with almost normal or nearly normal non-abelian subgroups is finitely generated by Corollary 2.4, it follows from Theorems 2.9 and 4.3 that in order to complete the study of mixed nilpotent groups with such properties, it is now enough to consider the case of groups whose commutator subgroup is periodic-by-(infinite cyclic).

Theorem 4.5. Let $G$ be a nilpotent group such that all non-trivial commutators have infinite order and the commutator subgroup $G'$ is periodic-by-cyclic. Then all non-abelian subgroups of $G$ are nearly normal if and only if $G'$ is finitely generated.

Proof. If every non-abelian subgroup of $G$ is nearly normal, it follows from Corollary 2.4 that $G'$ is finitely generated. Conversely, if $G'$ is finitely generated, then it is finite-by-cyclic and hence all non-abelian subgroups of $G$ are nearly normal by Lemma 3.6.  

Lemma 4.6. Let $G = \langle x, y \rangle$ be a 2-generator nilpotent group of class 2. If the commutator $[x, y]$ has infinite order, then $G$ is torsion-free.
Proof. Let $T$ be the subgroup consisting of all elements of finite order of $G$. Then $\tilde{G} = G/T$ is a torsion-free non-abelian group, and so $\tilde{G}/Z(\tilde{G})$ is a free abelian group of rank 2. It follows that all factors of the normal series

$$\{1\} \triangleleft \langle [x, y] \rangle \triangleleft \langle x, [x, y] \rangle \triangleleft G$$

must be infinite, and hence $G$ is torsion-free. □

Let $G$ be a group, and let $x$ and $y$ be elements of $G$ such that $xy \neq yx$. We shall say that $[x, y]$ is a maximal commutator if the subgroup $\langle [x, y] \rangle$ is maximal among all cyclic subgroups generated by commutators of $G$.

Lemma 4.7. Let $G$ be a nilpotent group such that all non-trivial commutators have infinite order and the commutator subgroup $G'$ is periodic-by-(infinite cyclic). Then $G$ has class 2. Moreover, if $x$ and $y$ are elements of $G$ such that $[x, y]$ is a maximal commutator, then $G = \langle x, y \rangle C_G(\langle x, y \rangle)$.

Proof. As $G'$ is periodic-by-(infinite cyclic), the subgroup $[G', G]$ is periodic, so that $[G', G] = \{1\}$ and $G$ has class 2. Let $g$ be any element of $G$. Then

$$[G, g] = \{[u, g] \mid u \in G\},$$

so that $[G, g]$ is torsion-free and hence either $[G, g] = \{1\}$ or $[G, g]$ is infinite cyclic. Clearly, $C_G(x)$ is normal in $G$ and

$$G/C_G(x) \simeq [G, x]$$

is infinite cyclic, so that $G = \langle z \rangle \ltimes C_G(x)$ for some $z \in G$, and $y = z^m a$ with $m \neq 0$ and $a \in C_G(x)$. Thus

$$[x, y] = [x, z^m a] = [x, z]^m$$

and so $m = \pm 1$. It follows that $G = \langle y \rangle \ltimes C_G(x)$. Moreover, if $X = \langle x, y \rangle$, the group

$$C_G(x)/C_G(X) \simeq [C_G(x), y]$$

is infinite cyclic, and hence $C_G(x) = \langle g \rangle \ltimes C_G(X)$ for some $g \in G$. Then $x = g^n b$ with $n \neq 0$ and $b \in C_G(X)$, so that

$$[x, y] = [g^nb, y] = [g, y]^n$$

and $n = \pm 1$. Therefore $C_G(x) = \langle x \rangle \ltimes C_G(X)$ and

$$G = \langle y \rangle \ltimes (\langle x \rangle \ltimes C_G(X)) = X C_G(X).$$

The lemma is proved. □
Theorem 4.8. Let $G$ be a nilpotent group such that all non-trivial commutators have infinite order and the commutator subgroup $G'$ is periodic-by-(infinite cyclic). Then $G$ has finite conjugacy classes of non-abelian subgroups if and only if $G'$ is infinite cyclic and $G/(G')^n$ is central-by-finite for each positive integer $n$.

Proof. Suppose that all non-abelian subgroups of $G$ are almost normal. Let $u$ and $v$ be elements of $G$ such that $uv \neq vu$. Since $G$ has class 2 by Lemma 4.7 and the commutator $[u, v]$ has infinite order, the subgroup $(u, v)$ is torsion-free by Lemma 4.6. It follows from Lemma 2.7 that $G$ contains a subgroup $K$ of finite index such that $K' \leq (u, v) \leq K$, so that $K'$ is torsion-free (and hence infinite cyclic). Among all non-abelian subgroups of finite index of $G$ with torsion-free commutator subgroup, choose a maximal one $M$. Assume for a contradiction that $G'$ is not infinite cyclic, so that $M \neq G$ and replacing $G$ by a suitable subgroup we may suppose without loss of generality that $M$ is a maximal subgroup of $G$. Let $g$ be any element of $G \setminus M$. Then $G = M\langle g \rangle$ and $G' = M'[M, g]$. As $G'$ is not cyclic, $[M, g]$ is not contained in $M'$; moreover,

$$[M, g] = \{ [a, g] \mid a \in M \}$$

is torsion-free and so infinite cyclic. It follows that there exist elements $x \in M$ and $y \in G \setminus M$ such that $[x, y]$ is a maximal commutator of $G$; in particular $[M, y] = \langle [x, y] \rangle$. Put $X = \langle x, y \rangle$, so that $G = XCG(X)$ by Lemma 4.7. Assume that $CG(X)$ is not contained in $M'$; then $G = MCG(X)$ and $y = az$, with $a \in M$ and $z \in CG(X)$, so that $[x, y] = [x, a]$ belongs to $M'$, a contradiction. Therefore $CG(X)$ is a subgroup of $M$, so that $G'(X)'$ is cyclic and there are elements $h$ and $k$ of $CG(X)$ such that $CG(X)' = \langle [h, k] \rangle$ (see [8]). Thus

$$G' = X'C_G(X)' = \langle [x, y], [h, k] \rangle,$$

and for each element $g$ of $G'$ there exist integers $r$ and $s$ such that

$$g = [x, y]^r[h, k]^s = [x^r, y][h^s, k] = [x^r h^s, y k].$$

It follows that $G'$ is torsion-free, and hence infinite cyclic. This contradiction proves that $G'$ is infinite cyclic. Let $T$ be the subgroup consisting of all elements of finite order of $G$. Application of Theorem 3.7(b) yields that for each positive integer $n$ the factor group $G/(G')^n T$ is central-by-finite. As $G' \cap (G')^n T = (G')^n$, it follows that $G/(G')^n$ is likewise central-by-finite.

Conversely, if $G'$ is infinite cyclic and $G/(G')^n$ is central-by-finite for each positive integer $n$, then all non-abelian subgroup of $G$ are almost normal by Lemma 3.6. \qed

Lemma 4.9. Let $G$ be a locally polycyclic group whose non-abelian subgroups are nearly normal. If $G$ contains an element $x$ such that the centralizer $C_G(x)$ is an abelian normal subgroup of $G$ and $G/C_G(x)$ is polycyclic, then $G/Z(G)$ is polycyclic.

Proof. Let $E$ be a finitely generated subgroup of $G$ such that $G = EC_G(x)$. It can obviously be assumed that $G$ is not abelian, so that $E$ is not contained in $C_G(x)$. Thus $\langle E, x \rangle$ is a non-abelian subgroup of $G$ and hence the index $|\langle E, x \rangle^G : \langle E, x \rangle|$ is finite. It follows that $\langle E, x \rangle^G$ is finitely generated, and so $E^G$ is polycyclic. Therefore $G/C_G(E^G)$ is likewise polycyclic (see [10, Part 1, Theorem 3.27]). As $C_G(x) \cap C_G(E^G) \subseteq Z(G)$, the factor group $G/Z(G)$ is polycyclic. \qed
Our last result completes the description of nilpotent groups whose non-abelian subgroups either are almost normal or nearly normal.

**Theorem 4.10.** Let $G$ be a nilpotent group whose commutator subgroup $G'$ is periodic-by-(infinite cyclic). If $G$ contains a non-trivial commutator of finite order, then the following statements are equivalent:

(i) Every non-abelian subgroup of $G$ is almost normal.
(ii) Every non-abelian subgroup of $G$ is nearly normal.
(iii) $G$ has torsion-free rank 3, the subgroup $T$ consisting of all elements of finite order of $G$ is contained in $Z(G)$, $Z(G)/T$ is locally cyclic and $G/Z(G)$ is finitely generated.

**Proof.** It follows from Lemma 3.1 that (ii) is a consequence of (i). Suppose now that all non-abelian subgroups of $G$ are nearly normal. As $G'$ is infinite, the subgroup $T$ consisting of all elements of finite order of $G$ is contained in $Z(G)$ by Lemma 4.1. Let $x$ and $y$ be elements of $G$ such that $xy \neq yx$ and $z = [x, y]$ has finite order. Clearly, $x$ and $y$ have infinite order and $\langle x, y \rangle$ has class 2; moreover, $(x, y)/(x, y) \cap T$ cannot be cyclic, so that it is free abelian of rank 2 and

$$\langle x, y \rangle = \langle y \rangle \ltimes (\langle x \rangle \times \langle z \rangle).$$

At least one of the subgroups $G' \cap \langle x \rangle$ and $G' \cap \langle y \rangle$ must be trivial, and without loss of generality it can be assumed that $G' \cap \langle x \rangle = \langle 1 \rangle$. Put $C/T = C_{G/T}(xT)$, so that $\langle x, y \rangle G'$ is clearly contained in $C$. Assume that $C/\langle x, y \rangle G'$ has an element $c \langle x, y \rangle G'$ of infinite order. As

$$[x, yc] = [x, c][x, y] \in T \leq Z(G),$$

the subgroup $\langle x, yc \rangle$ has class at most 2. If $[x, yc] \neq 1$, the index $|\langle x, yc \rangle^G/\langle x, yc \rangle|$ is finite and the factor group $G/\langle x, yc \rangle^G$ has finite commutator subgroup; then $G' \cap \langle x, yc \rangle$ has finite index in $G'$ and hence there exist integers $r, s, t$ such that $[x, yc]^t x^s(yc)^t$ is an element of infinite order of $G'$. It follows that $x^s(yc)^t$ is a non-trivial element of $G'$, so that $t \neq 0$ and $(yc)^t = x^{-s}u$ with $u \in G'$; moreover, $(yc)^t = y^t c^t v$ for some $v \in G'$ and so $y^t c^t v = x^{-s} u$. Thus $c^t = y^{-r} x^{-s} uv^{-1}$ belongs to $\langle x, y \rangle G'$, and this contradiction shows that $[x, yc] = 1$. Therefore $z = [x, y] = [x, c]^{-1}$ and

$$\langle x, c \rangle = \langle c \rangle \ltimes (\langle x \rangle \times \langle z \rangle).$$

As above, the index $|G' : G' \cap \langle x, c \rangle|$ is finite, and so there are integers $r, s, t$ such that $z^r x^s c^t$ is an element of infinite order of $G'$; thus $x^s c^t$ is a non-trivial element of $G'$, so that $t \neq 0$ and $c^t$ belongs to $\langle x, y \rangle G'$. This new contradiction proves that the factor group $C/\langle x, y \rangle G'$ is periodic. Since the index $|G' : G' \cap \langle x, y \rangle|$ is finite, it follows that

$$r_0(C) = r_0(\langle x, y \rangle G') = r_0(\langle x, y \rangle) = 2.$$

Obviously, $G' T / T$ is contained in the centre of $G / T$, so that the map

$$\varphi : g \in G \mapsto [x, g] T \in G' T / T.$$
is a homomorphism with kernel $C$ and $G/C$ is cyclic. As $\tilde{G} = G/T$ is not abelian, $G$ cannot have torsion-free rank 2 so that $r_0(G) = 3$ and $\tilde{G}/Z(\tilde{G})$ is a torsion-free abelian group of rank 2. Thus $Z(\tilde{G})$ has rank 1 and in particular $Z(G)/T$ is locally cyclic. Moreover, as the commutator $[x, y]$ has finite order, the group $\langle x, y \rangle G'T/T$ is abelian and so the torsion-free group $C/T$ is likewise abelian; application of Lemma 4.9 yields that $\tilde{G}/Z(\tilde{G})$ is finitely generated. Put $Z(\tilde{G}) = A/T$. Then $G/A$ is a free abelian group of rank 2 and hence there are elements $u, v$ of $G$ such that $G = \langle u, v, A \rangle$. Clearly, $\langle A, u \rangle/T$ is abelian, so that $[A, u]$ is contained in $T$ and hence it is finite by Corollary 2.4. It follows that $A/C_A(u)$ is finite, and similarly it can be shown that $A/C_A(v)$ is finite. As

$$C_A(u) \cap C_A(v) = Z(G),$$

it follows that $A/Z(G)$ is finite and so $G/Z(G)$ is finitely generated.

Suppose finally that the group $G$ satisfies condition (iii). Then there exists a finitely generated subgroup $E$ of $G$ such that $G = E Z(G)$. In particular, $G' = E'$ is finitely generated and hence $T \cap G'$ is finite of order $n$, say. Let $X$ be any non-abelian subgroup of $G$, and let $x$ and $y$ be elements of $X$ such that $xy \neq yx$. If $[x, y]$ has infinite order, the subgroup $\langle [x, y] \rangle$ has finite index in $G'$, so that also the index $|X G' : X|$ is finite and $X$ is nearly normal in $G$. Assume now that $[x, y]$ has finite order, so that it belongs to $Z(G)$. Consider elements $x_1, y_1$ of $E$ and $z_1, z_2$ of $Z(G)$ such that $x = x_1 z_1$ and $y = y_1 z_2$. Clearly, $[x_1, y_1] = [x, y] \in T \cap G'$, so that

$$[x_1^n, y_1] = [x_1, y_1] = 1$$

and hence $\langle x_1^n, y_1^n \rangle$ is abelian; moreover, as $\langle x_1, y_1 \rangle$ is not abelian and $T$ lies in $Z(G)$, we have $\langle x_1 \rangle \cap \langle y_1 \rangle = \{1\}$ and so in particular $\langle x_1^n, y_1^n \rangle = \langle x_1^n \rangle \times \langle y_1^n \rangle$. Put $N = E \cap T$. Since $E' = G'$ is not contained in $T$, the factor group $E/N$ is not abelian, and so there exists an element $a$ of $E' \setminus N$ such that $aN \in Z(E/N)$. Then

$$[a, E] \leq E' \cap N = E' \cap T,$$

so that

$$[a^n, E] = [a, E]^n = \{1\}$$

and $a^n \in Z(E) \leq Z(G)$. If $\langle x_1^n, y_1^n \rangle \cap \langle a \rangle = \{1\}$, the subgroup

$$\langle x_1^n, y_1^n, a^n \rangle = \langle x_1^n \rangle \times \langle y_1^n \rangle \times \langle a^n \rangle$$

is free abelian of rank 3 and so the torsion-free nilpotent group $G/T$ would be abelian. This contradiction shows that there exist integers $r, s$ such that $1 \neq x_1^{nr} y_1^{ns} \in \langle a \rangle$. Moreover, as $a^n \in Z(G)$ and $Z(G)/T$ has rank 1, $r$ and $s$ can be chosen in such a way that $z_1^{nr}$ and $z_2^{ns}$ belong to $\langle a \rangle$. Thus

$$x_1^{nr} y_1^{ns} = x_1^{nr} z_1^{nr} y_1^{ns} z_2^{ns} = (x_1^{nr} y_1^{ns}) z_1^{nr} z_2^{ns}$$

belongs to $\langle a \rangle$, and $x_1^{nr} y_1^{ns} \neq 1$ has infinite order since $\langle x \rangle \cap \langle y \rangle = \{1\}$. It follows that $\langle x, y \rangle \cap G'$ is infinite, so that the index $|X G' : X| = |G' : X \cap G'|$ must be finite. Therefore all non-abelian subgroups of $G$ are nearly normal, and Lemma 3.8 yields that the group $G$ has finite conjugacy classes of non-abelian subgroups. \(\square\)
References