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Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations

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Abstract

In this article, the homotopy perturbation method proposed by J.- H. He is adopted for solving linear fractional partial differential equations. The fractional derivatives are described in the Caputo sense. Comparison of the results obtained by the homotopy perturbation method with those obtained by the variational iteration method reveals that the present methods are very effective and convenient.

Keywords: Variational iteration method; Homotopy perturbation method; Fractional differential equations; Time-fractional diffusion wave equation; Time-fractional wave equation; Caputo derivative

1. Introduction

Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems [1–11]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Recently, the Adomian decomposition method [12–21] and variational iteration method [21–32] have been used for solving a wide range of problems. In the present paper, we use the homotopy perturbation method, proposed by He [33,34], to construct an approximate solution to linear partial differential equations of fractional order.

The homotopy perturbation method is a new approach which searches for an analytical approximate solution of linear and nonlinear problems. The homotopy perturbation method has been applied to Volterra's integro-differential equation [35], to nonlinear oscillators [36], bifurcation of nonlinear problems [37], bifurcation of delay-differential equations [38], nonlinear wave equations [39] and boundary value problems [40], and to other fields [41–50]. Recently,

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the application of the method has been extended to quadratic Riccati differential equation of fractional order [51]. In [51], Odibat and Momani modified the homotopy perturbation method, which is presented in Section 2, to solve nonlinear differential equations of fractional order. This modification reduces the nonlinear fractional differential equations to a set of linear ordinary differential equations.

Our aim in this paper is to compare the homotopy perturbation method [33,34] with the variational iteration method [21] for solving the linear time-fractional partial differential equation of the form [11,21]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + L[x]u = q(x,t), \quad t > 0, x \in R, \tag{1.1}$$

where $L[x]$ is the linear differential operator

$$L[x] = a_0(x)u + a_1(x)\frac{\partial}{\partial x} + a_2(x)\frac{\partial^2}{\partial x^2} + a_3(x)\frac{\partial^3}{\partial x^3} + \cdots + a_n(x)\frac{\partial^n}{\partial x^n}, \tag{1.2}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0) = f(x), \quad 0 < \alpha \le 1,\\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty, t > 0, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), 1 < \alpha \le 2,\\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty, t > 0, \end{aligned} \tag{1.4}$$

where a_i $(i = 0, 1, \ldots, n)$, $f(x)$, $g(x)$, and $q(x,t)$ all are continuous functions and α is a parameter describing the order of the time-fractional derivative in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. Obviously, the integer-order linear partial differential equation can be viewed as a special case of the more general fractional-order linear partial differential equation by putting the time-fractional order of the derivative equal to unity. In other words, the ultimate behavior of the fractional system response must converge to the response of the integer order version of the equation.

There are several definitions of a fractional derivative of order $\alpha > 0$ [7–10]. The two most commonly used definitions are the Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. Riemann–Liouville fractional integration of order α is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0, x > 0. \tag{1.5}$$

The next two equations define Riemann–Liouville and Caputo fractional derivatives of order α, respectively,

$$D^\alpha f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}(J^{m-\alpha} f(x)), \tag{1.6}$$

$$D_*^\alpha f(x) = J^{m-\alpha}\left(\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x)\right), \tag{1.7}$$

where $m - 1 < \alpha \le m$ and $m \in N$. For now, the Caputo fractional derivative will be denoted by D_*^α to maintain a clear distinction with the Riemann–Liouville fractional derivative.

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [2]. In this paper, we consider the one-dimensional linear inhomogeneous fractional partial differential equations, where the unknown function $u(x,t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The fractional derivative is taken in the Caputo sense as follows:

Definition 1.1. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} \mathrm{d}\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N}. \end{cases} \tag{1.8}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

2. Variational iteration method

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in [21–32]. He [27] was the first to apply the variational iteration method to fractional differential equations. The variational iteration method for solving the time-fractional partial differential equation (1.1) has been considered in the work of Momani and Odibat [21]. For the sake of clarity, the present section gives a brief account of their approach. The variational iteration method requires that the time-fractional partial differential equation (1.1) be expressed in the form:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + a_0(x)u + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} + a_3(x)\frac{\partial^3 u}{\partial x^3} + \cdots + a_n(x)\frac{\partial^n u}{\partial x^n} = q(x,t), \quad t > 0, x \in R. \tag{2.1}$$

The correction functional for Eq. (2.1) can be approximately expressed as follows:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial^m}{\partial \xi^m} u_k(x,\xi) + a_0(x)\tilde{u}_k(x,\xi) + \cdots + a_n(x)\frac{\partial^n}{\partial x^n}\tilde{u}_k(x,\xi) - q(x,\xi) \right) \mathrm{d}\xi, \tag{2.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory; here $\tilde{u}_k$, $\frac{\partial \tilde{u}_k}{\partial x}, \ldots, \frac{\partial^n \tilde{u}_k}{\partial x^n}$ are considered as restricted variations. Making the above functional stationary, noticing that $\delta \tilde{u}_k = 0$,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \lambda(\xi) \left(\frac{\partial^m}{\partial \xi^m} u_k(x,\xi) - q(x,\xi) \right) \mathrm{d}\xi, \tag{2.3}$$

yields the following Lagrange multipliers

$$\lambda = -1, \quad \text{for } m = 1, \tag{2.4}$$

$$\lambda = \xi - t, \quad \text{for } m = 2. \tag{2.5}$$

Therefore, for $m = 1$, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) + a_0(x)u_k(x,\xi) + \cdots + a_n(x)\frac{\partial^n}{\partial x^n} u_k(x,\xi) - q(x,\xi) \right) \mathrm{d}\xi, \tag{2.6}$$

and for $m = 2$, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t) \times \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) + a_0(x)u_k(x,\xi) + \cdots + a_n(x)\frac{\partial^n}{\partial x^n} u_k(x,\xi) - q(x,\xi) \right) \mathrm{d}\xi. \tag{2.7}$$

If we begin with the initial approximation $u_0(x,t) = f(x)$, in the case of $m = 1$, and the initial approximation $u_0(x,t) = f(x) + tg(x)$, in the case of $m = 2$, then the approximations $u_n(x,t)$, for $n \geq 1$, can be completely determined. Finally, we approximate the solution of the space-fractional Burger's equation (2.1) $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ by the Nth term $u_N(x,t)$.

3. Modified homotopy perturbation method

The homotopy perturbation method, which provides an analytical approximate solution, is applied to various nonlinear problems [33–50]. In this section, we recapitulate a modification of the homotopy perturbation method introduced in [51]. To illustrate the basic ideas of the new modification, we consider the following nonlinear differential equation of fractional order:

$$D_*^{\alpha} u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, m-1 < \alpha \le m, \tag{3.1}$$

where L is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operator which also might include other fractional derivatives of order less than α, f is a known analytic function, and D_*^{α} is the Caputo fractional derivative of order α, subject to the initial conditions

$$u^k(0) = c_k, \quad k = 0, 1, 2, \ldots, m-1. \tag{3.2}$$

In view of the homotopy technique, we can construct the following homotopy:

$$u^{(m)} + L(u) - f(t) = p\left[u^{(m)} - N(u) - D_*^{\alpha} u\right], \quad p \in [0, 1], \tag{3.3}$$

or

$$u^{(m)} - f(t) = p\left[u^{(m)} - L(u) - N(u) - D_*^{\alpha} u\right], \quad p \in [0, 1]. \tag{3.4}$$

The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (3.3) becomes the linearized equation

$$\frac{\mathrm{d}^m u}{\mathrm{d}t^m} + L(u) = f(t), \tag{3.5}$$

and Eq. (3.4) becomes the linearized equation

$$\frac{\mathrm{d}^m u}{\mathrm{d}t^m} = f(t), \tag{3.6}$$

and when it is one, Eq. (3.3) or Eq. (3.4) turns out to be the original fractional differential equation (3.1). The basic assumption is that the solution of Eq. (3.3) or Eq. (3.4) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.3) or Eq. (3.4), and equating the terms with identical powers of p, we can obtain a series of linear equations of the form

$$\begin{aligned}
&p^0 : \frac{\mathrm{d}^m u_0}{\mathrm{d}t^m} + L_0(u_0) = f(t), \quad u^k(0) = c_k, \\
&p^1 : \frac{\mathrm{d}^m u_1}{\mathrm{d}t^m} + L_1(u_0, u_1) = \frac{\mathrm{d}^m u_0}{\mathrm{d}t^m} - N_0(u_0) - D_*^{\alpha} u_0, \quad u^k(0) = 0, \\
&p^2 : \frac{\mathrm{d}^m u_2}{\mathrm{d}t^m} + L_2(u_0, u_1, u_2) = \frac{\mathrm{d}^m u_1}{\mathrm{d}t^m} - N_1(u_0, u_1) - D_*^{\alpha} u_1, \quad u^k(0) = 0, \\
&p^3 : \frac{\mathrm{d}^m u_3}{\mathrm{d}t^m} + L_3(u_0, u_1, u_2, u_3) = \frac{\mathrm{d}^m u_2}{\mathrm{d}t^m} - N_2(u_0, u_1, u_2) - D_*^{\alpha} u_2, \quad u^k(0) = 0, \\
&\vdots
\end{aligned} \tag{3.8}$$

or the form

$$\begin{aligned}
p^0 &: \frac{\mathrm{d}^m u_0}{\mathrm{d}t^m} = f(t), \quad u^k(0) = c_k,\\
p^1 &: \frac{\mathrm{d}^m u_1}{\mathrm{d}t^m} = \frac{\mathrm{d}^m u_0}{\mathrm{d}t^m} - L_0(u_0) - N_0(u_0) - D_*^\alpha u_0, \quad u^k(0) = 0,\\
p^2 &: \frac{\mathrm{d}^m u_2}{\mathrm{d}t^m} = \frac{\mathrm{d}^m u_1}{\mathrm{d}t^m} - L_1(u_0, u_1) - N_1(u_0, u_1) - D_*^\alpha u_1, \quad u^k(0) = 0,\\
p^3 &: \frac{\mathrm{d}^m u_3}{\mathrm{d}t^m} = \frac{\mathrm{d}^m u_2}{\mathrm{d}t^m} - L_2(u_0, u_1, u_2) - N_2(u_0, u_1, u_2) - D_*^\alpha u_2, \quad u^k(0) = 0,\\
&\vdots
\end{aligned} \tag{3.9}$$

respectively, where the terms $L_0, L_1, L_2, \ldots$ and $N_0, N_1, N_2, \ldots$ satisfy the following equations

$$L(u_0 + pu_1 + p^2u_2 + \cdots) = L_0(u_0) + pL_1(u_0, u_1) + p^2L_2(u_0, u_1, u_2) + \cdots$$
$$N(u_0 + pu_1 + p^2u_2 + \cdots) = N_0(u_0) + pN_1(u_0, u_1) + p^2N_2(u_0, u_1, u_2) + \cdots.$$

Setting $p = 1$ in Eq. (3.7) yields the solution of Eq. (3.1). It obvious that the linear equations in (3.8) or (3.9) are easy to solve, and the components u_n, $n \geq 0$ of the homotopy perturbation method can be completely determined, and the series solutions are thus entirely determined.

Finally, we approximate the solution $u(t) = \sum_{n=0}^{\infty} u_n(t)$ by the truncated series

$$\phi_N(t) = \sum_{n=0}^{N-1} u_n(t). \tag{3.10}$$

4. Applications and results

In this section, we first implement the modified homotopy perturbation method to solve the linear fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + L[x]u = q(x, t), \quad t > 0, x \in R, \tag{4.1}$$

where $m - 1 < \alpha \leq m$, subject to the initial conditions

$$\frac{\partial^k}{\partial t^k} u(x, 0) = f_k(x), \quad k = 0, 1, 2, \ldots, m - 1. \tag{4.2}$$

In view of the modified homotopy perturbation technique, we can construct the following homotopy

$$\frac{\partial u^m}{\partial t^m} - q_1(x, t) = p\left[\frac{\partial u^m}{\partial t^m} - L[x]u - \frac{\partial^\alpha u}{\partial t^\alpha} + q_2(x, t)\right], \quad p \in [0, 1], \tag{4.3}$$

where

$$q(x, t) = q_1(x, t) + q_2(x, t). \tag{4.4}$$

Here the function $q(x, t)$ is divided into two parts, namely $q_1(x, t)$ and $q_2(x, t)$. The suggestion is that only the part $q_1(x, t)$ be assigned to the zeroth component u_0, whereas the remaining part $q_2(x, t)$ be combined with u_1.

The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (4.3) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = q_1(x, t), \tag{4.5}$$

and when it is one, Eq. (4.3) turns out to be the original fractional differential Eq. (4.1). Substituting Eq. (3.7) into Eq. (4.3), and equating the terms with identical powers of p, we obtain the following linear equations:

$$\begin{aligned}
&p^0: \frac{\partial^m u_0}{\partial t^m} = q_1(x,t), \qquad \frac{\partial^k}{\partial t^k}u_0(x,0) = f_k(x), \quad k = 0,1,2,\ldots,m-1,\\
&p^1: \frac{\partial^m u_1}{\partial t^m} = \frac{\partial^m u_0}{\partial t^m} - L[x]u_0 - \frac{\partial^\alpha u_0}{\partial t^\alpha} + q_2(x,t), \qquad \frac{\partial^k}{\partial t^k}u_1(x,0) = 0, \quad k = 0,1,2,\ldots,m-1,\\
&p^2: \frac{\partial^m u_2}{\partial t^m} = \frac{\partial^m u_1}{\partial t^m} - L[x]u_1 - \frac{\partial^\alpha u_1}{\partial t^\alpha}, \qquad \frac{\partial^k}{\partial t^k}u_2(x,0) = 0, \quad k = 0,1,2,\ldots,m-1,\\
&p^3: \frac{\partial^m u_3}{\partial t^m} = \frac{\partial^m u_2}{\partial t^m} - L[x]u_2 - \frac{\partial^\alpha u_2}{\partial t^\alpha}, \qquad \frac{\partial^k}{\partial t^k}u_3(x,0) = 0, \quad k = 0,1,2,\ldots,m-1,\\
&\vdots
\end{aligned} \tag{4.6}$$

which are easy to solve. Now, solving the linear equations given in (4.6) for $u_0, u_1, u_2, \ldots$ and setting $p = 1$ in Eq. (3.7) yields the solution of Eq. (4.1). It is clear that the homotopy perturbation method reduces a nonlinear differential equation to a series of linear differential equations, while in our case the modified homotopy perturbation method reduces the fractional linear differential equation to a series of ordinary linear differential equations.

To incorporate our discussion above, three special cases of the fractional evolution equation (1.1) will be studied. All the results are calculated by using the symbolic calculus software Mathematica.

Example 4.1. Consider the linear time-fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, x \in R, 0 < \alpha \le 1, \tag{4.7}$$

subject to the initial condition

$$u(x,0) = \sin x. \tag{4.8}$$

According to the homotopy (4.3), substituting the initial condition (4.8) into (4.6), we obtain the following set of linear partial differential equations

$$\begin{aligned}
&\frac{\partial u_0}{\partial t} = 0, \qquad u_0(x,0) = \sin(x),\\
&\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial^2 x} - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \qquad u_1(x,0) = 0,\\
&\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial^2 x} - \frac{\partial^\alpha u_1}{\partial t^\alpha}, \qquad u_2(x,0) = 0,\\
&\frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial^2 x} - \frac{\partial^\alpha u_2}{\partial t^\alpha}, \qquad u_3(x,0) = 0.\\
&\vdots
\end{aligned} \tag{4.9}$$

Consequently, solving the above equations for u_0, u_1, u_2 and u_3, the first few components of the homotopy perturbation solution for Eq. (4.7) are derived as follows:

$$\begin{aligned}
&u_0(x,t) = \sin(x),\\
&u_1(x,t) = -t\sin(x),\\
&u_2(x,t) = \left(-t + \frac{1}{2}t^2 + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\sin(x),\\
&u_3(x,t) = \left(-t + t^2 - \frac{1}{6}t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right)\sin(x),\\
&\vdots
\end{aligned}$$

and so on; in this manner the rest of components of the homotopy perturbation solution can be obtained. The fourth-term approximate solution for Eq. (4.7) is given by

$$u(x,t)=\left(1-3t+\frac{3}{2}t^2-\frac{1}{6}t^3+\frac{3t^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}-\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right)\sin(x). \tag{4.10}$$

To solve the problem using the variational iteration method, we use formula (2.6) to construct the iteration formula for Eq. (4.7) as follows:

$$u_{k+1}(x,t)=u_k(x,t)-\int_0^t\left(\frac{\partial^\alpha}{\partial\xi^\alpha}u_k(x,\xi)-\frac{\partial^2}{\partial x^2}u_k(x,\xi)\right)\mathrm{d}\xi. \tag{4.11}$$

By the above variational iteration formula, begin with $u_0=\sin x$, we can obtain the following approximations

$$\begin{aligned}
u_0(x,t)&=\sin(x),\\
u_1(x,t)&=(1-t)\sin(x),\\
u_2(x,t)&=\left(1+\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}-2t+\frac{1}{2}t^2\right)\sin(x),\\
u_3(x,t)&=\left(1-\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}+\frac{3t^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}-3t+\frac{3}{2}t^2-\frac{1}{6}t^3\right)\sin(x),\\
&\vdots
\end{aligned}$$

and so on; in the same manner the rest of components of the iteration formula (4.11) can be obtained using the Mathematica package.

It is obvious that the fourth-term approximate solution (4.10) obtained using the modified homotopy perturbation method is the same as the fourth-order term of the variational iteration solution.

Example 4.2. We next consider the linear time-fractional wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2},\quad t>0, x\in R,\ 1<\alpha\le 2, \tag{4.12}$$

subject to the initial conditions

$$u(x,0)=x,\qquad \frac{\partial u(x,0)}{\partial t}=x^2. \tag{4.13}$$

According to the homotopy (4.3), substituting the initial conditions (4.13) into (4.6), we obtain the following set of linear partial differential equations:

$$\begin{aligned}
&\frac{\partial^2 u_0}{\partial t^2}=0,\quad u_0(x,0)=x,\qquad \frac{\partial}{\partial t}u_0(x,0)=x^2,\\
&\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial^2 u_0}{\partial t^2}-\frac{1}{2}x^2\frac{\partial^2 u_0}{\partial^2 x}-\frac{\partial^\alpha u_0}{\partial t^\alpha},\quad u_1(x,0)=0,\qquad \frac{\partial}{\partial t}u_1(x,0)=0,\\
&\frac{\partial^2 u_2}{\partial t}=\frac{\partial^2 u_1}{\partial t^2}-\frac{1}{2}x^2\frac{\partial^2 u_1}{\partial^2 x}-\frac{\partial^\alpha u_1}{\partial t^\alpha},\quad u_2(x,0)=0,\qquad \frac{\partial}{\partial t}u_2(x,0)=0,\\
&\frac{\partial^2 u_3}{\partial t^2}=\frac{\partial^2 u_2}{\partial t^2}-\frac{1}{2}x^2\frac{\partial^2 u_2}{\partial^2 x}-\frac{\partial^\alpha u_2}{\partial t^\alpha},\quad u_3(x,0)=0,\qquad \frac{\partial}{\partial t}u_3(x,0)=0.\\
&\vdots
\end{aligned} \tag{4.14}$$

Consequently, solving the above equations for u_0, u_1, u_2 and u_3, the first few components of the homotopy perturbation solution for Eq. (4.12) are derived as follows

$$u_0(x,t)=x+x^2t,$$

$$u_1(x,t) = x^2\frac{t^3}{3!},$$
$$u_2(x,t) = x^2\left(\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right),$$
$$u_3(x,t) = x + x^2\left(\frac{t^3}{3!}+\frac{2t^5}{5!}+\frac{t^7}{7!}-\frac{2t^{5-\alpha}}{\Gamma(6-\alpha)}-\frac{2t^{7-\alpha}}{\Gamma(8-\alpha)}+\frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right),$$
$$\vdots$$

and so on; in this manner the rest of components of the homotopy perturbation solution can be obtained. The fourth-term approximate solution for Eq. (4.12) is given by

$$u(x,t) = x + x^2\left(t+\frac{t^3}{2}+\frac{t^5}{40}+\frac{t^7}{7!}-\frac{3t^{5-\alpha}}{\Gamma(6-\alpha)}-\frac{2t^{7-\alpha}}{\Gamma(8-\alpha)}+\frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right), \tag{4.15}$$

which is exactly the same fourth-order approximate solution obtained in [20] using the variational iteration method.

Example 4.3. In this example, we consider the one-dimensional linear inhomogeneous time-fractional equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t + 2x^2 + 2, \quad t > 0, x \in R, 0 < \alpha \le 1, \tag{4.16}$$

subject to the initial condition

$$u(x,0) = x^2. \tag{4.17}$$

If we use the homotopy

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial u}{\partial t} - x\frac{\partial}{\partial x}u - \frac{\partial^2}{\partial^2 x}u - \frac{\partial^\alpha u}{\partial t^\alpha} + 2t + 2x^2 + 2\right], \quad p \in [0,1], \tag{4.18}$$

we obtain the following set of linear partial differential equations

$$\begin{aligned}
&\frac{\partial u_0}{\partial t} = 0, \quad u_0(x,0) = x^2,\\
&\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} - x\frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial^2 x} - \frac{\partial^\alpha u_0}{\partial t^\alpha} + 2t + 2x^2 + 2, \quad u_1(x,0) = 0\\
&\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} - x\frac{\partial u_1}{\partial x} - \frac{\partial^2 u_1}{\partial^2 x} - \frac{\partial^\alpha u_1}{\partial t^\alpha}, \quad u_2(x,0) = 0,\\
&\frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial t} - x\frac{\partial u_2}{\partial x} - \frac{\partial^2 u_2}{\partial^2 x} - \frac{\partial^\alpha u_2}{\partial t^\alpha}, \quad u_3(x,0) = 0.\\
&\vdots
\end{aligned} \tag{4.19}$$

Consequently, solving the above equations for u_0, u_1, u_2 and u_3, the first few components of the homotopy perturbation solution for Eq. (4.16) are derived as follows:

$$u_0(x,t) = x^2,$$
$$u_1(x,t) = t^2,$$
$$u_2(x,t) = t^2 - 2\frac{t^{3-\alpha}}{\Gamma(4-\alpha)},$$
$$u_3(x,t) = t^2 - 4\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + 2\frac{t^{4-2\alpha}}{\Gamma(5-2\alpha)},$$
$$\vdots$$

and so on; in this manner the rest of components of the homotopy perturbation solution can be obtained. The fourth-term approximate solution for Eq. (4.16) is given by

$$u(x,t) = x^2 + 3t^2 - 6\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + 2\frac{t^{4-2\alpha}}{\Gamma(5-2\alpha)}, \tag{4.20}$$

which is exactly the same fourth-order approximate solution obtained in [20] using the variational iteration method.

5. Conclusions

In this paper, we compared the homotopy perturbation method and the variational iteration method as applied to linear inhomogeneous fractional partial differential equations. For illustration purposes, we consider three different examples. It may be concluded that the two methods are powerful and efficient techniques for finding exact as well as approximate solutions for wide classes of linear inhomogeneous fractional partial differential equations. They provide more realistic series solutions that converge very rapidly in real physical problems. The implementation of the modified homotopy perturbation method reduces the fractional differential equation to a set of ordinary differential equations, which are easy to solve.

The study shows that the techniques require less computational work than existing approaches while supplying quantitatively reliable results. Also the high agreement of the numerical results so obtained between the homotopy perturbation method and the variational iteration method in all examples reinforces the conclusion that the efficiency of the two methods and related phenomena give the methods much wider applicability.

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