On the algebras generated by Toeplitz operators with piecewise continuous symbols

Nikolai Vasilevski

Departamento de Matemáticas, CINVESTAV, Apartado Postal 14-740, 07000, México, D.F., Mexico

Received 25 February 2012; received in revised form 17 April 2012; accepted 26 April 2012

Communicated by M.A. Kaashoek

Dedicated to the memory of Professor Israel Gohberg, a great mathematician and personality

Abstract

We explain an apparent disagreement between the fact that the Fredholm symbol algebras of two different $C^*$-algebras generated by Toeplitz operators with piecewise continuous symbols, acting on the Hardy space and the Bergman space, have the same Fredholm symbol algebras and the same symbol homomorphism on generating operator, and the fact that the initial generators of these algebras are not unitary equivalent modulo compact operators.

© 2012 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.

Keywords: Toeplitz operators; Hardy space; Bergman space; Piecewise continuous symbol

1. Introduction

In their celebrated paper [2], Gohberg and Krupnik studied the $C^*$-algebra generated by discrete Toeplitz operators (which are unitary equivalent to the Toeplitz operators acting on the Hardy space) with piecewise continuous symbols and described its Fredholm symbol algebra. This paper and the new approach to the problem developed therein originated a flow of papers devoted to the study of numerous algebras generated by one-dimensional singular integral operators, Toeplitz and convolution operators, whose defining data (coefficients, symbols, etc.) were discontinuous in different senses.

E-mail address: nvasilev@math.cinvestav.mx.

0019-3577/S - see front matter © 2012 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.indag.2012.04.004
The initial study of algebras generated by Toeplitz operators acting on the Bergman space was inspired by the corresponding theory for Toeplitz operators acting on the Hardy space. In particular, for the case of continuous defining symbols, it turned out that both algebras have the same Fredholm symbol (or Calkin algebra) and the symbol homomorphism is generated by the same mapping of generating operators. The natural explanation of this fact came later, when it was observed that the generating operators of both algebras are unitary equivalent modulo compact operators (see Theorem 1 for details).

The \( C^* \)-algebra generated by Toeplitz operators with piecewise continuous symbols and acting on the Bergman space was described in [11] (see also [4,5,7,12] for further detailed study of this algebra). We note that, for the piecewise continuous generating symbols, the situation remains the same: both operator algebras, which are generated by Toeplitz operators acting on the Hardy space and on the Bergman space, have the same Fredholm symbol algebra and the symbol homomorphism is generated by the same mapping of generating operators (see Theorem 11 for details). Nevertheless, as was found out later on, the initial generators of these algebras are not anymore unitary equivalent modulo compact operators. That is, the coincidence of the Fredholm symbol algebras and the symbol homomorphisms on generators looks now strangely and needs at least an additional explanation.

The challenging question: why is it so and what is the deep reason for such a phenomenon has been never studied, and it seems to be interesting and important to clarify the situation and to explain the reason of this coincidence.

In the paper, we answer this question. The main result is given in Section 5 and says: although the initial generators for the piecewise continuous case are not unitary equivalent modulo compact operators, the two \( C^* \)-algebras: \( \mathcal{T}_H \), generated by Toeplitz operators acting on the Hardy space, and \( \mathcal{T}_B \), generated by Toeplitz operators acting on the Bergman space, are unitary equivalent; the initial generators of either one of these algebras are unitary equivalent to certain elements of the another algebra. These two algebras have the same Fredholm symbol algebra and in both cases the symbol homomorphism is generated by the same mapping of the corresponding generators, but, when compared, the symbol homomorphisms differ on the parameterization of the auxiliary segments \([0, 1]\). The explicit connection between the parameterizations is also given.

2. Preliminaries

We introduce first a number of Hilbert spaces and unitary operators we will be dealing with.

Let \( S^1 \) be the unit circle in \( \mathbb{C} \), introduce the space \( L_2(S^1) \) with the standard arc-length measure \( d\theta = \frac{dt}{it} \), where \( t = e^{i\theta} \in S^1 \). The discrete Fourier transform \( \mathcal{F} : L_2(S^1) \to l_2 \):

\[
\mathcal{F} : f \longmapsto \{f_n\}_{n \in \mathbb{Z}}, \quad \text{where} \quad f_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t) t^{-n} \frac{dt}{it},
\]

is a unitary operator, and its inverse \( \mathcal{F}^{-1} = \mathcal{F}^* : l_2 \to L_2(S^1) \) is given by

\[
\mathcal{F}^{-1} : (f_n)_{n \in \mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} f_n t^n.
\]

The classical Hardy space \( H^2 = H^2(S^1) \) is the (closed) subspace of \( L_2(S^1) \) which consists of all functions \( f \) whose all negative Fourier coefficients are zero: \( f_n = 0 \), for all \( n < 0 \), or, in an alternative form, \( H^2 = H^2(S^1) \) consists of all functions admitting the analytic continuation...
on the unit disk \( \mathbb{D} \). We denote by \( P \) the orthogonal Szegő projection of \( L_2(S^1) \) onto the Hardy space \( H^2 \).

Let \( l_2^+ \) be the subspace of \( l_2 \) consisting of all sequences \( \{c_n\} \) such that \( c_n = 0 \), for all \( n < 0 \); we will identify later on \( l_2^+ \) with the one-sided \( l_2(\mathbb{Z}_+) \) space. Note that the operator

\[
R_H = \mathcal{F}|_{H^2} : H^2 \longrightarrow l_2^+
\]

is unitary as well.

Introduce now the space \( L_2(\mathbb{D}) \) with the standard Lebesgue plain measure \( d\nu(z) = dx dy, z = x + iy \), and the classical Bergman space \( \mathcal{A}^2 = \mathcal{A}^2(\mathbb{D}) \) being the (closed) subspace of \( L_2(\mathbb{D}) \), which consists of all functions analytic in \( \mathbb{D} \). We denote by \( B \) the orthogonal Bergman projection of \( L_2(\mathbb{D}) \) onto the Bergman space \( \mathcal{A}^2 \).

Following [13, Chapter 4], introduce:

- the unitary operator \( U_1 = I \otimes \mathcal{F} \) mapping \( L_2(\mathbb{D}) = L_2([0, 1), r dr) \otimes L_2(S^1, \frac{dt}{\pi}) \) onto \( L_2([0, 1), r dr) \otimes l_2 \);
- unitary operator \( U_2 \) acting on \( L_2([0, 1), r dr) \otimes l_2 \) by the rule

\[
U_2 : \{c_n(r)\}_{n \in \mathbb{Z}} \mapsto \{(u_n c_n)(r)\}_{n \in \mathbb{Z}},
\]

where the operator \( u_n : L_2([0, 1), r dr) \longrightarrow L_2([0, 1), r dr), \ n \in \mathbb{Z}_+, \) is given by

\[
(u_n f)(r) = \frac{1}{\sqrt{n+1}} r^{-\frac{n}{2\pi+1}} f(r^\frac{1}{\pi+1});
\]

- the isometric embedding \( R_0 : l_2^+ \longrightarrow L_2([0, 1), r dr) \otimes l_2 \) acting by

\[
R_0 : \{c_n\}_{n \in \mathbb{Z}_+} \mapsto \sqrt{2} \{\chi_+(n) c_n\}_{n \in \mathbb{Z}},
\]

where \( \chi_+(n) \) is the characteristic function of \( \mathbb{Z}_+ \).

Then (see [13, Chapter 4] for details) the operator \( R = R_0^* U_2 U_1 \) maps \( L_2(\mathbb{D}) \) onto \( l_2^+ \), and the restriction

\[
R|_{\mathcal{A}^2} : \mathcal{A}^2 \longrightarrow l_2^+
\]

is an isometric isomorphism. The adjoint operator

\[
R^* = U_1^* U_2^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2 \subset L_2(\mathbb{D})
\]

is an isometric isomorphism of \( l_2^+ \) onto the subspace \( \mathcal{A}^2 \) of the space \( L_2(\mathbb{D}) \). Moreover

\[
RR^* = I : l_2^+ \longrightarrow l_2^+
\]

\[
R^* R = B : L_2(\mathbb{D}) \longrightarrow \mathcal{A}^2.
\]

That is, the operator

\[
R_B = R|_{\mathcal{A}^2} : \mathcal{A}^2 \longrightarrow l_2^+
\]

is unitary. Furthermore we have

\[
R_B : \varphi(z) \mapsto \left\{ \frac{\sqrt{2(n+1)}}{\sqrt{2\pi}} \int_{\mathbb{D}} \varphi(z) \overline{z}^n d\nu(z) \right\}_{n \in \mathbb{Z}_+},
\]

\[
R_B^* : \{c_n\}_{n \in \mathbb{Z}_+} \mapsto \varphi(z) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}_+} \sqrt{2(n+1)} c_n z^n.
\]
We introduce as well the isomorphisms

\[ W = R_B^* R_B : \mathcal{A}_2 \rightarrow H^2, \]
\[ W^* = R_B^* R_H : H^2 \rightarrow \mathcal{A}_2. \]

Given a function (symbol) \( a(t) \in L_\infty(S^1) \) for the Hardy space case, and \( a(z) \in L_\infty(\mathbb{D}) \) for the Bergman space case, we introduce the corresponding Toeplitz operators in a usual way: multiplication followed by the projection, i.e. \( T^H_a f = P(af), f \in H^2 \), for the Hardy space case, and \( T^B_a \varphi = B(a\varphi), \varphi \in \mathcal{A}_2 \), for the Bergman space case.

Let \( a(t) \in L_\infty(S^1) \subset L_2(S^1) \), and let

\[ a(t) = \sum_{n \in \mathbb{Z}} a_n t^n \left( = \sum_{n \in \mathbb{Z}} a'_n \frac{t^n}{\sqrt{2\pi}} \right), \]

where \( a'_n = \sqrt{2\pi} a_n, \ n \in \mathbb{Z} \), are the Fourier coefficients of the function \( a \). Then the operator \( R_B T^H_a R_B^* \) acting on the discrete space \( l^+_2 \) is nothing but the classical discrete Toeplitz operator already considered by O. Toeplitz himself, and defined by the matrix \((a_{n-k})_n, k \in \mathbb{Z}_+ \). We denote this operator by \( T(a) \).

The function \( a(t) \) of the form (2) can be extended into the unit disk to the function which depends only on the angular part \( t \) of \( z = rt \), i.e. to the function \( a(rt) = a(t) \). Thus for such a function we have well defined Toeplitz operator \( T^B_a \).

Direct computations show:

\[ R_B T^B_a R_B^* = Ra R_B^* : \{c_k\}_{k \in \mathbb{Z}_+} \xrightarrow{R_B^*} \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}_+} \sqrt{2(k+1)} c_k z^k \]

\[ \xrightarrow{a} \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}_+} \sqrt{2(k+1)} c_k r^k a(t) t^k \]

\[ \xrightarrow{R} \left\{ \frac{\sqrt{2(n+1)}}{2\pi} \sum_{k \in \mathbb{Z}_+} \sqrt{2(k+1)} c_k \int_{\mathbb{D}} r^k a(t) t^k r^n 1^{-n} d\mu \right\}_{n \in \mathbb{Z}_+} \]

\[ = \left\{ \frac{2\sqrt{(n+1)}}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}_+} \sqrt{(k+1)} c_k \int_0^1 r^{k+n+1} dr \frac{1}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}_+} \]

\[ \times \left\{ \int_{S^1} a(t) t^{k-n} dt \right\}_{n \in \mathbb{Z}_+} \]

\[ = \left\{ \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}_+} \frac{2\sqrt{(n+1)}(k+1)}{(n+1) + (k+1)} a'_{n-k} c_k \right\}_{n \in \mathbb{Z}_+} \]

\[ = \left\{ \sum_{k \in \mathbb{Z}_+} \frac{2\sqrt{(n+1)}(k+1)}{(n+1) + (k+1)} a_{n-k} c_k \right\}_{n \in \mathbb{Z}_+} . \]

Denote by \( T_*(a) \) the operator \( R_B T^B_a R_B^* \) acting on \( l^+_2 = l_2(\mathbb{Z}_+) \); it has the following matrix representation

\[ T_*(a) = \begin{pmatrix} a_{n-k} c_k \end{pmatrix}_{n \in \mathbb{Z}_+} . \]
We note that for a generic function \( a(t) \in L_\infty(S^1) \) the operator \( T_*(a) \) is neither discrete Toeplitz operator nor a compact perturbation of such an operator.

### 3. Some specific Toeplitz operators

Let \( a(t) = t \), then the operator \( T(t) = R_H T_t^H R_H^* \) is defined by the matrix \((a_{n-k})_{n,k \in \mathbb{Z}_+}\), where \( a_{n-k} = 1 \) if \( n-k = 1 \) and \( a_{n-k} = 0 \) otherwise. That is, \( T(t) \) is just the unilateral forward shift operator

\[
T(t) : (c_0, c_1, \ldots, c_k, \ldots) \mapsto (0, c_0, \ldots, c_{k-1}, \ldots).
\]

At the same time the operator \( T_*(t) = R_B T_t^B R_B^* \) is a unilateral forward weighted shift operator

\[
T_*(t) : (c_0, c_1, \ldots, c_k, \ldots) \mapsto \left(0, \frac{2 \sqrt{2}}{3} c_0, \ldots, \frac{2 \sqrt{k(k+1)}}{2k+1} c_{k-1}, \ldots \right).
\]

That is, the difference

\[
(T_*(t) - T(t)) \{c_k\} = \left(\frac{2 \sqrt{k(k+1)}}{2k+1} - 1\right) \{c_{k-1}\}
\]

is compact, since the sequence \( \left\{\frac{2 \sqrt{k(k+1)}}{2k+1} - 1\right\} \) tends to 0 as \( k \to \infty \). Passing to adjoin operators, we have

\[
R_B T_t^B R_B^* - R_H T_t^H R_H^* = T_*(\bar{t}) - T(\bar{t}) = k,
\]

where \( k \) is compact. Both projections \( B \) and \( P \) commute with multiplication operators by \( t \) and by \( \bar{t} \) modulo compact operators, thus for all polynomials \( p(t, \bar{t}) = \sum_{k=-n}^{\infty} a_k t^k \) we have

\[
R_B T_{p(t, \bar{t})}^B R_B^* - R_H T_{p(t, \bar{t})}^H R_H^* = T_*(p(t, \bar{t})) - T(p(t, \bar{t})) = k_p
\]

where \( k_p \) is compact.

Given now a continuous function \( a(t) \), we approximate it uniformly by a sequence of polynomials \( p_n(t, \bar{t}), n \in \mathbb{N} \). Passing then to the limit in (3), we have

\[
R_B T_{a(t)}^B R_B^* - R_H T_{a(t)}^H R_H^* = T_*(a(t)) - T(a(t)) = k_a,
\]

where \( k_a \) is compact.

Thus we come to the known statement (see, for example, [1]).

**Theorem 1.** Let \( a(z) \in C(\overline{\mathbb{D}}) \) and \( a_0(t) \) be its restriction onto \( S^1 \), or let \( a_0(t) \in C(S^1) \) and \( a(z) \) be its extension to a continuous function on \( \overline{\mathbb{D}} \). Then

\[
W T_a^B W^* = T_{a_0}^H + K_H,
\]

\[
W^* T_{a_0}^B W = T_a^B + K_B,
\]

where the operators \( K_H \) and \( K_B \) are compact.
**Proof.** Follows directly from the statement that the operator $T^B_{a(z)} - T^B_{a_0} \left( \frac{z}{\pi} \right)$ is compact, the representations $W = R^*_H R_B$, $W^* = R^*_B R_H$, and equality (4). □

Consider now another special case of the function $a(t)$. Namely, let $a(t) = \ell(t) = \ell(e^{i\theta}) = \frac{1}{\pi}(\pi - \theta)$. The function $\ell(t) = \ell(e^{i\theta})$ is continuous in $S^1 \setminus \{1\}$, $\ell(1 \pm 0) = \pm 1$, and it is known (see, for example, [10]) that

$$\ell(t) = \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_{R^+} \lambda^{-\frac{1}{2}} \phi(\lambda) d\lambda.$$  

For this specific case we have $T(\ell) = (\ell_{nk})_{n,k \in \mathbb{Z}_+}$, where

$$\ell_{nk} = \begin{cases} 
\frac{1}{\pi i} \frac{1}{n-k}, & n \neq k \\
0, & n = k,
\end{cases}$$

while the matrix entries of the operator $T^*_{\ell} = (\ell^*_{nk})_{n,k \in \mathbb{Z}_+}$ are as follows

$$\ell^*_{nk} = \begin{cases} 
\frac{1}{\pi i} \frac{2\sqrt{(n+1)(k+1)}}{(n+1)^2 - (k+1)^2}, & n \neq k \\
0, & n = k.
\end{cases}$$

Note that the difference $T^*_{\ell} - T(\ell)$ is not anymore compact.

4. Mellin convolution algebra and the discrete Toeplitz operator algebra

The results of this section are mostly known (see [6,8–10] for further details), and the presented proofs are given for the sake of completeness.

Recall first, that for a function $\phi(\lambda) \in L^\infty(R)$ the Mellin convolution operator $M^0(\phi)$ is defined on the space $L^2(R_+)$ as follows

$$M^0(\phi) = M^{-1} \phi(\lambda) M,$$

where the Mellin transform $M : L^2(R_+) \to L^2(R)$ and its inverse $M^{-1} = M^* : L^2(R) \to L^2(R_+)$ are given by

$$(M\phi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{R^+} x^{-i\lambda - \frac{1}{2}} \phi(x) dx,$$

$$(M^{-1}\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{R} x^{i\lambda - \frac{1}{2}} \psi(\lambda) d\lambda.$$

The Mellin convolution operator admits the following integral representation

$$(M^0(\phi)\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{R^+} k \left( \frac{x}{t} \right) \phi(t) \frac{dt}{t},$$

where the symbol-function $\phi$ and the kernel $k$ are connected by $\phi(\lambda) = (Mk)(\lambda)$.

**Example 2 (Singular Integral Operator).** Let

$$S\phi(x) = \frac{1}{\pi i} \int_{R^+} \frac{\phi(t)}{t-x} dt = \frac{1}{\sqrt{2\pi}} \int_{R^+} \sqrt{\frac{2\pi}{\pi i}} \frac{1}{1 - \frac{t}{x}} \phi(t) \frac{dt}{t}.$$
By Gradshteyn and Ryzhik [3, formula 3.222.2], we have
\[ M \left( \sqrt{\frac{2\pi}{\pi i}} \frac{1}{1-x} \right)(\lambda) = \frac{1}{\pi i} \int_{\mathbb{R}^+} x^{-i\lambda - \frac{i}{2}} \frac{1}{1-x} \, dx = \coth \left( \frac{\lambda + i}{2} \right) = \tanh \pi \lambda. \]

Thus \( S = M^{-1} \tanh(\pi \lambda) M \).

We denote by \( \mathcal{R}(\mathbb{C}, S) \) the unital C*-algebra generated by a single operator \( S \). The real valued function \( \tanh \pi \lambda \) is continuous on the compact set \( \mathbb{R} = [-\infty, +\infty] \) and monotone. That is, it separates the points of \( \mathbb{R} \). Thus by the Stone–Weierstrass theorem it, together with the identity function, generates the C*-algebra \( C(\mathbb{R}) \).

**Corollary 3.** The C*-algebra \( \mathcal{R}(\mathbb{C}, S) \) is isomorphic and isometric to \( C(\mathbb{R}) \); it consists of all operators of the form \( M^0(\phi) \), where \( \phi \in C(\mathbb{R}) \). The above isomorphism is given by the formula
\[ M^0(\phi) \in \mathcal{R}(\mathbb{C}, S) \mapsto \phi \in C(\mathbb{R}). \]

Furthermore, each operator \( M^0(\phi) \) admits the following canonical representation
\[ M^0(\phi) = \phi(-\infty) \frac{1}{2} (I - S) + \phi(+\infty) \frac{1}{2} (I + S) + M^0(\phi_0) \]
\[ = \frac{1}{2} \left[ \phi(+\infty) + \phi(-\infty) \right] I + \frac{1}{2} \left[ \phi(+\infty) - \phi(-\infty) \right] S + M^0(\phi_0), \quad (6) \]

where
\[ \phi_0(\lambda) = \phi(\lambda) - \left[ \phi(-\infty) \frac{1}{2} (1 - \tanh \pi \lambda) + \phi(+\infty) \frac{1}{2} (1 + \tanh \pi \lambda) \right] \in C_0(\mathbb{R}). \]

Here we denote by \( C_0(\mathbb{R}) \) the closed two-sided ideal of \( C(\mathbb{R}) \) which consists of all functions \( \phi_0(\lambda) \) satisfying the condition \( \phi(-\infty) = \phi(+\infty) = 0 \). The corresponding ideal of the algebra \( \mathcal{R}(\mathbb{C}, S) \) we denote by \( N \).

**Example 4.** Let
\[ (S_1 \varphi)(x) = \frac{1}{\pi i} \int_{\mathbb{R}^+} \frac{2\sqrt{t} \varphi(t)}{t + x} \frac{dt}{t-x}. \]

By Gradshteyn and Ryzhik [3, formula 3.223.2], we have
\[ M \left( \sqrt{\frac{2\pi}{\pi i}} \frac{2\sqrt{x}}{1-x^2} \right)(\lambda) = \frac{1}{\pi i} \int_{\mathbb{R}^+} x^{-i\lambda - \frac{i}{2}} \frac{2\sqrt{x}}{1-x^2} \, dx = \frac{2}{\pi i} \int_{\mathbb{R}^+} x^{-i\lambda} \frac{1}{1-x^2} \, dx \]
\[ = \frac{1}{i} \left( \csc(1-i\lambda) + \cot(1-i\lambda) \right) = \tanh \frac{\pi \lambda}{2}. \]

Thus \( S_1 = M^{-1} \tanh \left( \frac{\pi \lambda}{2} \right) M \).

The formula
\[ \tanh \pi \lambda = \frac{2 \tanh \frac{\pi \lambda}{2}}{1 + \tanh^2 \frac{\pi \lambda}{2}} \]
implies that $S = 2S_1(1 + S_1^2)^{-1}$, and that $S_1 = c(S)$, where the function
\[
c(x) = \frac{1 - \sqrt{1 - x^2}}{x}
\]
is continuous on $\text{sp } S = [-1, 1]$.

**Corollary 5.** The unital $C^*$-algebra $R(C, S_1)$ generated by the operator $S_1$ coincides with the algebra $R(C, S)$. It is generated either by $S$ or by $S_1$. In either case, each one of these operators belongs to the algebra generated by the other operator.

We denote by $PS(S^1, 1)$ the algebra of all piecewise continuous functions $a(t)$ on $S^1$ which are continuous in $S^1 \setminus \{1\}$ and have one sided limit values $a(1 + 0)$ and $a(1 - 0)$ at the point $1 \in S^1$. And let $T_H(PS(S^1, 1))$ stand for the $C^*$-algebra generated by all Toeplitz operators $T^H_a$, with symbols $a \in PS(S^1, 1)$, acting on the Hardy space $H^2$.

We are interested in this section in the discrete analog of this algebra, i.e. in the $C^*$-algebra $T_d = T_d(PS(S^1, 1)) = R_H T_H(PS(S^1, 1)) R^*_H$, which is generated by all discrete Toeplitz operators $T(a) = R_H T^H_a R^*_H$, with $a \in PS(S^1, 1)$.

There is an amazing connection (see [8,10]) between the Mellin convolution algebra $R(C, S)$ and the local (at the point 1) algebra of the algebra $T_d$, which we describe now in more detail. It is well known that the algebra $T_d$ contains the ideal $K$ of all compact operators. We denote by $	ext{Sym } T_d = \widehat{T}_d = T_d/K$ its Fredholm symbol or Calkin algebra, and let $\pi : T_d \longrightarrow \widehat{T}_d$ be the natural projection.

It is a common knowledge now that the algebra $\pi(T_d(C(S^1))) \cong C(S^1)$ is a central commutative subalgebra of $\widehat{T}_d$, and that the local algebras $\widehat{T}_d(t)$, for all points $t \in S^1 \setminus \{1\}$, are isomorphic and isometric to $C$. Recall that, for $t \in S^1$, the local algebra at the point $t$ is defined as follows: $\widehat{T}_d(t) = \widehat{T}_d/J(t)$, where $J(t)$ is the two-sided closed ideal in $\widehat{T}_d$ generated by all elements of the central subalgebra $\pi(T_d(C(S^1)))$ whose Gelfand images in $C(S^1)$ vanish at the point $t$.

The natural projection
\[
\pi(t) : T_d \longrightarrow \widehat{T}_d \longrightarrow \widehat{T}_d(t)
\]
identifies elements of $T_d$ that are locally equivalent at the point $t$, and for all $t \in S^1 \setminus \{1\}$ is generated by the following mapping of generators of $T_d$:
\[
\pi(t) : T(a) \in T_d \longmapsto a(t) \in C.
\]

For the jump point $1 \in S^1$ the situation is more delicate. To characterize the local algebra $\widehat{T}_d(1)$, we proceed following [10].

Introduce the isometry $E : l^+_2 \longrightarrow L_2(\mathbb{R}_+)$ by the rule
\[
E : \{f_k\}_{k \in \mathbb{Z}_+} \longmapsto f(x) = \sum_{k \in \mathbb{Z}_+} f_k \chi_{[k,k+1)}(x),
\]
and its adjoint $E^* : L_2(\mathbb{R}_+) \longrightarrow l^+_2$, where
\[
E^* : f(x) \longmapsto \{f_k\}_{k \in \mathbb{Z}_+}, \quad \text{with } f_k = \int_k^{k+1} f(x) dx.
\]
Then we obviously have
\[ E^*E = I : l^+_2 \to l^+_2, \]
\[ EE^* = L : L_2(\mathbb{R}^+) \to \text{Im} \, E, \]
with \( L \) being the orthogonal projection.
For each \( r \in \mathbb{R}^+ \), introduce as well the unitary dilation operator
\[ Z_r : f(x) \in L_2(\mathbb{R}^+) \longmapsto \frac{1}{\sqrt{r}} f\left(\frac{x}{r}\right) \in L_2(\mathbb{R}^+). \]
The next result, providing the connection between the algebras \( T_d \) and \( \mathcal{R}(\mathbb{C}, S) \), is proved in [8,10].

**Lemma 6.** For each operator \( T \in T_d \), the strong limit
\[ H(T) = s - \lim_{r \to \infty} Z_r^* LEAE^*LZ_r = s - \lim_{r \to \infty} Z_r^* EAE^*Z_r \]
exists and belongs to \( \mathcal{R}(\mathbb{C}, S) \). The mapping
\[ H : T \in T_d \to H(T) \in \mathcal{R}(\mathbb{C}, S) \]
is a homomorphism, and, in particular,
\[ H(T(a)) = a(1 + 0) \frac{1}{2}(I - S) + a(1 - 0) \frac{1}{2}(I + S), \tag{8} \]
\[ H(K) = 0, \quad \text{if} \ K \text{ is compact.} \tag{9} \]
A simple observation, based on (9) and \( H(T(a)) = 0 \) whenever \( a(1 + 0) = a(1 - 0) = 0 \), implies that, in fact, \( H(T) \) depends only on the image of \( T \) in the local algebra \( \mathcal{T}_d(1) \). That is, the mapping \( H \) admits the decomposition
\[ H : T_d \xrightarrow{\pi(1)} \mathcal{T}_d(1) \xrightarrow{\sigma} \mathcal{R}(\mathbb{C}, S). \]
Moreover, by a deep result from [10], the mapping \( \sigma \) is an isomorphism, each operator \( G(\phi) = E^*M^0(\phi)E \), with \( \phi \in C(\mathbb{R}) \), belongs to the algebra \( T_d \), and the inverse isomorphism is given by
\[ \sigma^{-1} : M^0(\phi) \mapsto \pi(1)\left(E^*LM^0(\phi)E\right) = \pi(1)\left(E^*M^0(\phi)E\right). \tag{10} \]

**Example 7** ([9,10]). We have that \( G(\tanh \pi \lambda) = E^*SE = T(\varrho) \), where the function
\[ \varrho(t) = \varrho(e^{i\theta}) = -4 \sin^2 \frac{\theta}{2} \sum_{k \in \mathbb{Z}} \frac{\text{sign}(k + \frac{1}{2})}{(\theta + 2\pi k)^2}, \quad \theta \in [0, 2\pi), \]
is continuous in \( S^1 \setminus \{1\} \) and \( \varrho(1 \pm 0) = \mp 1 \).

**Corollary 8.** The algebra \( T_d = T_d(P S(S^1, 1)) \) consists of all operators of the form
\[ T = T(a) + G(\phi_0) + K, \tag{11} \]
where \( a \in P S(S^1, 1), \phi_0 \in C_0(\mathbb{R}) \), and \( K \) is compact.
Proof. Follows from the above considerations, representation (6) and the previous example. □

Corollary 9. The Fredholm symbol algebra \( \text{Sym} \mathcal{T}_d(PC(S^1, 1)) = T_d(PC(S^1, 1))/\mathcal{K} \) is isomorphic and isometric to \( C(\Gamma_0) \), where \( \Gamma_0 = \mathbb{S}^1 \cup \mathbb{R} \) is the disjoint union of \( \mathbb{S}^1 \), the unit circle cut by the point 1, and \( \mathbb{R} = [-\infty, +\infty] \), the end points of which are identified by the rule
\[
1 + 0 \in \mathbb{S}^1 \equiv -\infty \in \mathbb{R}, \quad 1 - 0 \in \mathbb{S}^1 \equiv +\infty \in \mathbb{R}.
\]
The symbol homomorphism \( \text{sym} : \mathcal{T}_d(PC(S^1, 1)) \rightarrow \text{Sym} \mathcal{T}_d(PC(S^1, 1)) = C(\Gamma_0) \) assigns to each operator \( T \) of the form (11) the following function
\[
\text{sym}T = \begin{cases} a(t), & t \in \mathbb{S}^1 \\ a(1 + 0) \frac{1}{2} (1 - \tanh \pi \lambda) + a(1 - 0) \frac{1}{2} (1 + \tanh \pi \lambda) + \phi_0(\lambda), & \lambda \in \mathbb{R}. \end{cases}
\]

Proof. Follows from the description of all local algebras \( \mathcal{T}_d(t), \ t \in S^1 \), Corollary 3, and formulas (8) and (10). □

Example 10. We introduce the functions \( s(\lambda) = \tanh \pi \lambda, s_1(\lambda) = \tanh \frac{\pi \lambda}{2}, \) and \( s_0(\lambda) = s(\lambda) - s_1(\lambda); \) and let \( S_0 = M^0(s_0(\lambda)) = S - S_1 \). As \( s_0(\lambda) \in C_0(\mathbb{R}) \), and so \( S_0 \in \mathcal{N} \), by Roch [10, Proposition 2], we have that the difference
\[
G(s_0) - \left( \frac{1}{\sqrt{2\pi}} k_0 \left( \frac{n + 1}{k + 1} \right) \frac{1}{k + 1} \right)_{n,k \in \mathbb{Z}^2}
\]
is compact on \( l_2^+ \).

Here
\[
k_0 = M^{-1}s_0 = \frac{\sqrt{2\pi}}{\pi i} \left( \frac{1}{1 - x} - \frac{2\sqrt{x}}{1 - x^2} \right)
\]
Thus,
\[
G(s_0) = \left( \frac{1}{\pi i} \frac{1}{k - n} \right)_{n,k \in \mathbb{Z}^2} - \left( \frac{1}{\pi i} \frac{2\sqrt{(n + 1)(k + 1)}}{(k + 1)^2 - (n + 1)^2} \right)_{n,k \in \mathbb{Z}^2} + K
\]
where \( K \) is compact. That is, although the difference \( T_\ell - T(\ell) \) is not compact, the operator \( T_\ell = T(\ell) + G(s_0) - K \) belongs to the algebra \( \mathcal{T}_d(PC(S^1, 1)) \).

5. Toeplitz operators with piecewise continuous symbols acting on the Bergman and on the Hardy spaces

The aim of this section is to compare the two \( C^* \)-algebras, both of which are generated by Toeplitz operators with piecewise continuous symbols. In the first case the operators act on the Hardy space \( H^2(S^1) \), while in the second case the operators act on the Bergman space \( \mathcal{A}^2(\mathbb{D}) \).

We start from the Hardy space case. Let \( A = \{t_1, t_2, \ldots, t_m\} \) be a finite set of disjoint points of the unit circle \( S^1 \). Denote by \( PC(S^1, A) \) the algebra of piecewise continuous functions \( a(t) \) on \( S^1 \), which are continuous in \( S^1 \setminus A \) and have one-sided limit values \( a(t_k + 0) \) and \( a(t_k - 0) \) at the points \( t_k \in A \). Introduce the \( C^* \)-algebra \( \mathcal{T}_H(PC(S^1, A)) \) which is generated by all Toeplitz operators \( T_\ell \) acting on the Hardy space \( H^2(S^1) \) and having symbols \( a(t) \in PC(S^1, A) \). We denote by \( \text{Sym} \mathcal{T}_H(PC(S^1, A)) \) its Fredholm symbol or Calkin algebra \( \mathcal{T}_H(PC(S^1, A))/\mathcal{K} \).
For the Bergman space case, for each point \( t_k \in \Lambda \), let \( l_k \) be a line segment starting at \( t_k \) and lying on the radius to the point \( t_k \), and let \( L = \bigcup_{k=1}^{m} l_k \) be the union of these segments. Denote by \( PC(\mathbb{D}, L) \) the algebra of piecewise continuous functions \( a(t) \) on \( \mathbb{D} \), i.e. the functions that are continuous in \( \mathbb{D} \setminus L \) and have one-sided limit values \( a(z+0) \) and \( a(z-0) \) at the points \( z \in L \). Introduce the \( C^* \)-algebra \( T_B(\mathbb{D}, L) \) which is generated by all Toeplitz operators \( T_a \) acting on the Bergman space \( A^2(\mathbb{D}) \) and having symbols \( a(z) \in PC(\mathbb{D}, L) \). We denote by \( \text{Sym} \, T_B(\mathbb{D}, L) \) its Fredholm symbol or Calkin algebra \( T_B(\mathbb{D}, L)/K \).

Note that, for each \( a(z) \in PC(\mathbb{D}, L) \), the function \( a(t) = a(\tilde{z}) \) belongs to \( PC(S^1, \Lambda) \), and the function \( b(z) = a(z) - a(\tilde{z}) \) is continuous in each point of the boundary \( S^1 \), moreover \( b(z)|_{S^1} \equiv 0 \). Thus the Toeplitz operator \( T_B = T_a(z) - T_a(\tilde{z}) \) is compact, and the algebra \( T_B(\mathbb{D}, L) \) can be considered as generated by all Toeplitz operators \( T_a(\tilde{z}) \) with \( a(t) \in PC(S^1, \Lambda) \), or, in other words, \( T_B(\mathbb{D}, L) = T_B(PC(S^1, \Lambda)). \)

It is known that both algebras \( T_H(PC(S^1, \Lambda)) \) and \( T_B(PC(\mathbb{D}, L)) = T_B(PC(S^1, \Lambda)) \) have the same Fredholm symbol algebra, whose description was given in [2] for the Hardy space case and in [11] for the Bergman space case.

To present this description, we introduce some notation. Denote by \( \hat{S}^1 \) the compactification of the unit circle cut by points \( t_k \in \Lambda \), that is \( \hat{S}^1 \) coincides with the disjoint union of closed circular arcs \( [t_k + 0, t_{k+1} - 0] \), where \( k = 1, 2, \ldots, m, t_{m+1} = t_1 \). Let \( \Delta = \bigcup_{k=1}^{m} [0, 1]_k \) be the disjoint union of segments \( [0, 1] \) parameterized by \( k = 1, 2, \ldots, m \). Let finally \( \Gamma = \hat{S}^1 \cup \Delta \) be the union of \( \hat{S}^1 \) and \( \Delta \), whose endpoints are identified as follows

\[
\begin{align*}
    t_k + 0 &\in [t_k + 0, t_{k+1} - 0] \equiv 0 \in [0, 1]_k \\
    t_k - 0 &\in [t_{k-1} + 0, t_k - 0] \equiv 1 \in [0, 1]_k,
\end{align*}
\]

where \( k = 1, 2, \ldots, m, m + 1 \equiv 1 \).

**Theorem 11.** Both Fredholm symbol algebras of the \( C^* \)-algebras \( T_H(PC(S^1, \Lambda)) \) and \( T_B(PC(\mathbb{D}, L)) = T_B(PC(S^1, \Lambda)) \) are isomorphic and isometric to \( C(\Gamma) \). In both cases, the symbol homomorphism

\[
sym : T(PC(S^1, \Lambda)) \longrightarrow C(\Gamma)
\]

is generated by the following mapping of generators

\[
sym : T_a \longmapsto \begin{cases} a(t), & t \in \hat{S}^1, \\ a(t_k + 0)(1 - x) + a(t_k - 0)x, & t_k \in \Lambda, \ x \in [0, 1]_k, \end{cases}
\]

where \( a(t) \in PC(S^1, \Lambda) \).

Recall that by Theorem 1, for each continuous function \( a(t) \in C(S^1) \), the Toeplitz operators \( T_a^B \) and \( T_a^H \), acting on the Bergman and the Hardy spaces, respectively, are connected as follows

\[
W T_a^B W^* - T_a^H = K,
\]

where the unitary operator \( W : A^2(\mathbb{D}) \longrightarrow H^2(S^1) \) is given by (1) and \( K \) is compact. At the same time, for a piecewise continuous function \( a \in PC(S^1, \Lambda) \), the difference \( W T_a^B W^* - T_a^H \) is not compact, in general.

From the point of view of the identical Fredholm symbol algebras, the first fact looks quite natural, while the second one looks somewhat strange and needs additional explanations.
In the rest of the section, we explain this apparent disagreement studying the precise connections between Toeplitz operator algebras \( T_H(PC(S^1, \Lambda)) \) and \( T_B(PC(\mathbb{D}, L)) = T_B(PC(S^1, \Lambda)) \), and between their Fredholm symbol algebras.

The best “meeting point” for Toeplitz operators \( T^a_B \) and \( T^a_H \) is their discrete versions \( T_n(a) \) and \( T(a) \), which act on the same discrete space \( l_2^\mathbb{N} \).

We start with a model case of a single point of discontinuity, \( \Lambda = \{1\} \), writing again in this case \( PC(S^1, 1) \).

Note first that, as the algebra \( T_d(PC(S^1, 1)) = R_H T_H(PC(S^1, 1)) R_H^* \) is unitary equivalent to the algebra \( T_H(PC(S^1, 1)) \), both these algebras possess the same Fredholm symbol algebra and the same homomorphism \( \text{sym} \), defined on generators by (12). At the same time the algebra \( T_d(PC(S^1, 1)) \) has an alternative description of its Fredholm symbol algebra \( \text{Sym} T_d(PC(S^1, 1)) \) and the homomorphism \( \text{sym} \) given by Corollary 9.

Moreover, both descriptions of \( \text{Sym} T_d(PC(S^1, 1)) \) coincide under the identification

\[
\lambda \in \mathbb{R} \mapsto x = \frac{1 + s(\lambda)}{2} = \frac{e^{\pi \lambda}}{e^{\pi \lambda} + e^{-\pi \lambda}} \in [0, 1].
\]

As it easy to see, each function \( a(t) \in PC(S^1, 1) \) admits the unique representation

\[
a(t) = c(t) + \alpha \ell(t),
\]

where \( c(t) \in C(S^1), \alpha \in \mathbb{C} \), and the function \( \ell(t) \) is given by (5). As Toeplitz operators \( T(c) \) and \( T_n(c) \) are unitary equivalent modulo a compact operator, the homomorphism \( \text{sym} \) maps them to the same element of the algebra \( \text{Sym} T_d(PC(S^1, 1)) \); thus, what is left, is to analyze the images of \( T(\ell) \) and \( T_n(\ell) \) in \( \text{Sym} T_d(PC(S^1, 1)) \).

Observe first that

\[
\text{sym}T(\ell) = \begin{cases} 
\ell(t), & t \in \hat{S}^1 \\
-s(\lambda), & \lambda \in \mathbb{R},
\end{cases}
\]

or

\[
\text{sym}T(\ell) = \begin{cases} 
\ell(t), & t \in \hat{S}^1 \\
1 \cdot (1 - x) + (-1) \cdot x = 1 - 2x, & x \in [0, 1].
\end{cases}
\]

Then, by Example 10,

\[
\text{sym}T_n(\ell) = \begin{cases} 
\ell(t), & t \in \hat{S}^1 \\
-s(\lambda) + s_0(\lambda), & \lambda \in \mathbb{R},
\end{cases}
\]

where \(-s(\lambda) + s_0(\lambda) = -s_1(\lambda) = -\tanh \frac{\pi \lambda}{2} \), or

\[
\text{sym}T_n(\ell) = \begin{cases} 
\ell(t), & t \in \hat{S}^1 \\
1 \cdot (1 - y) + (-1) \cdot y = 1 - 2y, & y \in [0, 1],
\end{cases}
\]

where

\[
y = \frac{1 + s_1(\lambda)}{2} = \frac{e^{\frac{\pi}{2} \lambda}}{e^{\frac{\pi}{2} \lambda} + e^{-\frac{\pi}{2} \lambda}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{1 - x}}, \quad x \in [0, 1]. \tag{13}
\]

Inverting the last formula, we have

\[
x = \frac{y^2}{y^2 + (1 - y)^2}, \quad y \in [0, 1]. \tag{14}
\]
That is, the symbol of the operator $T(\ell)$ has, in terms of the parameter $y$, the following expression
\[
\text{sym}T(\ell) = \begin{cases} 
\ell(t), & t \in \mathbb{S}^1 \\
(1 - 2y)(y^2 + (1 - y)^2)^{-1}, & y \in [0, 1],
\end{cases}
\]
which in particular means that the operator $T(\ell)$ belongs to the algebra $R_B T_B(\text{PC}(S^1, 1)) R_B^*$ generated by the operators $T_*(c)$ with $c(t) \in C(S^1)$ and $T_*(\ell)$.

Thus the $C^*$-algebra
\[
T_d(\text{PC}(S^1, 1)) = R_H T_H(\text{PC}(S^1, 1)) R_H^* = R_B T_B(\text{PC}(S^1, 1)) R_B^*
\]
is generated either by all operators $T(a)$, where $a(t) \in \text{PC}(S^1, 1)$, or by all operators $T_*(a)$, where $a(t) \in \text{PC}(S^1, 1)$.

The Fredholm symbol algebra $\text{Sym} T_d(\text{PC}(S^1, 1)) = T_d(\text{PC}(S^1, 1))/\mathcal{K}$ is isomorphic and isometric to $C(\Gamma)$, and the symbol homomorphism
\[
\text{sym} : T_d(\text{PC}(S^1, 1)) \longrightarrow C(\Gamma)
\]
is generated by
\[
\text{sym} : T(a) \longmapsto \begin{cases} 
a(t), & t \in \mathbb{S}^1 \\
a(1 + 0)(1 - x) + a(1 - 0)x, & x \in [0, 1],
\end{cases}
\]
for first system of generators, or by
\[
\text{sym} : T_*(a) \longmapsto \begin{cases} 
a(t), & t \in \mathbb{S}^1 \\
a(1 + 0)(1 - y) + a(1 - 0)y, & y \in [0, 1],
\end{cases}
\]
for the second system of generators, and both these descriptions agree under the reparameterization of the segment $[0, 1]$, given by formulas (13) and (14).

We summarize the above in the next theorem.

**Theorem 12.** Two $C^*$-algebras we are interested in: $T_H(\text{PC}(S^1, 1))$, generated by Toeplitz operators acting on the Hardy space, and $T_B(\text{PC}(S^1, 1))$, generated by Toeplitz operators acting on the Bergman space, are unitary equivalent.

They have the same Fredholm symbol algebra $C(\Gamma)$, but the symbol homomorphism, which is generated by the same mapping (12) of the corresponding generators, differs on the parameterization of the segment $[0, 1]$. The connection between these two parameterizations is given by (13) and (14).

Now for each function $a(t) \in \text{PC}(S^1, 1)$ the symbol of the operator $W T_a^B W^* - T_a^H = R_H^*(T_*(a) - T(a)) R_H$ is given by
\[
\text{sym}(W T_a^B W^* - T_a^H) = \begin{cases} 
0, & t \in \mathbb{S}^1 \\
f(a, x, y), & x, y = y(x) \in [0, 1],
\end{cases}
\]
where
\[
f(a, x, y) = [a(1 + 0)(1 - y) + a(1 - 0)y] - [a(1 + 0)(1 - x) + a(1 - 0)x] = [a(1 + 0) - a(1 - 0)] \sqrt{x(1 - x)} \frac{\sqrt{x} - \sqrt{(1 - x)}}{\sqrt{x} + \sqrt{(1 - x)}}.
\]
Thus the difference $W T_a^B W^* - T_a^H$ is compact if and only if the function $a(t)$ is continuous.
The extension of the above results from the model case of a single point of discontinuity to a general case of an arbitrary finite set $\Lambda$ of points of discontinuity is quite standard. To describe the corresponding Fredholm symbol algebras, one needs to apply the local principle, localizing by points of $S^1$, and then to observe that all local algebras at the points of discontinuity are isomorphic, which can be easily done by means of the rotation (along the unit circle) operators.

The main qualitative feature remains valid in a general case of a finite set of points of discontinuity, and the result is as follows.

**Corollary 13.** The $C^*$-algebras $T_H(\mathcal{P}C(S^1, \Lambda))$, generated by Toeplitz operators acting on the Hardy space, and $T_B(\mathcal{P}C(S^1, \Lambda))$, generated by Toeplitz operators acting on the Bergman space, are unitary equivalent, and have the same Fredholm symbol algebra $C(\Gamma)$. The symbol homomorphism is generated by the same mapping (12) of generators, but, when compared, differs on the parameterizations of the corresponding auxiliary segments $[0, 1]$. The connection between these two different parameterizations is given by (13) and (14). The difference $W^*_a W - T^H_a$ is compact if and only if the function $a(t)$ is continuous.

As a concluding remark we mention that each one of the above two approaches to the same (via the unitary equivalence) $C^*$-algebra generated by Toeplitz operators with piecewise continuous symbols has its own advantages. In particular, for example, the Bergman space approach to this algebra permits to single out many its interesting and unexpected elements (see, for details, [4,5,12]): the Toeplitz operators whose symbols are drastically different from the symbols of the initial generators, which are completely hidden in the Hardy space approach. Among these operators are Toeplitz operators with symbols having two different limit values along the lines of discontinuity, but these lines have completely different shapes compared with discontinuity lines of initial generators; Toeplitz operators having any finite or even infinite number of different limit values at the points of $\Lambda$ when approaching them from the interior of the unit disk, etc. On the other hand, for the Hardy space approach, the connection with Mellin convolution operators is much more transparent.

**Acknowledgment**

The author was partially supported by the CONACYT Project 102800, Mexico.

**References**


