J. Symbolic Computation (2000) **29**, 545–583
doi:10.1006/jsco.1999.0323
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Non-general Type Surfaces in $\mathbf{P}^4$: Some Remarks on Bounds and Constructions

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1. Introduction

The fast implementation of Buchberger's algorithm in modern computer algebra systems allows the computation of complicated examples in algebraic geometry. During the last couple of years such computations have helped to predict and check many theorems in algebraic geometry. Vice versa, inspired by complicated examples coming from algebraic geometry, computer algebra developers have refined their algorithms and implementations. In this paper we present some typical applications of computer algebra to projective algebraic geometry. We focus on one specific problem, namely the classification of non-general type surfaces in $\mathbf{P}^4$. Let us start with an introduction to this problem.

If $S \subset \mathbf{P}^n$, $n \geq 6$, is a smooth surface, then its secant variety $\mathrm{Sec}(S)$ does not fill up $\mathbf{P}^n$, and we may embed S into $\mathbf{P}^{n-1}$ via a linear projection from a point off $\mathrm{Sec}(S)$. For $n = 5$, however, the situation is different due to the following classical theorem.

Theorem 1.1. (Severi, 1901) *Let $S \subset \mathbf{P}^5$ be a smooth, non-degenerate surface. Then the following are equivalent:*

(i) $\mathrm{Sec}(S)$ *does not fill up* $\mathbf{P}^5$.
(ii) S *is the Veronese surface.*

So every smooth, projective surface can be embedded in $\mathbf{P}^5$, but we expect constraints on the numerical invariants of a smooth surface in $\mathbf{P}^4$. Indeed, the invariants of such a surface have to satisfy the double point formula 3.2. This formula is a key ingredient in the proof of the following theorem.

Theorem 1.2. (Ellingsrud and Peskine, 1989) *There exists an integer d_0 such that every smooth surface in $\mathbf{P}^4$ of degree $d > d_0$ is of general type.*

In particular, only finitely many components of the Hilbert scheme of surfaces in $\mathbf{P}^4$ contain smooth surfaces S with Kodaira dimension $\kappa(S) \leq 1$.

The problem of classifying these finitely many families is divided into two parts. The first task is the following problem.

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Problem 1.3. Find the true d_0.

The proof of Theorem 1.2 is based on a counting argument. A particularly transparent approach to the counting argument via ideas of Green (1998) on generic initial ideals is due to Braun and Fløystad (1994). Whereas an explicit calculation along the lines of Ellingsrud and Peskine (1989) shows that, roughly, $d_0 \leq 10\,000$, the arguments of Braun and Fløystad (1994) give that $d_0 \leq 105$. This result was slightly improved by Cook (1996) who showed that $d_0 \leq 76$. Further improvements are announced in Braun and Cook (1997) and Cook (1997), but these two papers contain serious mistakes. One motivation for our paper was to understand and clarify the approach of Braun, Cook and Fløystad. As a result we prove the following theorem.

Theorem 1.4. *Let $S \subset \mathbf{P}^4$ be a smooth surface which is not of general type. Then the degree of S is smaller or equal to* 52.

A more thorough study of the arguments of Braun, Cook and Fløystad would quite probably lead to a further slight improvement of the degree bound. We believe, however, that for a substantial improvement new ideas are needed. The conjectured bound is 15 (there are examples known in degree 15, see Aure *et al.*, 1997).

The second task one faces is the following problem.

Problem 1.5. Classify the non-general type surfaces in $\mathbf{P}^4$ degree by degree.

One reason for the recent progress in the classification of smooth surfaces in $\mathbf{P}^4$ of small degree is the finer study of the adjunction mapping by Sommese (1979), Van de Ven (1979), Sommese and Van de Ven (1987), and Reider (1988). In our context adjunction theory is used as a tool for determining where a given surface stands in the Enriques–Kodaira classification (see Barth *et al.*, 1984, for this classification). Classically, the adjunction process was introduced by Castelnuovo and Enriques (1971) in order to study curves on ruled surfaces. The Italian geometers around the turn of the century also started the classification of smooth surfaces in $\mathbf{P}^4$ of low degree. Further classification results are due to Roth (1937), who used the adjunction mapping to get surfaces with smaller invariants already known to him. Nowadays, through the effort of several mathematicians, a complete classification of smooth surfaces in $\mathbf{P}^4$ was worked out up to degree 10. Note that in the degree 10 classification adjunction theory played a minor role. Here a new approach by Popescu and Ranestad (1996) using the relations between multisecants, linear systems, syzygies and linkage proved to be more effective.

Another tool needed in order to attack Problem 1.5 is an effective method for the construction of surfaces. Besides working with general linear projections, which only gives the Veronese surface by Severi's theorem, there are two other classical construction methods. One is to verify that a certain linear system of projective dimension four on a certain abstract surface is very ample. This works especially well for rational, abelian and bielliptic surfaces. The other is to apply liaison (see Peskine and Szpiro, 1974) to a local complete intersection surface already known (presumably of lower degree). With a few exceptions these methods failed to produce examples in degree ≥ 11. In the case of liaison this is mainly due to the fact that the surfaces to be constructed tend to be minimal in their even liaison class (see Lazarsfeld and Rao, 1983), whereas if we consider linear systems of curves on minimal surfaces, the base points have to be in a special position.

Such configurations are hard to find. On the other hand, the syzygy type approach of Decker *et al.* (1993) yielded many examples and provided in fact a method to construct all examples known so far in a unified way (see also Popescu, 1993; Schreyer, 1996; Aure *et al.*, 1997; Abo *et al.*, 1998). The method of Decker *et al.* relies on symbolic computation and goes well together with adjunction theory and the approach of Popescu and Ranestad (1996) to classification. A second goal of our paper is to explain the computational details behind the construction method. We will, however, not comment further on the status quo of the classification.

So, our paper is divided into two main parts: a theoretical and a practical application of Gröbner bases.

The first part of the paper deals with the degree bound. In Section 2 we explain how to compute the Euler–Poincaré characteristic of a projective variety via generic initial ideals. This provides the counting argument needed in the proof of the theorem of Ellingsrud and Peskine as enhanced by Braun and Fløystad. We review this proof in Section 3. In Section 4 we improve the arguments of Braun, Cook and Fløystad and show Theorem 1.4.

In the second part of the paper we explain the construction method of Decker *et al.* with special emphasis on the computational aspects. The general outline of the construction method is presented in Section 5. Here we also start to discuss the computational aspects. Before giving further details we illustrate the construction method with some examples in Section 6. In Section 7 we present an algorithm to verify the smoothness of a locally Cohen–Macaulay surface in $\mathbf{P}^4$ which is considerably faster than the Jacobian Criterion. Section 8 is concerned with computational aspects of the adjunction process, whereas corresponding examples can be found in Section 9.

2. Counting

In this section we explain the counting argument behind the proof of the theorem of Ellingsrud and Peskine as enhanced by Braun and Fløystad. More precisely, we explain how to compute the Euler–Poincaré characteristic of a projective variety via generic initial ideals with special emphasis on points in the plane and on space curves. We start by recalling some basic facts on generic initial ideals (see Eisenbud, 1994, and Green, 1998, for details and proofs not given here).

We consider the polynomial ring $R = \mathbf{C}[x_0, \dots, x_n]$ and denote by $>$ a multiplicative monomial order on R with $x_0 > \cdots > x_n$. If $I \subset R$ is an ideal then its *initial ideal* $\mathrm{in}_>(I)$ is the monomial ideal generated by the initial monomials $\mathrm{in}_>(f)$, $f \in I$, with respect to $>$. $\mathrm{in}_>(I)$ carries important information on I. For example, we have the following theorem.

Theorem 2.1. (Macaulay, 1927) *If $I \subset \mathbf{C}[x_0, \dots, x_n]$ is a homogeneous ideal, then $\mathrm{in}_>(I)$ has the same Hilbert function as I.*

In generic coordinates much more can be said. Let us first recall the following basic definition.

Definition 2.2. A monomial ideal $I \subset \mathbf{C}[x_0, \dots, x_n]$ is said to be *Borel-fixed* if $x_i m \in I$, m a monomial in $\mathbf{C}[x_0, \dots, x_n]$, implies $x_j m \in I$ for all $j \leq i$.

Note that I is Borel-fixed iff I is invariant under the Borel-subgroup of lower triangular matrices of $\mathrm{GL}(n+1, \mathbf{C})$ acting on $\mathbf{C}[x_0, \ldots, x_n]$ in the standard way.

Let now

$$I = \bigoplus_{k \geq 0} I_k \subset \mathbf{C}[x_0, \ldots, x_n]$$

be an arbitrary homogeneous ideal and denote by $X \subset \mathbf{P}^n = \mathbf{P}^n(\mathbf{C})$ the subscheme defined by I. Then the homogeneous ideal I_X of X is obtained by saturation:

$$I_X = I^{\mathrm{sat}} := (I : \langle x_0, \ldots, x_n \rangle^\infty).$$

Theorem 2.3. (Galligo's Theorem) (Galligo, 1974; Bayer and Stillman, 1987a) *For any multiplicative monomial order $>$ on $\mathbf{C}[x_0, \ldots, x_n]$ as above there is a non-empty Zariski open subset $U_> \subset \mathrm{GL}(n+1, \mathbf{C})$ such that the initial ideal $\mathrm{in}_>(g(I))$ is constant and Borel-fixed for $g \in U_>$.*

For our purposes it is convenient to fix the (graded) reverse lexicographic order $>_{\mathrm{rlex}}$ induced by $x_0 > \cdots > x_n$ on R.

Definition 2.4. $\mathrm{gin}(I) := \mathrm{in}_{>_{\mathrm{rlex}}}(g(I))$, $g \in U_{>_{\mathrm{rlex}}}$, is called the *generic initial ideal* of I.

$>_{\mathrm{rlex}}$ is particularly well-suited for studying hyperplane sections:

Theorem 2.5. (Properties of generic initial ideals) (Bayer and Stillman, 1987b)

(i) The homogenous ideal I_X of X has the generic initial ideal

$$\mathrm{gin}(I^{\mathrm{sat}}) = (\mathrm{gin}(I) : x_n^\infty).$$

In particular, I is saturated iff no minimal generator of $\mathrm{gin}(I)$ involves x_n.

(ii) The homogeneous ideal of the general hyperplane section $X \cap H$ of X has the generic initial ideal

$$\mathrm{gin}(I_{X\cap H}) = (\mathrm{gin}(I) : x_{n-1}^\infty) \cap \mathbf{C}[x_0, \ldots, x_{n-1}].$$

(iii) We have

$$(\mathrm{gin}(I) : x_{n-\alpha}^\infty) = (\mathrm{gin}(I) : \langle x_{n-\alpha} \cdot \ldots \cdot x_n \rangle^\infty), \quad 0 \leq \alpha \leq n,$$

$$\operatorname{depth} R/I = \operatorname{depth} R/\mathrm{gin}(I) = \max\{\alpha \mid \mathrm{gin}(I) = (\mathrm{gin}(I) : x_{n+1-\alpha}^\infty)\},$$

and

$$\dim R/I = \dim R/\mathrm{gin}(I) = \max\{\alpha \mid \langle 1 \rangle \neq (\mathrm{gin}(I) : x_{n+1-\alpha}^\infty)\}.$$

(iv) The Castelnuovo–Mumford regularity of I is equal to the highest degree of a minimal generator of $\mathrm{gin}(I)$.

Schreyer's algorithm for computing syzygies (see Schreyer, 1980 or Eisenbud, 1994) gives the following theorem.

Theorem 2.6. *Suppose that the coordinates are chosen so that $\mathrm{in}(I) = \mathrm{gin}(I)$. Let $g_1, \ldots, g_N$ be a minimal Gröbner basis of I. For $1 \leq i \leq N$ set*

$$d_i := \deg g_i \qquad \text{and} \qquad \alpha_i := \min\{\alpha \mid \mathrm{in}(g_i) \in \mathbf{C}[x_0, \ldots, x_\alpha]\}.$$

Then I has a free resolution of type

$$0 \leftarrow I \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n+1} \leftarrow 0$$

with

$$F_k = \bigoplus_{i=1}^{N} \binom{\alpha_i}{k-1} R(-d_i - k + 1), \qquad 1 \le k \le n+1.$$

Proof. We sort the polynomials $g_1, \dots, g_N$ so that their initial terms are ordered by degree refined by $>_{\text{rlex}}$. Then g_1 has smallest degree, and $\alpha_1 = 0$ by Borel-fixedness. F_1 is a free R-module of rank N. Let $e_1, \dots, e_N$ be a basis of F_1 and consider the map

$$\Phi : F_1 \to R, \qquad e_i \mapsto g_i.$$

We sort the monomials in F_1 by the Schreyer order $>$ induced by $>_{\text{rlex}}$ and Φ:

$$\begin{array}{lll} me_i > me_j & \text{iff} & \text{in}(mg_i) >_{\text{rlex}} \text{in}(mg_j), \text{ or} \\ & & \text{in}(mg_i) = \text{in}(mg_j) \text{ but } i > j. \end{array}$$

Buchberger's S-pair test

$$m_{ji} g_i - m_{ij} g_j = \sum f_\mu^{(ij)} g_\mu, \quad i > j,$$

gives syzygies

$$m_{ji} e_i - m_{ij} e_j - \sum f_\mu^{(ij)} e_\mu, \quad i > j,$$

with initial monomials

$$x_\alpha e_i, \qquad 2 \le i \le N, 0 \le \alpha < \alpha_i,$$

by the definition of $>$ and Borel-fixedness. These syzygies form a Gröbner basis of the first syzygy module of I, that is, of ker Φ. Continuing in this way gives the desired resolution.□

Definition 2.7. The resolution above is called the *standard resolution* (or *S-resolution*) of I.

Remark 2.8. For a Borel-fixed monomial ideal this algorithm computes the minimal free resolution which has been analyzed by Eliahu and Kervaire (1990). In the case of an arbitrary homogeneous ideal I the S-resolution is rarely minimal. It has, however, the minimal possible length and no terms above the Castelnuovo–Mumford regularity of I by properties (iii) and (iv) of generic initial ideals. See Bayer *et al.* (1999) for some generalizations.

Corollary 2.9. (Cook, 1998, Theorem 4) *Suppose that I is in general coordinates, that is, that* $\text{in}(I) = \text{gin}(I)$. *Let I be generated by forms of degree $\le r$, and let $g_1, \dots, g_N$ be a minimal Gröbner basis of I. Then every minimal generator m of* $\text{gin}(I)$ *in degree $r+1$ satisfies*

$$x_\alpha \cdot \text{in}(g_i) >_{\text{rlex}} m$$

for some g_i of degree r, and for some $0 \le \alpha < \alpha_i$.

Proof. A minimal generator of degree $r+1$ of $\text{gin}(I)$ arises as the initial monomial of a remainder in Buchberger's S-pair test

$$m_{ji}g_i - m_{ij}g_j = \sum_{\mu=1}^{t} f_\mu^{(ij)} g_\mu + g_{t+1}$$

since I is generated by forms of degree $\leq r$. Arguing as above gives

$$x_\alpha \cdot \mathrm{in}(g_i) >_{\mathrm{rlex}} \mathrm{in}(m_{ji}g_i - m_{ij}g_j) \geq_{\mathrm{rlex}} \mathrm{in}(g_{t+1})$$

and thus the result.□

NOTATION 2.10. If $J \subset \mathbf{C}[x_0, \dots, x_n]$ is an ideal, then $J_{\leq r}$ (resp. $J_{<r}$) denotes the ideal generated by the elements of J of degree $\leq r$ (resp. $< r$).

DEFINITION 2.11. By abuse of language we say that $\mathrm{gin}(I)$ has a *gap* in degree r if $\mathrm{gin}(I)$ has no minimal generator in degree r.

THEOREM 2.12. (CRYSTALLIZATION PRINCIPLE) (Green, 1998) *If* $\mathrm{gin}(I)$ *has a gap in degree* r *then*

$$\mathrm{gin}(I_{<r}) = (\mathrm{gin}(I))_{<r}.$$

PROOF. The S-resolution of $I_{<r}$ is a subcomplex of the S-resolution of I since the entries of the syzygy matrices in the S-resolution of I of negative degrees are zero.□

In order to compute the Hilbert function of R/I or equivalently of $R/\mathrm{gin}(I)$ we stratify the set of monomials not in $\mathrm{gin}(I)$ by studying the generic initial ideal of successive hyperplane sections of X. For $0 \leq \alpha \leq \dim X$ consider the monomial ideal

$$J_\alpha := J_\alpha(I) := (\mathrm{gin}(I) : x_{n-\alpha}^\infty).$$

Then $J_\alpha = (\mathrm{gin}(I) : \langle x_{n-\alpha} \cdot \ldots \cdot x_n \rangle^\infty)$ by property (iii) of generic initial ideals. It follows from properties (i) and (ii) of generic initial ideals that $J_\alpha \cap \mathbf{C}[x_0, \dots, x_{n-\alpha}] = \mathrm{gin}(I_{X \cap H_1 \cap \dots \cap H_\alpha})$ is the generic initial ideal of the intersection of X with α general hyperplanes $H_1, \dots, H_\alpha$. The J_α define a filtration

$$\mathrm{gin}(I) =: J_{-1} \subset J_0 \subset J_1 \subset \dots \subset J_{\dim X} \subset \langle 1 \rangle =: J_{\dim X + 1}$$

(with $J_0 = J_{-1}$ if I is saturated). For $-1 \leq \alpha \leq \dim X$ let

$$M_\alpha := M_\alpha(I) := \{ \text{ monomials } m \in J_{\alpha+1} \cap \mathbf{C}[x_0, \dots, x_{n-\alpha-1}] \mid m \notin J_\alpha \}.$$

Then M_α is a finite set of monomials. Indeed, for each minimal generator m of $J_{\alpha+1}$ there is by definition an integer N such that $x_{n-\alpha-1}^N m \in \mathrm{gin}(I) \subset J_\alpha$. It follows that

$$\langle x_0, \dots, x_{n-\alpha-1} \rangle^N m \subset \mathrm{gin}(I) \subset J_\alpha$$

since $\mathrm{gin}(I)$ is Borel-fixed. The same argument shows that

$$(m\mathbf{C}[x_{n-\alpha}, \dots, x_n]) \cap \mathrm{gin}(I) = \langle 0 \rangle$$

if $m \notin J_\alpha$. Therefore the above filtration induces a decomposition

$$\mathbf{C}[x_0, \dots, x_n]/\mathrm{gin}(I) \cong \bigoplus_{\alpha=-1}^{\dim X} \bigoplus_{m \in M_\alpha} m\mathbf{C}[x_{n-\alpha}, \dots, x_n] \subset \mathbf{C}[x_0, \dots, x_n]$$

as a **C**-vector space. For the Hilbert function h_X of X this gives

$$h_X(t) = \sum_{\alpha=0}^{\dim X} \sum_{m \in M_\alpha} \binom{t- \mid m \mid +\alpha}{\alpha}$$

for $t \gg 0$. Here $\mid m \mid$ denotes the degree of the monomial m.

PROPOSITION 2.13. (FORMULA FOR χ) *With notations as above*

$$\chi(\mathcal{O}_X) = \sum_{\alpha=0}^{\dim X} (-1)^\alpha \sum_{m \in M_\alpha} \binom{\mid m \mid -1}{\alpha}.$$

PROOF. $\chi(\mathcal{O}_X)$ is the value of the Hilbert polynomial at $t = 0$.$\square$

NOTATION 2.14. For $0 \leq \alpha \leq \dim X$ let $M_\alpha(X) := M_\alpha(I_X)$.

REMARK 2.15. (i) By property (iii) of generic initial ideals X is arithmetically Cohen–Macaulay iff $M_\alpha(X) = \emptyset$, $0 \leq \alpha < \dim X$.

(ii) Let $X \cap H$ be a general hyperplane section. Then

$$M_\alpha(X \cap H) = M_{\alpha+1}(X), \qquad 0 \leq \alpha \leq \dim X - 1.$$

Bayer (1982) and Green (1998) visualize a monomial ideal by its *diagrams* in each given degree. For example, consider the generic initial ideal of a space curve C. By property (i) of generic initial ideals $\text{gin}(I_C)$ is generated by monomials of type $x_0^a x_1^b x_2^c$. Therefore we can picture all monomials in $\text{gin}(I_C)$ by one diagram as follows. Consider integers $a \geq b \geq 0$. Then all monomials of the form $x_0^a x_1^b x_2^c$, $c \geq 0$, are represented by the bth entry in the $(a+b)$th row of the diagram. More precisely, there are three possible types of entries. We put an empty circle iff $x_0^a x_1^b x_2^c$ is not in $\text{gin}(I_C)$ for every $c \geq 0$ (then $x_0^a x_1^b \in M_1(C)$). We put a circle containing an integer $c > 0$ iff $x_0^a x_1^b x_2^c \in \text{gin}(I_C)$ but $x_0^a x_1^b x_2^{c-1} \notin \text{gin}(I_C)$ (then $x_0^a x_1^b, \ldots, x_0^a x_1^b x_2^{c-1} \in M_0(C)$). Finally, we put an X iff $x_0^a x_1^b$ is in $\text{gin}(I_C)$. If the ith row of the diagram contains X's only, all information on $\text{gin}(I_C)$ can be read off from the first i rows of the diagram (we always suppose that rows not in the picture contain X's only). For a fixed $a \geq 0$, all entries representing monomials of type $x_0^a x_1^b x_2^c$, $b \geq 0, c \geq 0$, form the ath *diagonal* of the diagram. As another example, consider the diagram of the generic initial ideal of a set of points in the plane which is defined in the same way. In this case the entries are empty circles and X's only. In order to obtain the diagram of the generic initial ideal of a general plane section Γ of a space curve C from that one of $\text{gin}(I_C)$ we have to replace each circle containing a positive number by an X (this follows from property (ii) of generic initial ideals).

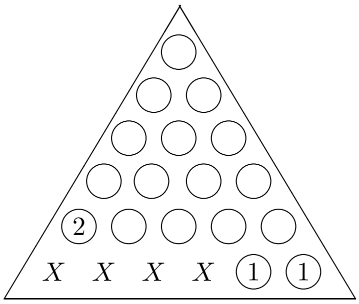
EXAMPLE 2.16. Let C be a general hyperplane section of a K3-surface $S \subset \mathbf{P}^4$ with degree $d = 14$ and sectional genus $\pi = 19$ as constructed by Popescu (1993). Then, as one can check,

$$\text{gin}(I_C) = \langle x_0^5, x_0^4 x_1, x_0^3 x_1^2, x_0^2 x_1^3, x_0 x_1^5, x_1^6, x_0 x_1^4 x_2, x_1^5 x_2, x_0^4 x_2^2 \rangle$$

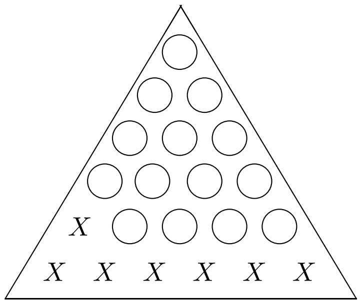
and

$$\text{gin}(I_\Gamma) = \langle x_0^4, x_0^3 x_1^2, x_0^2 x_1^3, x_0 x_1^4, x_1^5 \rangle.$$

The corresponding diagrams are:



$\mathrm{gin}(I_C)$



$\mathrm{gin}(I_\Gamma)$

In the spirit of Green (1998) we define:

Definition 2.17. In general, every monomial contained in one of the $M_\alpha(X)$ is called a *zero* of X. Monomials in $M_{\dim X}$ are called *ordinary zeros*, those in $M_{\dim X-1}$ *sporadic zeros*, those in $M_{\dim X-2}$ *sporadic sporadic zeros*, and so on.

For a set Γ of d points in the plane $\mathbf{P}^2$ the zeros provide one way of encoding the *numerical character* of Γ. By property (i) of generic initial ideals and Borel-fixedness $\mathrm{gin}(I_\Gamma)$ is of the form

$$\mathrm{gin}(I_\Gamma) = \langle x_0^s, x_0^{s-1}x_1^{\lambda_{s-1}}, \ldots, x_0x_1^{\lambda_1}, x_1^{\lambda_0}\rangle$$

with

$$\lambda_0 > \cdots > \lambda_{s-1} \geq 1.$$

Then s is the minimal degree of a hypersurface containing Γ and

$$M_0(\Gamma) = \{x_0^a x_1^b \mid 0 \leq a \leq s-1, 0 \leq b \leq \lambda_a - 1\}.$$

The sequence $n_i := \lambda_i + i$, $0 \leq i \leq s-1$, is the numerical character of Γ as introduced by Gruson and Peskine (1977).

Definition 2.18. $\lambda_0, \ldots, \lambda_{s-1}$ are called the *GP-invariants*, and

$$\nu_i := \lambda_{s-i-1} - \lambda_{s-i}, \qquad i = 1, \ldots, s-1,$$

the *GP-differences* of Γ.

In terms of the diagram s is the number of diagonals containing at least one circle and λ_i is the number of circles in the ith diagonal, $0 \leq i \leq s-1$. The degree of Γ can be read off from the diagram of Γ by counting the number of circles.

Remark 2.19. The formula for $\chi(\mathcal{O}_\Gamma)$ reads

$$d = \chi(\mathcal{O}_\Gamma) = \sum_{i=0}^{s-1} \lambda_i.$$

In particular, two sets of points in the plane have the same GP-invariants iff they have the same Hilbert function.

Definition 2.20. Γ is said to be in *uniform position* if every two subsets of Γ containing the same number of points have the same Hilbert function.

By applying the crystallization principle to a possible gap one can prove the following theorem.

Theorem 2.21. (Connectedness of the GP-invariants) (Gruson and Peskine, 1977) *If Γ is a set of points in the plane in uniform position, then the GP-invariants of Γ are* connected*, that is,*

$$\lambda_i - 1 \geq \lambda_{i+1} \geq \lambda_i - 2, \qquad 0 \leq i \leq s-2.$$

In other words, the GP-differences of Γ take values in $\{1, 2\}$ only.

Remark 2.22. By Harris' uniform position principle (Harris, 1980; Arbarello *et al.*, 1985) Theorem 2.21 applies in particular to a general plane section Γ of an integral space curve C.

The arithmetic genus π of C can be read off the diagram of C.

Remark 2.23. Let π be the arithmetic genus of C, and let $\lambda_0 > \cdots > \lambda_{s-1}$ be the GP-invariants of Γ. Then the formula for $\chi(\mathcal{O}_C) = 1 - \pi$ gives

$$\begin{aligned}\pi &= 1 + \sum_{m \in M_1(C)} (\mid m \mid -1) \quad - \sum_{m \in M_0(C)} 1 \\ &= 1 + \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)\lambda_i \right) \quad - \sum_{m \in M_0(C)} 1.\end{aligned}$$

In order to derive an upper bound for π we need a result of Laudal which can also be seen as a corollary to a more general result of Strano (see Theorem 4.5 below).

Theorem 2.24. (Laudal's Lemma) (Laudal, 1977; Gruson and Peskine, 1982; Strano, 1987; Green, 1998) *If the general hyperplane section $\Gamma \subset \mathbf{P}^2$ of an integral curve $C \subset \mathbf{P}^3$ of degree $d > s^2 + 1$ lies on a hypersurface of degree s then the same holds for C.*

Theorem 2.25. (Halphen's Bound) (Gruson and Peskine, 1977) *Let $C \subset \mathbf{P}^3$ be an integral space curve of degree d and arithmetic genus π. Let $s > 0$ be an integer such that $d > s(s-1)$ and $(I_C)_{<s} = 0$. Then*

$$\pi \leq 1 + \frac{d^2 + s(s-4)d}{2s} - \frac{r(s-r)(s-1)}{2s},$$

where

$$d + r \equiv 0 \ (\mathrm{mod}\ s) \qquad \text{with} \quad 0 \leq r < s.$$

Equality holds iff C is linked to a plane curve of degree r by a complete intersection of type $(s, \frac{d+r}{s})$.

PROOF. (1) Write $t = \frac{d+r}{s}$. Then $t - s + 1 \geq 1$ since $d > s(s-1)$.

(2) Denote by $\mathcal{C}(d,s)$ the set of all curves as in the assertion (with d and s fixed). If $C \in \mathcal{C}(d,s)$ write $\lambda_0(C), \ldots, \lambda_{s(C)-1}(C)$ for the GP-invariants of the general hyperplane section of C and $\nu_i(C) := \lambda_{s(C)-i-1} - \lambda_{s(C)-i}$, $i = 1, \ldots, s(C)-1$, for the corresponding GP-differences. Then $s(C) \geq s$ by the assumption and Laudal's lemma. Our goal is to maximize

$$1 + \sum_{m \in M_1(C)} (\mid m \mid -1) - \sum_{m \in M_0(C)} 1, \qquad C \in \mathcal{C}(d,s).$$

The first sum gets bigger if we replace an ordinary zero of low degree by an ordinary zero of higher degree. Thus the maximum is obtained for a curve $C \in \mathcal{C}(d,s)$ satisfying the following conditions: $s(C) = s$, at most one of the GP-differences $\nu_i(C)$ is 1, and C has no sporadic zeros. In fact, for each pair (d,s) there is precisely one connected s-tuple of GP-invariants satisfying the above condition (depending on the remainder r). This is defined by $\lambda_{s-1}(C) = t - s + 1$ (compare (1)) and the sequence of GP-differences given by $\nu_i(C) = 2$ if $i \neq s - r$ and $\nu_i(C) = 1$ if $i = s - r$ (note that in the case $r = 0$ all GP-differences are 2). For a curve C with these GP-invariants

$$\sum_{m \in M_1(C)} (\mid m \mid -1) = \frac{d^2 + s(s-4)d}{2s} - \frac{r(s-r)(s-1)}{2s}.$$

Every curve $C \subset \mathbf{P}^3$ linked to a plane curve of degree r by a complete intersection of type (s,t) has no sporadic zeros and GP-invariants as above. Indeed, such a curve has syzygies of type

$$0 \leftarrow I_C \leftarrow R(-s) \oplus R(-t) \oplus R(-y) \leftarrow R(-y-1) \oplus R(-y-r) \leftarrow 0,$$

where $y = t + s - r - 1$ (if $r = 0$ the curve C is a complete intersection of type (s,t) and the terms of type $R(-y)$ cancel out). So there are no sporadic zeros by Remark 2.15(i), and the GP-invariants $\lambda_0(C), \ldots, \lambda_{s-1}(C)$ are as claimed by connectedness. Conversely, the S-resolution of a curve with these invariants and no sporadic zeros minimalizes to a resolution of the type above because otherwise we would obtain an exact subcomplex as in the crystallization principle. This subcomplex resolves an ideal of codimension 1 which contradicts the irreducibility of the generators of I_C of smallest degree.□

One can dispense with the assumption $d > s(s-1)$.

COROLLARY 2.26. (Ellingsrud and Peskine, 1989) *Let $C \subset \mathbf{P}^3$ be an integral space curve of degree d and arithmetic genus π. Let $s > 0$ be an integer such that $(I_C)_{<s} = 0$. Then*

$$\pi \leq 1 + \frac{d^2 + s(s-4)d}{2s} =: G(d,s).$$

PROOF. If $d \leq s(s-1)$ let $t = \frac{d+r}{s}$ as above. Then

$$d > t(t-1), \qquad G(d,t) \leq G(d,s) \qquad \text{and} \qquad t < s.$$

The assertion follows from Theorem 2.25.□

3. Bounds

In this section we review the proof of the theorem of Ellingsrud and Peskine as enhanced by Braun and Fløystad.

NOTATION 3.1. S will denote a smooth surface in $\mathbf{P}^4 = \mathbf{P}^4(\mathbf{C})$ and

— H its hyperplane class,
— $d = H^2$ its degree,
— $K = K_S$ its canonical class,
— $\pi = \frac{1}{2}H.(H+K)+1$ its sectional genus,
— $\chi(\mathcal{O}_S) = 1-q+p_g$ its Euler–Poincaré characteristic.

Severi's Theorem 1.1 tells us that if we project a smooth non-degenerate surface in $\mathbf{P}^5$ other then the Veronese surface from a general point off the surface then its image in $\mathbf{P}^4$ will have double points. The formula computing the number of these points yields constraints on the invariants of a smooth surface in $\mathbf{P}^4$.

PROPOSITION 3.2. (DOUBLE POINT FORMULA) *The numerical invariants of a smooth surface $S \subset \mathbf{P}^4$ satisfy*

$$d^2 - 5d - 10(\pi - 1) + 2(6\chi(\mathcal{O}_S) - K^2) = 0.$$

PROOF. See, for example, Hartshorne (1977).□

The significance of the double point formula for the proof of the theorem of Ellingsrud and Peskine comes from Halphen's bound for π and the following two inequalities for K^2 and $6\chi(\mathcal{O}_S) - K^2$ respectively.

REMARK 3.3. Suppose that S is not of general type. Then by the Enriques–Kodaira classification $K^2 \leq 0$ and $\chi(\mathcal{O}_S) \geq 0$ except for rational surfaces with $K^2 \geq 1$ or irrational ruled surfaces.

- If S is a rational surface in $\mathbf{P}^4$ with $K^2 \geq 1$ then its degree is bounded by 5. Indeed, $H.-K \geq 3$ since $-K$ is effective. Severi's theorem implies
$$5 \geq h^0(\mathcal{O}_S(1)) \geq \chi(\mathcal{O}_S(1)) = \frac{H.(H-K)}{2} + 1 \geq \frac{(d+3)}{2} + 1$$
unless S is the Veronese surface in which case $d = 4$.
- Irrational ruled surfaces satisfy $K^2 \leq 8\chi(\mathcal{O}_S) \leq 0$.

Thus in order to prove the theorem of Ellingsrud and Peskine we may suppose that

$$K^2 \leq 0 \qquad \text{and} \qquad 6\chi(\mathcal{O}_S) - K^2 \geq 0.$$

In fact, the assumption not of general type is only needed to establish these two inequalities.

In order to combine surface theory with results on space curves like Halphen's bound we need an analogue to Laudal's lemma for surfaces in $\mathbf{P}^4$.

Theorem 3.4. (Roth's Lemma) (Roth, 1937; Mezzetti and Raspanti, 1993) *If the general hyperplane section $C \subset \mathbf{P}^3$ of a smooth surface $S \subset \mathbf{P}^4$ of degree $d > s^2 - s + 2$ lies on a hypersurface of degree s then the same holds for S.*

Notation 3.5. If $X \subset \mathbf{P}^n$ is a subscheme, then

$$s_X := \min\{k \in \mathbf{Z} \mid (I_X)_k \neq 0\}$$

denotes the minimal degree of a hypersurface containing X.

Remark 3.6. Let

$$C = S \cap \mathbf{P}^3, \qquad \Gamma = S \cap \mathbf{P}^2 = C \cap \mathbf{P}^2$$

be a general hyperplane and plane section of S, respectively. Then

$$s_S \geq s_C \geq s_\Gamma.$$

If $d > (s_S - 1)^2 + 1$ then equality holds by the lemmas of Roth and Laudal.

Here we give some information on s_S.

Proposition 3.7. (Ellingsrud and Peskine, 1989) *If $6\chi(\mathcal{O}_S) - K^2 \geq 0$, $s \in \{5, 6, 7, 8\}$, and $d > D(s) := 5(s+1)(s-2)/(s-4)$ then S lies on a hypersurface of degree s.*

Proof. If S does not lie on a hypersurface of degree s then the double point formula, Roth's lemma and Halphen's bound imply that

$$d^2 - 5d \leq \frac{10(d^2 + (s+1)(s-3)d)}{2(s+1)}. \square$$

Remark 3.8. (i) We have $D(5) = 90$, $D(6) = 70$, $D(7) = 200/3$, and $D(8) = 135/2$.
(ii) For $d \geq 9$ the assumption in Proposition 3.7 does not imply the assumption of Roth's lemma.
(iii) By classification (Roth, 1937; Aure, 1987; Koelblen, 1992) the degree of smooth, non-general type surfaces in $\mathbf{P}^4$ contained in a hypersurface of degree 3 is bounded by 8.

As a final key ingredient in the proof of the theorem of Ellingsrud and Peskine we derive an upper bound for the number of sporadic zeros of C.

Remark 3.9. The proof of Halphen's bound gives

$$\mu := \sum_{m \in M_0(C)} 1 \leq G(d, s_\Gamma) - \pi.$$

Proposition 3.10. (Ellingsrud and Peskine, 1989) *If $d > (s_S - 1)^2 + 1$ then $s := s_S = s_C = s_\Gamma$ and*

$$\mu \leq G(d, s) - \pi \leq \frac{d(s-1)^2}{2s}.$$

PROOF. Since S is contained in a hypersurface V of degree s the inclusion $S \subset V$ induces a section σ of the twisted conormal bundle $\mathcal{N}_S^*(s)$. $\gamma := 2s(G(d,s) - \pi)$ is the second Chern class of this twisted bundle, and the zero locus of σ is defined by the partials of the defining equation of V. It follows that $\gamma \leq d(s-1)^2$ (see Ellingsrud and Peskine, 1989). Note that if σ does not vanish in the expected codimension 2 then the contribution of the divisorial part gives an even better bound.□

We are now ready to show the key lemma in the proof of the theorem of Ellingsrud and Peskine.

PROPOSITION 3.11. (Ellingsrud and Peskine, 1989; Braun and Fløystad, 1994) *For every integer $s > 0$ there is a cubic polynomial $P_s(d)$ with leading term $\frac{d^3}{6s^2}$ such that every smooth surface $S \subset \mathbf{P}^4$ with $s = s_S$ and degree $d > (s-1)^2 + 1$ satisfies*

$$\chi(\mathcal{O}_S) \geq P_s(d).$$

PROOF. Let S be a surface as in the assertion with C and Γ as in Remark 3.6. Then

$$s := s_S = s_C = s_\Gamma$$

since $d > (s-1)^2 + 1$. Let $\lambda_0 > \cdots > \lambda_{s-1}$ be the GP-invariants of Γ and write $M_i = M_i(S)$, $i = 0, 1, 2$, for the spaces of monomials as in Section 2. Then

$$M_2 = M_1(C) = M_0(\Gamma) \qquad \text{and} \qquad M_1 = M_0(C).$$

We want to bound

$$\chi(\mathcal{O}_S) = \sum_{m \in M_2} \binom{|\, m \,| - 1}{2} \quad - \sum_{m \in M_1} (|\, m \,| - 1) \quad + \sum_{m \in M_0} 1$$

from below. Let us consider each summand separately.

(1) The first summand is smallest, if the d monomials of M_2 have smallest possible degrees, that is, if at most one of the GP-differences of Γ is 2. In that case λ_0 is the smallest integer $\geq \frac{d}{s} + \frac{s-1}{2}$. Hence

$$\sum_{m \in M_2} \binom{|\, m \,| - 1}{2} = \sum_{i=0}^{s-1} \left(\binom{\lambda_i + i - 1}{3} - \binom{i-1}{3} \right) \geq s \binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4}.$$

(2) Next consider the second summand. The Castelnuovo–Mumford regularity of I_C is at most $d-1$ (see Gruson *et al.*, 1983). It follows from property (iv) of generic initial ideals that the degree of each sporadic zero of C is bounded by $d-2$. Together with the bound for the number of sporadic zeros of C from Proposition 3.10 this gives

$$- \sum_{m \in M_1} (|\, m \,| - 1) \geq -\mu(d-3) \geq -\frac{(s-1)^2}{2s} d(d-3).$$

(3) The third summand in the formula for $\chi(\mathcal{O}_S)$ can be neglected since it is non-negative.

By summing up we obtain the desired polynomial

$$P_s(d) = s \binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4} - \frac{(s-1)^2}{2s} d(d-3)$$

with leading term $\frac{d^3}{6s^2}$.□

PROOF OF THE THEOREM OF ELLINGSRUD AND PESKINE. By Remark 3.3 we may suppose that $K^2 \leq 0$ and that $6\chi(\mathcal{O}_S) - K^2 \geq 0$. By Proposition 3.7 $d \leq 90$ or S is contained in a hypersurface of degree 5. We thus may and will assume that $s := s_S \leq 5$ and that $d > (s-1)^2 + 1$. Now

$$\begin{aligned} 0 \geq 2K^2 &= d^2 - 5d - 10(\pi - 1) + 12\chi(\mathcal{O}_S) \\ &\geq d^2 - 5d - 10(G(d,s) - 1) + 12P_s(d) \end{aligned}$$

by the double point formula, Halphen's bound and Proposition 3.11. The right-hand side is a cubic polynomial in d with positive leading coefficient $\frac{2}{s^2}$. So d is bounded.□

Before establishing a first effective degree bound we derive another upper bound for μ.

LEMMA 3.12. (Braun and Fløystad, 1994) *If* $6\chi(\mathcal{O}_S) - K^2 \geq 0$ *then*

$$\mu \leq G(d, s_\Gamma) - (d^2 - 5d + 10)/10.$$

PROOF. The assumption and the double point formula give a lower bound for π:

$$\pi \geq (d^2 - 5d + 10)/10.$$

The assertion follows by Remark 3.9.□

REMARK 3.13. Rewriting the proof of Proposition 3.11 with the bound above gives a cubic polynomial for smooth surfaces $S \subset \mathbf{P}^4$ with $6\chi(\mathcal{O}_S) - K^2 \geq 0$ and $s = s_\Gamma$. This time the leading coefficient is $\frac{1}{6s^2} - \frac{1}{2s} + \frac{1}{10}$. Note that this coefficient is negative in the case $s = 4$.

Let us compare the two upper bounds for μ.

REMARK 3.14. Taking the minimum of $G(d,s) - (d^2 - 5d + 10)/10$ and $d(s-1)^2/(2s)$ gives, for example, $9d/8$ if $s = 4$ and $d > 25$ ($d(20+d)/40$ if $d \leq 25$), d if $s = 5$ and $d(90-d)/60$ if $s = 6$.

In the following let $S \subset \mathbf{P}^4$ be a smooth surface with $K^2 \leq 0$ and $6\chi(\mathcal{O}_S) - K^2 \geq 0$.

REMARK 3.15. Let us derive a first effective degree bound. We slightly rewrite the estimates in the proof of the theorem of Ellingsrud and Peskine. Using

$$\pi \leq G(d,s) - \mu$$

(see Remark 3.9) instead of $\pi \leq G(d,s)$ and collecting terms involving sporadic zeros gives

$$\begin{aligned} 0 &\geq d^2 - 5d - 10(G(d,s) - 1) + 12\left(s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4}\right) - \sum_{m \in M_1} (12 \mid m \mid -22) \\ &\geq d^2 - 5d - \frac{10}{2s}(d^2 + s(s-4)d) + 12\left(s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4}\right) - \mu(12d - 46). \end{aligned}$$

Now take for simplicity Remark 3.8(iii) into account, insert the upper bound for μ from Remark 3.14 in cases $s = 4$ and $s = 5$ and compute that $d_0 \leq 147$.

In order to improve this bound along the above arguments it is crucial to improve the rough estimate for

$$A := \sum_{m \in M_1} (12 \mid m \mid -22).$$

Here are some easy observations (in the following replace d/s by the smallest integer $\leq d/s$ if necessary).

REMARK 3.16. In this remark we always may assume that $s := s_\Gamma = s_C = s_S$ (the degrees considered are large enough).

(i) For fixed d and s the maximal possible A is achieved if there are as many sporadic zeros of C as possible, and if all these arise from one minimal generator of $\text{gin}(I_C)$ in highest possible degree, that is, if $x_1^{\lambda_0} x_2^{\mu}$ is a minimal generator of $\text{gin}(I_C)$. In this case

$$A \leq \sum_{i=\frac{d}{s}+s-1}^{\frac{d}{s}+s-1+\mu-1} (12i - 22):$$

since $\lambda_0 \leq \frac{d}{s} + s - 1$ by the connectedness of the GP-invariants of Γ. Already this coarse estimate allows us to exclude the existence of certain surfaces. For example, there are no smooth non-general type surfaces $S \subset \mathbf{P}^4$ with $s = 6$ and degree $71 \leq d \leq 90$ (insert $\mu \leq \frac{71(90-71)}{60} \leq 22$ above). It follows from Proposition 3.7 that $d \leq 70$ or S is contained in a hypersurface of degree 5.

(ii) If $s \in \{4, 5\}$, however, the degree of $x_1^{\lambda_0} x_2^{\mu}$ might be bigger than $d - 1$. By taking regularity into account one gets the estimate

$$A \leq \sum_{i=\frac{d}{s}+s-1}^{d-2} (12i - 22) + \sum_{i=\frac{d}{s}+s-1}^{\mu-d+\frac{2d}{s}+2s-2} (12i - 22):$$

use up the maximum number of sporadic zeros by picking besides $x_1^{\lambda_0} x_2^{d-1-\lambda_0}$ with $x_0 x_1^{\lambda_1} x_2^{\mu-d+2d/s+2s-1}$ a second minimal generator of $\text{gin}(I_C)$ of highest possible degree (taking $x_1^{\lambda_0+1} x_2^{\mu-d+2d/s+2s-1}$ is not a possibility by Theorem 4.3 below). Putting things together implies that $d_0 \leq 80$ (and $d_0 \leq 68$ if $s = 4$). This is the bound established in Cook (1996).

REMARK 3.17. (i) By a brute force check Cook (1996) actually obtains that $d_0 \leq 76$ and that $s = s_\Gamma \leq 8$: since we already know that $d_0 \leq 80$ there are only finitely many possibilities for the GP-invariants $\lambda_0, \ldots, \lambda_{s-1}$ of I_Γ left (in particular $s \leq 12$). Now bound A as in (i) above if $s \geq 6$ and as in (ii) above if $s = 5$ and evaluate in the remaining cases directly the estimates

$$\mu \leq \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)\lambda_i \right) - B,$$

with

$$B = \frac{d^2 - 5d}{10},$$

and

$$0 \geq d^2 - 5d - 10\sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right)$$
$$+12\sum_{i=0}^{s-1}\left(\binom{\lambda_i + i - 1}{3} - \binom{i-1}{3}\right) - A$$

(see Remark 2.23, the proofs of Lemma 3.12 and Proposition 3.11, and Remark 3.13).

(ii) Cook (1996) actually claims that $s_S \leq 8$, but one has to take the assumptions of the lemmas of Laudal and Roth into account.

(iii) For later use we note another result of the brute force check: $s_\Gamma \leq 5$ or $d \leq 50$.

4. Improving the Degree Bound

In this section we provide further information on the configuration of the sporadic zeros of a curve arising as the hyperplane section of a smooth surface in $\mathbf{P}^4$. As a consequence we show that $d_0 \leq 52$.

Let first $C \subset \mathbf{P}^3$ be an arbitrary integral space curve (and Γ a general plane section). For $N \geq 0$ write

$$J^{(N)} := (\mathrm{gin}(I_C) : x_2^N) \cap \mathbf{C}[x_0, x_1, x_2].$$

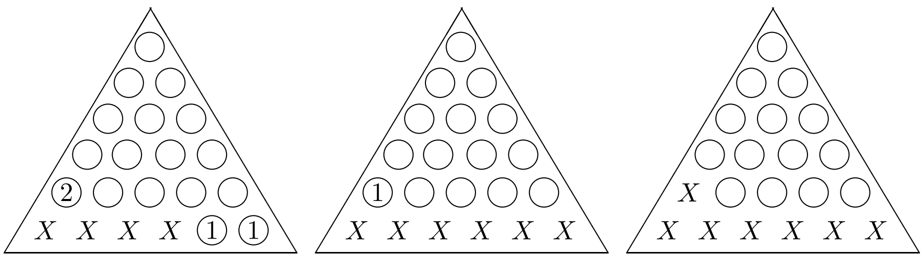
Then

$$\mathrm{gin}(I_\Gamma) = J^{(\infty)} := \bigcup_{N \geq 0} J^{(N)}.$$

Counting circles in the diagonals of the diagram of $J^{(\infty)}$ gives the GP-invariants of Γ. Hence the following definition generalizes the notion of GP-invariants from points in the plane to space curves.

Definition 4.1. (Cook, 1998) For $N \geq 0$ let s_N be the number of diagonals in the diagram of $J^{(N)}$ containing circles and write $\mu_i(N)$, $0 \leq i \leq s_N - 1$, for the number of circles in the ith diagonal of $J^{(N)}$. Then the $\mu_i(N)$ are called the *space curve invariants* of C.

Example 4.2. In Example 2.16 we have the following diagrams:



$\mathrm{gin}(I_C)$ $\quad J^{(1)}$ $\quad J^{(2)}$

So

$$s_0 = s_1 = 5, \qquad s_N = 4 \qquad \text{for } N \geq 2,$$

and

$$\begin{array}{lllll} \mu_0(0)=6, & \mu_1(0)=5, & \mu_2(0)=3, & \mu_3(0)=2, & \mu_4(0)=1, \\ \mu_0(1)=5, & \mu_1(1)=4, & \mu_2(1)=3, & \mu_3(1)=2, & \mu_4(1)=1, \\ \mu_0(N)=5, & \mu_1(N)=4, & \mu_2(N)=3, & \mu_3(N)=2 & \text{for } N\geq 2. \end{array}$$

Theorem 4.3. (Connectedness of the space curve invariants) (Cook, 1998) *If C is an integral space curve with $s_C = s_\Gamma$, then the invariants of C are* connected*, that is*

$$\mu_i(N)-1 \geq \mu_{i+1}(N) \geq \mu_i(N)-2, \qquad N\geq 0, \quad 0\leq i\leq s_N-2.$$

Remark 4.4. Cook claims the above result without the assumption $s_C = s_\Gamma$. As pointed out to us by Iustin Coandă, her proof has a gap: Lemma 8 in Cook (1998) is not true. A counterexample is the ideal

$$J := \langle x_0, x_1\rangle^{s-1}\cdot(x_0+x_2) + \langle x_0,x_1\rangle^{s-1}\cdot x_2^3 + \langle x_1^{s+2}, x_1^{s+1}x_2, x_1^s x_2^2\rangle.$$

The mistake in the proof is on page 230: $\mathrm{gin}(L)$ is not a Borel-fixed ideal.

The proof of the theorem with the additional hypothesis is, however, considerably simpler. It only depends on Theorem 4 and Lemmas 6 and 7 of Cook's paper, and on Harris' uniform position principle. For the convenience of the reader we give the complete proof.

Proof of Theorem 4.3. The first inequality

$$i+\mu_i(N) \geq i+1+\mu_{i+1}(N)$$

is just a consequence of Borel-fixedness. Suppose that the second inequality is not satisfied, that is, that

$$i+\mu_i(N) > i+2+\mu_{i+1}(N) =: k_0.$$

We assume that our coordinates are general. Write $I = I_C$ and consider the ideal

$$J := (\langle I, x_3\rangle : x_2^N) \cap \mathbf{C}[x_0,x_1,x_2].$$

Then

$$\mathrm{gin}(J) = \mathrm{gin}(\langle I,h\rangle : \langle x_0,\dots,x_3\rangle^N) \cap \mathbf{C}[x_0,x_1,x_2])$$

for a general hyperplane h, since $h = x_3$ is a general hyperplane, and

$$I_\Gamma \supset J \supset I\,|_{x_3=0}\ .$$

The ideal

$$K := J_{\leq k_0}$$

has the Borel-fixed initial ideal

$$\mathrm{in}(K) = \mathrm{gin}(K)$$

because the coordinates are general. Every minimal generator of $\mathrm{gin}(K)$ of degree $\leq k_0$ is divisible by x_0^{i+1} since $\mu_i(N)+i > i+2+\mu_{i+1}(N) = k_0$ and since $\mathrm{gin}(J)$ is Borel-fixed. Moreover, the minimal generators $x_0^a x_1^b x_2^c$ of $\mathrm{gin}(K)$ in degree k_0 satisfy $c>0$.

Claim. Every minimal generator of $\mathrm{gin}(K)$ of degree $k \geq k_0$ is divisible by x_2 and x_0^{i+1}.

We establish the claim by induction on k. Suppose that the claim holds for k. By Corollary 2.9 every minimal generator m of $\operatorname{gin}(K)$ in degree $k+1$ satisfies

$$x_\alpha \cdot n >_{\mathrm{rlex}} m$$

for some minimal generator $n \in \operatorname{gin}(K)$ of degree d, and for some $0 \leq \alpha < 2$. Hence $x_2|m$ since $x_2|n$ by assumption. Next suppose that $x_0^{i+1} \not| m$. Then $\tilde{m} := x_0^i \cdot x_1^{k+1-i} \in \operatorname{gin}(K)$ by Borel-fixedness. In fact, $\tilde{m}$ is a minimal generator of $\operatorname{gin}(K)$ since $x_0^{i+1}|\operatorname{gin}(K)_{\leq k}$. This contradicts $x_2|\tilde{m}$.

So $x_0^{i+1}|\operatorname{gin}(K)$. In fact, x_0^{i+1} is the maximal power which divides $\operatorname{gin}(K)$. So $K \subset \langle f \rangle$ with f a form of degree $i+1$. Since

$$i+1 < s_N = s_C = s_\Gamma \leq i+1+\mu_{i+1}(N),$$

I_Γ has a minimal generator of degree s_Γ which is divisible by f. Hence

$$\emptyset \subsetneq \Gamma \cap V(f) \subsetneq \Gamma \subset \mathbf{P}^2,$$

a contradiction to Harris' uniform position principle: for a general hyperplane H no subset of $\Gamma = C \cap H$ is distinguishable from another such set of the same degree.□

Theorem 4.3 gives constraints on the possible configurations of sporadic zeros of C. For the sake of completeness we mention another result in this direction.

Theorem 4.5. (Strano's Lemma) (Strano, 1988; Green, 1998) *Let C be an integral space curve with a sporadic zero in degree k. Then the ideal of a general plane section of C has a syzygy in degree $\leq k+2$.*

Besides Laudal's lemma one can deduce, for example, the following corollary also.

Corollary 4.6. (Strano, 1988; Green, 1998) *Let C be an integral space curve such that a general plane section of C has the Hilbert function of a complete intersection of type (a,b), $a,b \geq 2$. Then C is a complete intersection of type (a,b).*

In Remark 3.16 we obtained a worst-case estimate for A by supposing that $\operatorname{gin}(I_C)$ has one or two minimal generators of a very high degree. Let us analyze $\operatorname{gin}(I_C)$ in this case more carefully.

Notation 4.7. For a subscheme $X \subset \mathbf{P}^n$ let

$$t_X := \min\{k \in \mathbf{Z} \cup \{\infty\} \mid (I_X)_{\leq k} \text{ is not a principal ideal}\}.$$

Definition 4.8. We say that y minimal generators of $\operatorname{gin}(I_C)$ of degrees $r_1 \geq \cdots \geq r_y$ are *isolated*, if all other minimal generators of $\operatorname{gin}(I_C)$ are in degrees $\leq r_y - 2$, and if $r_y > t_C + 1$.

Lemma 4.9. *Let $C \subset \mathbf{P}^3$ be an integral space curve, and suppose that there are $1 \leq y <$ $\deg C$ isolated generators of $\operatorname{gin}(I_C)$ of degrees $r_1 \geq \cdots \geq r_y =: r$. Then $(I_C)_{<r}$ is the homogeneous ideal of a pure one-dimensional subscheme $Z \subset \mathbf{P}^3$ which decomposes as*

$$Z = C \cup Y,$$

where Y is a curve of degree y. Moreover,

$$\deg Y \cap C = 1 - p_a(Y) + \sum (r_i - 1).$$

Proof. The S-resolution of $(I_C)_{<r}$ is a subcomplex of the S-resolution of I_C since $\mathrm{gin}(I_C)$ has a gap in degree $r-1$ (compare the proof of the crystallization principle). Hence $(I_C)_{<r}$ is the homogeneous ideal of a pure one-dimensional subscheme Z since $(I_C)_{<r}$ is not a principal ideal. The y isolated generators of $\mathrm{gin}(I_C)$ in degrees $\geq r$ are monomials of type $x_0^a x_1^b x_2^c$ with $c \geq 1$ because otherwise such a generator would lie in the connected range around t_C. Hence $J := I_C/I_Z$ has a (not necessarily minimal) resolution of type

$$0 \leftarrow J \leftarrow \bigoplus_{i=1}^{y} R(-r_i) \leftarrow 2\bigoplus_{i=1}^{y} R(-r_i - 1) \leftarrow \bigoplus_{i=1}^{y} R(-r_i - 2) \leftarrow 0,$$

and $\deg Z = \deg C + y$. C is no component of the support of J since $y < \deg C$. So we have the desired decomposition $Z = C \cup Y$. Let $\mathcal{J}$ be the sheaf associated to J. Since

$$J \cong I_C/I_{C\cup Y} \cong I_C/I_C \cap I_Y \cong (I_C + I_Y)/I_Y$$

$\mathcal{J}$ is an ideal sheaf in $\mathcal{O}_Y$ and we obtain

$$\deg Y \cap C = 1 - p_a(Y) + \sum (r_i - 1)$$

from the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_Y \to \mathcal{O}_{C\cap Y} \to 0$$

and the resolution of J.□

For the rest of this section we suppose that C is a general hyperplane section of a smooth surface $S \subset \mathbf{P}^4$ of degree d, and with sectional genus π.

Proposition 4.10. *If $\mathrm{gin}(I_C)$ has a single isolated generator of degree r, or two isolated generators of degrees $r_1 =: r \geq r_2$ with $r_1 + r_2 > d+1$, then either S contains a plane curve of degree r, or $(I_S)_{<r}$ is a principal ideal.*

Proof. (1) We first apply Lemma 4.9 in order to show that C has an r-secant line. Let $Z = C \cup Y$ be as above.

(a) In the case of a single isolated generator $L = Y$ is an r-secant line.

(b) In the case of two isolated generators we may assume that $r_1 - r_2 \leq 1$ by (a). Y cannot be a plane conic because then the plane spanned by the conic would intersect C in more than d points, a contradiction to Bezout. So Y is a double line or consists of two skew lines.

In the case of a double line let L be the reduced part. Then

$$\text{length } C \cap L \geq (\text{length } C \cap Y)/2 = (r_1 + r_2)/2 - (1 + p_a(Y))/2$$

since this inequality holds locally: if (x, y) are local equations for L in a point p, and (x, y^2) are local equations in p for the double structure on L, then the exactness of

$$0 \leftarrow \mathcal{O}_{C,p}/\langle x, y\rangle \leftarrow \mathcal{O}_{C,p}/\langle x, y^2\rangle \xleftarrow{y} \mathcal{O}_{C,p}/\langle x, y\rangle$$

gives the desired inequality in p. The presentation of J in the proof of Lemma 4.9 gives

$$I_Y \subset I_L^2 + \langle g\rangle$$

with deg $g = r_1 - r_2 + 2$. Since Y is not a plane curve, equality holds and $p_a(Y) = r_2 - r_1 - 1$. Altogether

$$\text{length } C \cap L \geq (\text{length } C \cap Y)/2 = r_1 = r,$$

and equality holds by Bezout, since C is r_1-regular. Hence L is an r-secant line to C.

In the case $Y = L \cup L'$ with L, L' skew, one line has r_1 and the other r_2 intersection points with C. Thus again, there is an r-secant line to C.

(2) Now consider a component

$$\mathcal{L} \subset \{(L, H) \in \mathbf{G}(2,5) \times \check{\mathbf{P}}^4 \mid L \text{ is an } r\text{-secant of } S \cap H\}$$

which dominates $\check{\mathbf{P}}^4$, and its image $M \subset \mathbf{G}(2,5)$. $\mathcal{L}$ is of dimension 4. So M is two-dimensional since the fibre of $L \in M$ is $\mathbf{P}^2 \cong \{H \mid L \subset H\} \subset \check{\mathbf{P}}^4$.

Let $W = \bigcup_{L \in M} L \subset \mathbf{P}^4$. If W is a $\mathbf{P}^2$ then S intersects this $\mathbf{P}^2$ in a curve of degree r by (1). Otherwise W is the volume of minimal degree containing S. Suppose that $(I_S)_{<r}$ has another minimal generator. Then the two volumes intersect in dimension 2. So, at most, a curve $D \subset M$ could consist of r-secants. This contradicts the fact that $\mathcal{L}$ dominates M.□

Proposition 4.11. *A smooth surface $S \subset \mathbf{P}^4$ contains no plane curve of degree r with $\binom{r-1}{2} > G(d, s_\Gamma)$.*

Proof. Otherwise consider a general hyperplane H' through the plane curve D. Then

$$C' = H' \cap S = D \cup E,$$

where E is the union of reduced curves. C' is connected. Hence

$$\binom{r-1}{2} = p_a(D) \leq p_a(C') = p_a(C) \leq G(d, s_\Gamma),$$

a contradiction to our assumption.□

Proposition 4.12. *Let $S \subset \mathbf{P}^4$ be a smooth, non-general type surface of degree d. Suppose that the general hyperplane section C of S is r-regular with $r > (d+1)/2$. Moreover, suppose that $(I_S)_{<r}$ is a principal ideal. Then $d \leq 43$.*

Proof. Set $s = s_S$, $I = I_S$ and $M_i = M_i(S)$. By assumption and property (iv) of generic initial ideals all zeros of S have degree $\leq r - 1$. $I_{<r} = \langle I_s \rangle_{<r}$ implies that $\sum_{m \in M_0} 1$ gives a large contribution to $\chi(\mathcal{O}_S)$. There are at most

$$\binom{r+3}{4} - \binom{r-s+3}{4}$$

monomials in M_0 of degree $\leq r-1$. Not all of those actually are in M_0: for a monomial $m = x_0^a x_1^b \in M_2$ none of the monomials in $m\mathbf{C}[x_2, x_3]_{<r-|m|}$ is in M_0. For a monomial $m = x_0^a x_1^b x_2^c \in M_1$ the monomials in $m\mathbf{C}[x_3]_{<r-|m|}$ are not in M_0. Hence

$$\begin{aligned}\chi(\mathcal{O}_S) \geq & \sum_{m \in M_2} \left(\binom{\mid m \mid -1}{2} - \binom{r - \mid m \mid +1}{2} \right) \\ & - \sum_{m \in M_1} (\mid m \mid -1 + r - \mid m \mid) + \binom{r+3}{4} - \binom{r-s+3}{4}\end{aligned}$$

$$\geq \binom{r+3}{4} - \binom{r-s+3}{4} - d\binom{r+1}{2} + 2d + \sum_{m\in M_2} r(\mid m \mid -1) - \sum_{m\in M_1}(r-1)$$
$$\geq \binom{r+3}{4} - \binom{r-s+3}{4} - d\binom{r+1}{2} + 2d + r(\pi - 1).$$

The lower bound is weaker for smaller s. We treat the cases $s \geq 5$ and $s = 4$ separately.

In the case $s \geq 5$ we have $\pi - 1 \geq (d^2 - 5d)/10$ by Lemma 3.12. Substituting into the double point formula gives the desired

$$2K^2 \geq d^2 - 5d + (12r - 10)(\pi - 1) + 12\left(\binom{r+3}{4} - \binom{r-s+3}{4} - d\binom{r+1}{2}\right) + 24d$$
$$\geq 12\left(\binom{r+3}{4} - \binom{r-2}{4} - d\binom{r+1}{2} + r(d^2 - 5d)/10\right) + 24d > 0$$

for $d \geq 28$ since for $r = (d+2)/2 + x$ and $d = 28 + y$ the Taylor expansion in x and y of the lower bound has no negative terms and positive constant term.

In the case $s = 4$ we suppose that $d > 10$. We bound $\pi - 1 \leq G(d, 4) - 1 - 9d/8$ with Proposition 3.10 and obtain that $2K^2 > 0$ for $r > (d+1)/2$ and $d \geq 44$.

Note that the above expression is positive for all d if we suppose $r \geq d/2 + 3$ in the case $s \geq 5$, respectively $r \geq d/2 + 5$ in the case $s = 4$.□

Proof of Theorem 1.4. We combine the results above with explicit calculations for which we rely on Maple.

Let $S \subset \mathbf{P}^4$ be a smooth, non-general type surface of degree d. Our goal is to show that the degree d of S is bounded by 52.

(1) We already know from Remark 3.17(iii) that $s_\Gamma \leq 5$ or $d \leq 50$. So we may, and will, assume that

$$s := s_\Gamma = s_C \in \{4, 5\}.$$

(2) Suppose that S contains a plane curve of degree $r > (d+1)/2$. Then

$$\binom{r-1}{2} \leq G(d, s)$$

by Proposition 4.11. This implies $d \leq 31$ if $s = 5$. If $s = 4$, however, the above inequality is always fulfilled. In this case we use in Proposition 4.11 the sharper estimate

$$p_a(C) \leq G(d, s) - \mu$$

from Remark 3.9 (that is, we take the sporadic zeros into account). If, say, $\mu > d/2$ then

$$\binom{r-1}{2} \leq p_a(C) \leq G(d, s) - \mu < d^2/8 + 1 - d/2,$$

which is impossible for $d > 3$. Otherwise, by arguing naively as in Remark 3.16(i) with

$$A \leq \sum_{i=\frac{d}{4}+3}^{3\frac{d}{4}+2} (12i - 22),$$

we obtain

$$0 \geq \frac{d^3}{8} - 4d^2 - \frac{19}{2}d + 15,$$

and thus $d \leq 34$.

(3) Suppose that $\mathrm{gin}(I_C)$ has $y \leq 2$ isolated generators of degrees $r := r_1 \geq r_2 > (d+1)/2$. Then either S contains a plane curve of degree $r > (d+1)/2$, or $(I_S)_{<r}$ is a principal ideal by Proposition 4.10. Thus $d \leq 43$ by (2) and Proposition 4.12.

(4) It remains to treat the case that the largest generators of $\mathrm{gin}(I_C)$ of degree $r_1 \geq r_2 \geq r_3 \cdots$ satisfy $r_1 - r_2 \leq 1$ if $r_1 > (d+1)/2$ and $r_2 - r_3 \leq 1$ if $r_2 > (d+1)/2$ (in our worst cases $r_2 > (d+1)/2$ is achieved). A brute force check running through all of the finitely many still possible configurations of $\mathrm{gin}(I_C)$ gives $d_0 \leq 52$ (take the connectedness of the space curve invariants into account and use $B = \frac{d^2-9d}{8}$ in the case $s = 4$ for the estimate in Remark 3.17).$\square$

Remark 4.13. (i) By the criteria applied in the brute force check above we cannot exclude that

$$\langle x_0^5, x_0^4x_1^8, x_0^3x_1^9, x_0^2x_1^{10}, x_0x_1^{12}, x_1^{14}, x_0x_1^{11}x_2^{15}, x_1^{13}x_2^{14}, x_1^{12}x_2^{16}\rangle$$

is the generic initial ideal of a general hyperplane section of a non-general type surface in $\mathbf{P}^4$. Indeed, $s = 5$,

$$d = \sum_{i=0}^{4} \lambda_i = 12 + 11 + 10 + 9 + 8 = 50,$$

$$\mu = 15 + 14 + 16 = 45 = \sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right) - \frac{d^2 - 5d}{10},$$

and

$$2K^2 \geq d^2 - 5d - 10\sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right)$$
$$+12\sum_{i=0}^{s-1}\left(\binom{\lambda_i + i - 1}{3} - \binom{i-1}{3}\right) - \sum_{m\in M_1}(12 \mid m \mid -22) = 0.$$

(ii) Some of the remaining cases can be excluded with the help of Strano's lemma and some of its corollaries. We believe, however, that for a substantial improvement of the degree bound new ideas are needed.

5. Constructions

In this section we explain the construction method of Decker *et al.* (1993). A smooth surface $S \subset \mathbf{P}^4$ with given invariants d, π and $\chi(\mathcal{O}_S)$ is constructed as the degeneracy locus of a morphism $\phi : \mathcal{F} \to \mathcal{G}$ between vector bundles $\mathcal{F}$ and $\mathcal{G}$ with rank $\mathcal{G} =$ rank $\mathcal{F} + 1$. In order to construct $\mathcal{F}$ and $\mathcal{G}$ one has to analyze Beilinson's monad for the suitably twisted ideal sheaf $\mathcal{J}_S(m)$. Depending on the analysis $\mathcal{F}$ and $\mathcal{G}$ are constructed as direct sums of line bundles, twisted bundles of differentials, and/or syzygy bundles of the Hartshorne–Rao modules of S (or subbundles thereof). In many cases this construction is straightforward. In other, more subtle cases, the Hartshorne–Rao modules of S are rather special and their construction is non-trivial.

We first recall the general results we need. Let $\mathbf{P}^n$ be the projective space of lines in an $(n+1)$-dimensional $\mathbf{C}$-vector space V and denote by $R = \mathbf{C}[x_0, \ldots, x_n]$ its homogeneous coordinate ring.

Proposition 5.1. (Syzygy Bundles) *Let* $M = \bigoplus_{j\in\mathbf{Z}} M_j$ *be a graded* R*-module of finite length, and let*

$$0 \leftarrow M \leftarrow L_0 \xleftarrow{\alpha_1} L_1 \longleftarrow \cdots \xleftarrow{\alpha_{n+1}} L_{n+1} \leftarrow 0$$

be its minimal free resolution. Then for $1 \le k \le n-1$ *the sheafified syzygy module*

$$\mathcal{F}_k := Syz_k(M) := (\ker \alpha_k)^{\sim} = (\operatorname{im} \alpha_{k+1})^{\sim}$$

is a vector bundle on $\mathbf{P}^n$ *with the intermediate cohomology*

$$\bigoplus_{j\in\mathbf{Z}} H^i(\mathbf{P}^n, \mathcal{F}_k(j)) \cong \begin{cases} M & \text{if } i = k \\ 0 & \text{if } i \neq k, \quad 1 \le i \le n-1. \end{cases}$$

Conversely, any vector bundle $\mathcal{F}$ *on* $\mathbf{P}^n$ *with this intermediate cohomology is* stably equivalent *with* $\mathcal{F}_k$*, that is,*

$$\mathcal{F} \cong \mathcal{F}_k \oplus \mathcal{L}, \qquad \mathcal{L} \quad \text{a direct sum of line bundles.}$$

Example 5.2. (Bundles of Differentials) Consider $\mathbf{C}$ as a graded R-module sitting in degree 0. The minimal free resolution of $\mathbf{C}(k)$ is the *Koszul complex*

$$0 \leftarrow \mathbf{C}(k) \leftarrow \bigwedge^{0} V^* \otimes R(k) \leftarrow \cdots \leftarrow \bigwedge^{n+1} V^* \otimes R(k-n-1) \leftarrow 0.$$

The corresponding syzygy bundles are the twisted bundles of differentials,

$$Syz_k(\mathbf{C}(k)) \cong \Omega^k(k).$$

It follows from the sheafified Koszul complex that

$$\operatorname{Hom}(\Omega^k(k), \Omega^l(l)) \cong \bigwedge^{k-l} V, \qquad 0 \le k, l \le n,$$

the isomorphisms being given by contraction (see Beilinson, 1978).

Beilinson's theorem tells us that the derived category of coherent sheaves on $\mathbf{P}^n$ is generated by the $\Omega^k(k)$, $0 \le k \le n$.

Theorem 5.3. (Beilinson, 1978) (Monad Version) *For any coherent sheaf* $\mathcal{S}$ *on* $\mathbf{P}^n$ *there is a complex* $\mathcal{K}^{\cdot}$ *with*

$$\mathcal{K}^i \cong \bigoplus_j H^{i-j}(\mathbf{P}^n, \mathcal{S}(j)) \otimes \Omega^{-j}(-j),$$

such that

$$H^i(\mathcal{K}^{\cdot}) = \begin{cases} \mathcal{S} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Definition 5.4. $\mathcal{K}^{\cdot}$ above is called *Beilinson's monad* for $\mathcal{S}$.

For some applications it is useful to keep the following slightly weaker version of Beilinson's theorem in mind.

Theorem 5.5. (Beilinson, 1978) (Spectral Sequence Version) *For any coherent sheaf $\mathcal{S}$ on $\mathbf{P}^n$ there is a spectral sequence with E_1-terms*

$$E_1^{ji} = H^i\left(\mathbf{P}^n, \mathcal{S}(j)\right) \otimes \Omega^{-j}(-j)$$

converging to $\mathcal{S}$, that is, $E_\infty^{ji} = 0$ for $j+i \neq 0$ and $\bigoplus E_\infty^{-j,j}$ is the associated graded sheaf of a suitable filtration of $\mathcal{S}$.

Definition 5.6. The spectral sequence above is called *Beilinson's spectral sequence* for $\mathcal{S}$.

Remark 5.7. (i) The differentials

$$d_1^{ji} \in \operatorname{Hom}(H^i(\mathbf{P}^n, \mathcal{S}(j)) \otimes \Omega^{-j}(-j), H^i(\mathbf{P}^n, \mathcal{S}(j+1)) \otimes \Omega^{-j-1}(-j-1))$$

$$\cong \operatorname{Hom}(V^* \otimes H^i(\mathbf{P}^n, \mathcal{S}(j)),\ H^i(\mathbf{P}^n, \mathcal{S}(j+1)))$$

of the monad (spectral sequence) coincide with the natural multiplication maps.

(ii) The shape of the monad (spectral sequence) for $\mathcal{S}$ is determined by the dimensions

$$h^i\mathcal{S}(j) := H^i(\mathbf{P}^n, \mathcal{S}(j)), \qquad 0 \leq i \leq n,$$

in the range $-n \leq j \leq 0$. In order to construct a sheaf $\mathcal{S}$ with given cohomology via Beilinson's theorem one has to determine the differentials of the monad (spectral sequence).

(iii) It is often convenient to pick a twist m and apply Beilinson's theorem to $\mathcal{S}(m)$ instead of $\mathcal{S}$ itself.

We now specialize to the case $n = 4$ and present some information on the dimensions of the cohomology groups of the ideal sheaf of a smooth surface $S \subset \mathbf{P}^4$. First recall that for $i = 1, 2$ only finitely many of the dimensions $h^i\mathcal{J}_S(j)$, $j \in \mathbf{Z}$, are different from zero. In other words, the *Hartshorne–Rao modules* of S, that is, the graded R-modules

$$H^i_*\mathcal{J}_S = \bigoplus_{j \in \mathbf{Z}} H^i(\mathbf{P}^4, \mathcal{J}_S(j)), \qquad i = 1, 2,$$

are of finite length. Further information comes from the following proposition.

Proposition 5.8. (Riemann–Roch) *Let $S \subset \mathbf{P}^4$ be a smooth surface. Then*

$$\chi(\mathcal{J}_S(j)) = \chi(\mathcal{O}_{\mathbf{P}^4}(j)) - \binom{j+1}{2} d + j(\pi - 1) - 1 + q - p_g.$$

Before stating the next result let us recall from Remark 3.8(iii) that smooth, non-general type surfaces in $\mathbf{P}^4$ contained in a cubic hypersurface are classified.

Proposition 5.9. (Cohomology Table) (Decker *et al.*, 1993) *Let $S \subset \mathbf{P}^4$ be a smooth, non-general type surface which is not contained in any cubic hypersurface. Then we have* *the following table for the* $h^i\mathcal{J}_S(j)$*:*

$\uparrow i$

	0	0	0	0	0	0	
	$n'+1$	p_g	0	0	0	0	
	0	q	$h^2\mathcal{J}_S(1)$	$h^2\mathcal{J}_S(2)$	$h^2\mathcal{J}_S(3)$	$h^2\mathcal{J}_S(4)$	
	0	0	0	$h^1\mathcal{J}_S(2)$	$h^1\mathcal{J}_S(3)$	$h^1\mathcal{J}_S(4)$	
	0	0	0	0	0	$h^0\mathcal{J}_S(4)$	

$\longrightarrow j$

where

$$n' = \pi - q + p_g - 1.$$

In what follows we represent a zero in a cohomology table by an empty box.

We are now ready to explain the approach of Decker *et al.* (1993). We verify the existence of a family of smooth surfaces in $\mathbf{P}^4$ with given invariants by constructing an explicit example. In fact, we construct vector bundles $\mathcal{F}$ and $\mathcal{G}$ with rank $\mathcal{G}$ = rank $\mathcal{F}+1$ and a morphism $\phi : \mathcal{F} \to \mathcal{G}$ which drops rank along the desired surface S. If S has indeed the expected codimension 2, then S is locally Cohen–Macaulay and the Eagon–Northcott complex defined by the minors of ϕ identifies coker ϕ with the suitably twisted ideal sheaf of S,

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \to \mathcal{J}_S(m) \to 0.$$

Hopefully, S is smooth.

Construction Method 5.10. (Decker *et al.*, 1993)

0. Fix the numerical invariants d, π, and $\chi(\mathcal{O}_S) = 1 - q + p_g$ of the desired surface S.
1. Choose a plausible cohomology table for the ideal sheaf of S.
2. Pick a suitable twist m (usually $m = 4$ is a good choice) and consider the shape of Beilinson's monad $\mathcal{K}^{\cdot}$ for $\mathcal{J}_S(m)$. Quite frequently, for surfaces of low degree, the monad is simply a short exact sequence
$$0 \to \mathcal{K}^{-1} \to \mathcal{K}^0 \to \mathcal{J}_S(m) \to 0$$
(for example, if $m = 4$, this happens iff $q = 0$ and $\mathcal{J}_S$ is 5-regular). In this case we choose $\mathcal{F} = \mathcal{K}^{-1}$, $\mathcal{G} = \mathcal{K}^0$, and a generic $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ will give a smooth surface, or no smooth surface with such a cohomology table exists. If a smooth surface is obtained then the corresponding family of surfaces is unirational.
3. If Beilinson's monad has more terms we divide the H^1- and H^2-cohomology of $\mathcal{J}_S$ into two parts to be carried by $\mathcal{F}$ and $\mathcal{G}$ respectively. For an eventual analysis in step 6 later on it is convenient to divide the cohomology so that at least one of the bundles has no moduli.
4. Calculate vector bundles $\mathcal{F}$ and $\mathcal{G}$ with the desired H^1- and H^2-cohomology as direct sums of syzygy bundles and/or iterated syzygy bundles which are general in their particular moduli spaces. Alter $\mathcal{F}$ so that $\mathcal{F}$ also carries the H^3-cohomology of $\mathcal{J}_S$ (if $m = 4$ this amounts to add p_g copies of $\mathcal{O}(-1)$ as direct summands). Compare

the cohomology table of $\mathcal{F}$ and $\mathcal{G}$ with the chosen table for $\mathcal{J}_S$. If necessary, add direct sums of line bundles or replace $\mathcal{G}$ by a subbundle in order to adapt the H^0- and H^4-cohomology.

5. Compute $\operatorname{Hom}(\mathcal{F},\mathcal{G})$ and decide whether a general $\phi \in \operatorname{Hom}(\mathcal{F},\mathcal{G})$ gives rise to a smooth surface. If this is the case, the corresponding family of surfaces is unirational iff the moduli spaces of $\mathcal{F}$ and $\mathcal{G}$ are unirational.
6. If no smooth surface is obtained in this way analyze why not. Possible reasons are:
   (a) $\operatorname{Hom}(\mathcal{F},\mathcal{G}) = 0$.
   (b) A general $\phi \in \operatorname{Hom}(\mathcal{F},\mathcal{G})$ is not injective.
   (c) A general $\phi \in \operatorname{Hom}(\mathcal{F},\mathcal{G})$ does not vanish in expected codimension.
   (d) A general $\phi \in \operatorname{Hom}(\mathcal{F},\mathcal{G})$ defines a surface but always a singular one.

   In view of your analysis alter the construction of $\mathcal{F}$ and/or $\mathcal{G}$ in order to obtain special bundles in the particular moduli spaces with $\operatorname{Hom}(\mathcal{F},\mathcal{G})$ bigger (in practise this amounts to find special finite length modules which have additional syzygies).
7. Alter the construction of $\mathcal{F}$ and/or $\mathcal{G}$ in order to get a surface in a different family.

REMARK 5.11. In all examples known so far we can achieve that $\mathcal{F}$ has no moduli. Only in a few of these examples do we need to choose $\mathcal{G}$ special. The construction of the special bundles involves quite different ideas depending on the particular case.

In what follows we explain the single steps of the construction method in more detail.

For the explicit computation of the equations of S we note the following proposition.

PROPOSITION 5.12. *Let*

$$0 \leftarrow \mathcal{F} \longleftarrow \mathcal{F}_0 \xleftarrow{\alpha_1} \mathcal{F}_1$$

be a free presentation of $\mathcal{F}$ *and*

$$\mathcal{G}_{-2} \xleftarrow{\beta_{-1}} \mathcal{G}_{-1} \longleftarrow \mathcal{G} \leftarrow 0$$

be a free copresentation of $\mathcal{G}$. *Then*

(i) $\operatorname{Hom}(\mathcal{F},\mathcal{G}) \cong \{\tilde{\phi} \in \operatorname{Hom}(\mathcal{F}_0,\mathcal{G}_{-1}) \mid \tilde{\phi}\circ\alpha_1 = \beta_{-1}\circ\tilde{\phi} = 0\}$.

(ii) Let $\phi \in \operatorname{Hom}(\mathcal{F},\mathcal{G})$ *be represented by* $\tilde{\phi}$ *as in (i), and let*

$$\mathcal{G}_{-2} \xleftarrow{\beta_{-1}} \mathcal{G}_{-1} \xleftarrow{\beta_0} \mathcal{G}_0$$

be exact. Suppose that there exists an exact sequence

$$\mathcal{F}_0^* \xleftarrow{\tilde{\phi}^t} \mathcal{G}_{-1}^* \xleftarrow{(\beta_{-1}^t,\psi^t)} \mathcal{G}_{-2}^* \oplus \mathcal{O}(-m).$$

Then $\operatorname{im}(\psi\circ\beta_0)$ *is a twisted ideal sheaf* $\mathcal{J}(m)$.

For various steps in the construction we use computer algebra in an essential way. We do all computations over a finite prime field $\mathbf{F}_p$. In this way we avoid the well-known explosion of the coefficients in Buchberger's algorithm over the integers (rationals). In some cases we actually avoid working over a number field: if, for example, a kth root of unity has to be added, then we choose p so that a kth root of unity in $\mathbf{F}_p$ is known to us. Our main interest, however, is of course the existence of a surface over $\mathbf{C}$. Therefore we have to argue that the surface constructed explicitly over $\mathbf{F}_p$ is the restriction of a surface over a number field K: if $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ over $\mathbf{F}_p$ is the restriction of a homomorphism between vector bundles defined over a Zariski dense subset of Spec O_K, then there exists a surface defined over an eventually smaller Zariski dense subset of Spec O_K by the semi-continuity of the fibre dimension and since codim $S \leq 2$ in every point by the theorem of Hilbert–Burch. Thus, the general setup at this point is as follows: we have a surface S with desired invariants defined over $\mathbf{F}_p$ and we know that there exists

— a number field K and a prime $\wp$ in its ring of integers O_K such that $\mathbf{F}_p = O_K/\wp$, and
— a scheme $\mathcal{S} \subset \mathbf{P}^4_{\mathbf{Z}} \times \mathrm{Spec}\ (O_K)_\wp$ flat over Spec $(O_K)_\wp$ such that the special fibre $\mathcal{S} \otimes \mathbf{F}_p \cong S$.

If S is smooth then the general fibre $\mathcal{S} \otimes K$ is a smooth surface over a number field with the desired invariants since smoothness is an open property in the base Spec O_K. Once smoothness is established, we would like to spot the complex surface $\mathcal{S} \otimes \mathbf{C}$ in the Enriques–Kodaira classification by computations with S over the finite field. Here we use adjunction theory. Before discussing how to compute smoothness and the adjunction process explicitly, we present some examples.

Remark 5.13. When doing a construction over $\mathbf{F}_p$ with the computer we have to replace "a generic choice" in 5.10 by "a random choice" and hope for good luck. If we do not obtain a smooth surface for example in step 2, then we can only deduce that the existence of such a surface is rather unlikely.

6. Examples I

In this section we illustrate the method of Section 5 with some examples.

Example 6.1. (Alexander, 1988) One of the families constructed by Alexander consists of surfaces S with degree $d = 8$, sectional genus $\pi = 5$ and $p_g = q = 0$. Alexander verified the existence by showing that a certain linear system on a certain abstract surface is very ample (see Section 9). This proof is non-trivial. With the method of Section 5 the construction is straightforward: a plausible cohomology table for $\mathcal{J}_S$ is

i

5					
			2	2	
					5

j

By Riemann–Roch these are the smallest possible values for a smooth surface with the given invariants. The corresponding Beilinson monad for $\mathcal{J}_S(4)$ is of type

$$0 \to \mathcal{F} = 2\Omega^2(2) \xrightarrow{\phi} \mathcal{G} = 2\Omega^1(1) \oplus 5\mathcal{O} \to \mathcal{J}_S(4) \to 0.$$

That a general $\phi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ indeed gives rise to a smooth surface (with the desired invariants) can be checked in an explicit example.

In order to count parameters we note that

$$\operatorname{hom}(\mathcal{F}, \mathcal{G}) = 2 \cdot 2 \cdot 5 + 2 \cdot 5 \cdot \binom{5}{2} = 120,$$

and

$$\operatorname{end}(\mathcal{F}) + \operatorname{end}(\mathcal{G}) - 1 = 4 + (4 + 5 \cdot 5 + 2 \cdot 5 \cdot 5) - 1 = 82$$

(here we write $\operatorname{hom}(\mathcal{F}, \mathcal{G}) = \dim_{\mathbf{C}} \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ and so on). It follows that

$$\operatorname{hilb}_S(\mathbf{P}^4) = 120 - 82 = 38.$$

Up to projectivities we thus obtain a $(38 - 24) = 14$-dimensional, unirational family of surfaces.

Example 6.2. (Abo *et al.*, 1998) Another family of smooth surfaces S with degree $d = 8$ and $\pi = 5$ consists of irregular surfaces with $q = 1$ and $p_g = 0$. These surfaces had been falsely ruled out in the classification of degree 8 surfaces by Okonek (1986) and Ionescu (1988). With the method of Section 5 the construction is again straightforward (see Abo *et al.*, 1998, for more details): a plausible cohomology table for $\mathcal{J}_S$ is

$\uparrow i$

4					
	1	1			
			1	1	
					6

$\rightarrow j$

We construct a rank-5 vector bundle $\mathcal{G}$ which carrries the H^1- and H^2-cohomology of $\mathcal{J}_S$ as an iterated syzygy bundle (see Abo *et al.*, 1998, for the Beilinson monad of $\mathcal{G}$). In suitable coordinates a general finite length R-module with Hilbert function (1,1) can be written as

$$M = R/\langle x_0, \dots, x_3, x_4^2 \rangle.$$

Its Koszul resolution is of type

$$\begin{array}{ccccccccccc} & & & 4R(-1) & & 6R(-2) & & 4R(-3) & & R(-4) & \\ 0 \leftarrow M \leftarrow R & \nwarrow & & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & \nwarrow \\ & & & R(-2) & & 4R(-3) & & 6R(-4) & & 4R(-5) & & R(-6) \leftarrow 0. \end{array}$$

Let

$$\mathcal{K} := Syz_1(M(3))$$

and

$$\mathcal{G} := \ker\,(\mathcal{K} \xleftarrow{\psi} 5\mathcal{O}(1) \oplus 4\mathcal{O})$$

for a general $\psi \in \mathrm{Hom}(5\mathcal{O}(1) \oplus 4\mathcal{O}, \mathcal{K})$. Then

$$H^2_*\mathcal{G} \cong H^1_*\mathcal{K} \cong M(3)$$

and

$$H^1_*\mathcal{G} \cong \mathrm{coker}\,(H^0_*\mathcal{K} \leftarrow 5R(1) \oplus 4R) \cong M(1)$$

as desired. The minimal free resolution of $\mathcal{G}$ is of type

$$0 \leftarrow \mathcal{G} \leftarrow 10\mathcal{O}(-1) \leftarrow \begin{array}{c} 4\mathcal{O}(-2) \\ \oplus \\ 5\mathcal{O}(-3) \end{array} \leftarrow \begin{array}{c} \mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-4) \end{array} \leftarrow \mathcal{O}(-5) \leftarrow 0.$$

With $\mathcal{F} = 4\mathcal{O}(-1)$ we obtain the correct H^3–cohomology and adapt the H^0–cohomology at the same time. Note that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is globally generated. Thus in this case the criterion of Kleiman (1969) already implies that a general $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ drops rank along a smooth surface S.

Counting parameters gives

$$\mathrm{hilb}_S(\mathbf{P}^4) = 9 + 24.$$

Example 6.3. (Decker *et al.*, 1993) We construct smooth surfaces $S \subset \mathbf{P}^4$ with $d = \pi = 11$ and $p_g = p_a = 1$. A plausible cohomology table for the ideal sheaf $\mathcal{J}_S$ is

i

12	1					
		2				
				3	2	
						9

j

A check on Beilinson's monad for $\mathcal{J}_S(4)$ suggests picking

$$\mathcal{F} = \mathcal{O}(-1) \oplus 2\Omega^3(3) \qquad \text{and} \qquad \mathcal{G} = Syz_1(M),$$

where M is a graded, finite length R-module with Hilbert function (3,2). A general such module has syzygies of type

$$0 \leftarrow M \leftarrow 3R(1) \leftarrow 13R \leftarrow 20R(-1) \leftarrow \begin{array}{c} 10R(-2) \\ \oplus \\ 5R(-3) \end{array} \leftarrow 7R(-4) \leftarrow 2R(-5) \leftarrow 0.$$

Then any homomorphism $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is given by a commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & 0 \arrow[d] \\
10\mathcal{O}(-2) \oplus 5\mathcal{O}(-3) \arrow[d] & 2\mathcal{O}(-2) \arrow[l, "\phi_1"'] \arrow[d] \\
20\mathcal{O}(-1) \arrow[d] & \mathcal{O}(-1) \oplus 10\mathcal{O}(-1) \arrow[l, "(\phi_{01},\phi_{02})"'] \arrow[d] \\
\mathcal{G} \arrow[d] & \mathcal{F} \arrow[l, "\phi"'] \arrow[d] \\
0 & 0.
\end{tikzcd}$$

One can check in an explicit example that a general choice of ϕ_1 and ϕ_{01} gives rises to a smooth surface.

This time we obtain

$$\mathrm{hilb}_S(\mathbf{P}^4) = 19 + 24.$$

Other families with the desired invariants are obtained by choosing M more special: that is, with extra syzygies (see Popescu, 1993).

Example 6.4. (Decker *et al.*, 1993) We construct a family of smooth surfaces with $d = 11$, $\pi = 10$ and $p_g = q = 0$. A plausible cohomology table for the ideal sheaf of such a surface is

$\uparrow i$

10						
		2				
			1	5	5	
						5

$\longrightarrow j$

Thus we are tempted to take

$$\mathcal{F} = 2\Omega^3(3) \qquad \text{and} \qquad \mathcal{G} = Syz_1(M),$$

where M has Hilbert function (1,5,5). However, the general such module has syzygies of type

$$\begin{array}{cccccc}
0 \leftarrow M \leftarrow R(2) & & & & & \\
& \nwarrow & & & & \\
& 10R & & 15R(-1) & & \\
& & \nwarrow & \oplus & \nwarrow & \\
& & & 5R(-2) & & 26R(-3) \leftarrow 20R(-4) \leftarrow 5R(-5) \leftarrow 0,
\end{array}$$

so in the general case $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ (compare with the Koszul resolution of $2\Omega^3(3)$ as in Example 6.3). In order to obtain $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \neq 0$ we look for special quadrics $q = (q_1, \ldots, q_{10})$ which have extra syzygies.

One possibility is to choose the ten quadrics as follows (see Decker *et al.*, 1993, for details). Each elliptic normal curve $E \subset \mathbf{P}^4$ is cut out by five quadrics. Consider a 2-torsion translation scroll Q of E. Then Q actually contains three different elliptic normal curves of which it is a 2-torsion translation scroll. The ten quadrics given by a pair of these curves present a module M with syzygies of type

$$0 \leftarrow M \leftarrow R(2) \leftarrow 10R \leftarrow \begin{matrix}15R(-1)\\ \oplus \\ 10R(-2)\end{matrix} \leftarrow \begin{matrix}5R(-2)\\ \oplus \\ 26R(-3)\end{matrix} \leftarrow 20R(-4) \leftarrow 5R(-5) \leftarrow 0,$$

and now, as one can check in an example, a general $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ drops rank along a smooth surface.

Counting parameters gives a family of dimension $7 + 24$.

Example 6.5. (Schreyer, 1996) We explain how to construct further families with the same invariants and cohomology table as in 6.4. This time we are looking for modules M with syzygies of type

$$0 \leftarrow M \leftarrow R(2) \leftarrow 10R \leftarrow \begin{matrix}15R(-1)\\ \oplus \\ 7R(-2)\end{matrix} \leftarrow \begin{matrix}2R(-2)\\ \oplus \\ 26R(-3)\end{matrix} \leftarrow 20R(-4) \leftarrow 5R(-5) \leftarrow 0.$$

For the first syzygy bundle $\mathcal{G}$ corresponding to such a module there is up to an automorphism of $\mathcal{F} = 2\Omega^3(3)$ a unique morphism $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Indeed, such a morphism is induced by a commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & 0 \arrow[d] \\
2\mathcal{O}(-2) \oplus 26\mathcal{O}(-3) \arrow[d] & 2\mathcal{O}(-2) \arrow[l, "\phi_1"'] \arrow[d] \\
15\mathcal{O}(-1) \oplus 7\mathcal{O}(-2) \arrow[d] & 10\mathcal{O}(-1) \arrow[l, "\phi_0"'] \arrow[d] \\
\mathcal{G} \arrow[d] & \mathcal{F} \arrow[l, "\phi"'] \arrow[d] \\
0 & 0
\end{tikzcd}$$

with ϕ_1 an isomorphism onto the first summand. Thus if ϕ gives rise to a smooth surface, then the surface is completely determined by

$$H^1_* \mathcal{J}_S = M = [R/\langle q_1, \ldots, q_{10} \rangle](2).$$

The parameter space for cyclic, graded, finite length R-modules M with Hilbert function $(1, 5, 5, 0)$ is the Grassmanian $\mathbf{G}(10, H^0(\mathbf{P}^4, \mathcal{O}(2)))$, that is, a rational space of dimension 50. The expected codimension of the moduli space of those M's with two extra syzygies is

$$7 \cdot 2 = 14.$$

So in principle we expect a $50 - 14 = 12 + 24$-dimensional family of such surfaces.

Over the finite fields $\mathbf{F}_2$ or $\mathbf{F}_3$ the desired modules and surfaces can be found in reasonable time by brute force and trial. To check quadrics $q_1, \ldots, q_{10}$ for extra syzygies takes about 40 examples per second on a 400 MHz Pentium II machine with Macaulay2 (Grayson and Stillman, 1999). On the other hand, the chance to find an M with extra syzygies in $\mathbf{G}(10, H^0(\mathbf{P}^4, \mathcal{O}(2)))$ is roughly $1 : p^{14}$. So for $\mathbf{F}_3$ (we want to avoid funny Enriques surfaces in characteristic 2) we can hope to find an example in about 2 hours. Luckily one can describe the scheme of good M's as a codimension 7 subscheme of a rational variety, so we can find good points even within a couple of minutes (see Schreyer, 1996, for more details).

Once we have found a surface over our finite field we can deduce the existence of a surface over a number field if the variety of good M's is smooth in the given point of expected codimension 14 over the finite field (again see Schreyer, 1996, for more details). For the surfaces constructed with this method it is not clear to us whether their moduli space is unirational or not. The main point of the brute force trial method is that it does not implicitly assume the unirationality of the parameter space.

7. Smoothness

In this section we give a method to verify the smoothness of a locally Cohen–Macaulay surface $S \subset \mathbf{P}^4$ which is considerably faster than the Jacobian criterion.

Notation 7.1. $f_1, \ldots, f_N$ will be a set of homogeneous generators of $\mathcal{J}_S$ and $I := \langle f_1, \ldots, f_N \rangle$.

$$J := \left\langle \frac{\partial f_i}{\partial x_j} \mid 1 \leq i \leq N,\ 0 \leq j \leq 4 \right\rangle$$

is the Jacobian ideal of $f_1, \ldots, f_N$ and $I_k(J)$ the ideal of $k \times k$ minors of J. Moreover, if $f = f_i$ is one of the generators, then we write $I_k(f)$ for the ideal of $k \times k$-minors of J which involve the row corresponding to f and $J(f)$ for the Jacobian matrix of f.

The implicit function theorem gives the following result.

Theorem 7.2. (Jacobian Criterion) *A pure 2-codimensional subscheme $S \subset \mathbf{P}^4$ is smooth iff*

$$S \cap V(I_2(J)) = \emptyset,$$

that is, iff

$$I_2(J) + I \text{ is } \langle x_0, \ldots, x_4 \rangle\text{-primary.}$$

To check smoothness by this criterion means to compute the codimension of $I_2(J) + I$. This is expensive because

(1) the computation of the ideal $I_2(J)$ is large, and
(2) the computation of a Gröbner basis of $I_2(J) + I$ is large.

With the method below we replace (1) by the computation of only $10(N-1)$ (instead of $5N(N-1)$) minors and (2) by a Gröbner basis computation of an ideal of codimension 4 (instead of 5) which is much faster. In addition, we need one more Gröbner basis computation which is cheap.

Theorem 7.3. *Let $S \subset \mathbf{P}^4$ be a locally Cohen–Macaulay surface of degree d and sectional genus π. Let $f = f_i$ be one of the generators of $\mathcal{J}_S$ as above and write $e := \deg f$. Suppose that*

(i) $V((I_1(J))_{<e} + I) = \emptyset$,
(ii) $V(I_2(f) + I)$ *is finite and*

$$\deg V(I_2(f) + I) = \deg V(J(f) + I) = d^2 + e(e-4)d - 2e(\pi - 1).$$

Then S is smooth.

Proof. The crucial ingredient is that $\operatorname{codim} S \leq \dim S$.

By (i) S has at most hypersurface singularities. Hence the conormal bundle

$$\mathcal{N}^* = \mathcal{J}_S/\mathcal{J}_S^2$$

is locally free of rank 2. f induces a section σ of $\mathcal{N}^*(e)$. $J(f) + I$ describes the zero locus of the section $\tilde{\sigma}$ of $\Omega^1_{\mathbf{P}^4} \otimes \mathcal{O}_S(e)$ corresponding to σ via the exact sequence

$$0 \to \mathcal{J}_S/\mathcal{J}_S^2 \to \Omega^1_{\mathbf{P}^4} \otimes \mathcal{O}_S \to \Omega^1_S \to 0.$$

Sing S is contained in $V(I_2(f) + I)$ by the implicit function theorem. So S has at most finitely many singularities by (ii). The zero locus of $\tilde{\sigma}$ coincides with the zero locus of σ iff $\tilde{\sigma}$ does not vanish in the singular points of S, and both zero loci are finite by (ii). We have

$$\begin{aligned} \deg V(\sigma) &= c_2(\mathcal{N}^*(e)) = c_2(\mathcal{N}) + c_1(\mathcal{N}^*).eH + e^2 d \\ &= d^2 + (-5H - K).eH + e^2 d = d^2 + e(e-4)d - 2e(\pi - 1). \end{aligned}$$

So $V(I_2(f) + I) = V(J(f) + I)$ by (ii). For an arbitrary $g \in I_e$ we have

$$V(I_2(g)) \supset V(J(g)) \supset V(\sigma_g),$$

where σ_g denotes the section of $\mathcal{N}^*(e)$ induced by g. By semicontinuity

$$\deg V(I_2(f) + I) \geq \deg V(I_2(g) + I)$$

for g in a Zariski dense subset of I_e. Thus these g satisfy (ii) as well and we obtain

$$\operatorname{Sing} S \subset V((I_1(J))_{<e} + I) = \emptyset. \square$$

Remark 7.4. In order to check (ii) we have to compute a Gröbner basis of $I_2(f) + I$. This computation is easiest if we take f to be a generator of lowest possible degree. Frequently, in the examples known so far, I is generated by quintics and sextics, and the zero locus of the quintics alone is the surface S union some 6-secant lines. In that case, if S is smooth, a general $f \in I_5$ will satisfy our conditions (over an infinite ground field). Over a finite field it is possible but unlikely that no such f exists.

8. Adjunction Theory

In order to spot a surface given by explicit equations within the Enriques–Kodaira classification we invoke the adjunction process.

THEOREM 8.1. (Sommese, 1979; Van de Ven, 1979; Sommese and Van de Ven, 1987) *Let $S \subset \mathbf{P}^n$ be a smooth surface over* $\mathbf{C}$. *Then the adjoint linear system* $\mid H+K \mid$ *is non-special of dimension* $n' = \pi - q + p_g - 1$. *It defines a birational morphism*

$$\Phi = \Phi_{|H+K|} : S \to S' \subset \mathbf{P}^{n'}$$

onto a smooth surface S' *which blows down precisely all* (-1)*-curves on* S *which are lines in the given embedding, unless*

(1) *S is a plane, or the Veronese surface of degree* 4, *or S is ruled by lines, or*
(2) *S is a Del Pezzo surface, or a conic bundle, or*
(3) *S belongs to one of the following four families (with obvious notations):*
   (a) $S = \mathbf{P}^2(p_1, \ldots, p_7)$ *embedded by* $H \equiv 6L - \sum_{i=1}^{7} 2E_i$,
   (b) $S = \mathbf{P}^2(p_1, \ldots, p_8)$ *embedded by* $H \equiv 6L - \sum_{i=1}^{7} 2E_i - E_8$,
   (c) $S = \mathbf{P}^2(p_1, \ldots, p_8)$ *embedded by* $H \equiv 9L - \sum_{i=1}^{8} 3E_i$,
   (d) $S = \mathbf{P}(\mathcal{E})$, *where* $\mathcal{E}$ *is an indecomposable rank 2 bundle over an elliptic curve, and* $H \equiv 3B$, *where* B *is a section with* $B^2 = 1$ *on* S.

REMARK 8.2. (i) In the exceptional case (1) $\mid H+K \mid = \emptyset$ and $(H+K)^2 < 0$. In the exceptional case (2) $(H+K)^2 = 0$ and $\mid H+K \mid$ maps to a point, or defines the conic fibration (unless $d = 8$, $q = 1$ and $\pi = 3$, see Beltrametti and Sommese, 1995, 10.1). In all other cases $(H+K)^2 > 0$, but in the exceptional case (3) $\mid H+K \mid$ is simply too small to define a birational map.

(ii) In the general case

$$\deg S' = (H+K)^2 \qquad \text{and} \qquad 2\pi' - 2 = (H+K).(H+2K).$$

Over a finite field we do not know whether adjunction theory holds. Perhaps there is a larger list of exceptions. However, we have the following proposition.

PROPOSITION 8.3. *Let S be a surface over a field of arbitrary characteristic. Suppose that the adjoint linear system* $\mid H+K \mid$ *is base point free and that the image* $S' \subset \mathbf{P}^{n'}$ *under the adjunction map* $\Phi = \Phi_{|H+K|}$ *is a surface of expected degree* $(H+K)^2$, *expected sectional genus* $\pi' = \frac{1}{2}(H+K).(H+2K)+1$, *and with* $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'})$. *Then S' is smooth and* $\Phi : S \to S'$ *is a simultaneous blow down of the* $K'^2 - K^2$ *many* (-1)*-lines on* S.

PROOF. Φ is birational since S' has the expected degree. S' has at most isolated singularities since S' has the expected sectional genus. Let $\tilde{S}$ be the normalization of S' and $S \to \tilde{S}$ the induced map. Each irreducible curve $E \subset S$ contracted by Φ satisfies $E.(H+K) = 0$. Hence E^2 and $E.K$ are smaller than zero and E is a (-1)-curve. Moreover $H.E = -E.K = 1$. So $\tilde{S}$ is smooth and $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_S)$. S' non-normal would imply that $\chi(\mathcal{O}_{S'}) < \chi(\mathcal{O}_{\tilde{S}})$, a contradiction to our assumption.□

COROLLARY 8.4. *Let* $\mathcal{S} \to \operatorname{Spec}\,(O_K)_\wp$ *be a family as before Remark* 5.13. *If the Hilbert polynomial of the first adjoint surface of* $S = \mathcal{S} \otimes \mathbf{F}_p$ *is as expected, and if* $H^1(S, \mathcal{O}_S(-H)) = 0$ *then the adjunction map of the general fibre* $S_{\mathbf{C}} = \mathcal{S} \otimes \mathbf{C}$ *blows down the same number of* (-1)*-lines as the adjunction map of the special fibre* S.

Proof. K^2 is constant in smooth flat families of surfaces by Noether's formula. Hence $(H+K)^2 > 0$ both in the special and general fibre. In particular, $S_{\mathbf{C}}$ does not belong to one of the exceptional families in (1) or (2). The first adjoint surfaces of S and $S_{\mathbf{C}}$ lie in projective spaces of the same dimension and have the same Hilbert polynomial by the assumptions. So we obtain a family $\mathcal{S}' \to \text{Spec } (O_K)_{\wp}$ which is flat again and the result follows.$\square$

Thus it suffices to carry the adjunction process through for our surface explicitly given over $\mathbf{F}_p$.

Remark 8.5. The exceptional locus of the adjunction map $\Phi : S \to S'$ can be computed explicitly: the images under Φ of three disjoint hyperplane sections of S intersect precisely in the exceptional locus.

Hence for a surface in $\mathbf{P}^4$ we can keep track of the self-intersection number of the canonical divisors through the adjunction process since we know the initial value K^2 from the double point formula.

Adjunction Process 8.6.

1. Start with a smooth, non-degenerate surface $S \subset \mathbf{P}^n$ defined over a finite field and given by explicit equations in $x_0, \ldots, x_n$. Compute the Hilbert polynomial of S.
2. Compute a free presentation of ω_S as either
$$\omega_S \cong \mathcal{E}xt^{\text{codim } S}_{\mathbf{P}^n}(\mathcal{O}_S, \omega_{\mathbf{P}^n})$$
or
$$\omega_S \cong \mathcal{H}om_{\mathbf{P}^2}(pr_*\mathcal{O}_S, \omega_{\mathbf{P}^2}),$$
where $pr\colon S \to \mathbf{P}^2$ is a generic linear projection. Compute
$$n' + 1 = h^0(S, \omega_S(1)).$$
If $n' \leq 0$ then stop.
3. Compute a free presentation
$$0 \leftarrow \langle H^0(S, \omega_S(1)) \rangle \leftarrow R^{n'+1} \stackrel{\psi}{\leftarrow} F_1$$
of the submodule $\langle H^0(S, \omega_S(1)) \rangle \subset H^0_*(S, \omega_S)$ generated by the sections of $\omega_S(1)$. Let $y_0, \ldots, y_{n'}$ be the coordinates of $\mathbf{P}^{n'}$. Then $(y_0, \ldots, y_{n'}) \cdot \psi = 0$ defines the graph of Φ in $\mathbf{P}^n \times \mathbf{P}^{n'}$. Let J be the corrresponding ideal.
4. Projecting onto the second factor gives the image $\Phi(S)$: saturate J with respect to one of the old variables x_i. The elements of bihomogeneous type $(0, *)$ in $(J : x_i^\infty) = (J : \langle x_0, \ldots, x_n \rangle^\infty)$ define $\Phi(S)$. If the adjunction map is birational, then the elements of bihomogeneous type $(1, *)$ in $(J : x_i^\infty)$ give rise to the inverse rational map.
5. Compute the Hilbert polynomial of $\Phi(S)$. If $\Phi(S)$ is a curve then analyze the situation and stop. If Φ is not birational onto a smooth surface $\Phi(S)$ then analyze the situation and stop. Otherwise compute the number of (-1)-lines on the surface $\Phi(S)$, set $S = \Phi(S)$ and continue with step 1.

9. Examples II

We briefly discuss the adjunction process in two of the examples of Section 6.

NOTATION 9.1. With $S^d_\pi \subset \mathbf{P}^n$ we denote a surface of degree d and sectional genus π in $\mathbf{P}^n$. $C^d_\pi \subset \mathbf{P}^n$ stands similarly for a curve with the indicated invariants.

$$S^d_\pi \xrightarrow{\alpha} S^{d'}_{\pi'}$$

denotes an adjunction map which simultaneously blows down α (-1)-lines.

EXAMPLE 9.2. The adjunction process for a surface $S \subset \mathbf{P}^4$ as in Example 6.1 can be easily carried through by hand. The intersection matrix of S is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}.$$

So S has negative Kodaira dimension since $H.K = 0$ and $K^2 = -2$. It follows that S is rational since $q = 0$. From its explicit construction we know that S is cut out by quartics and quintics. Thus there are no 6-secants to S, and we deduce from the 6-secant formula of Le Barz (1981) that S has precisely one (-1)-line. Hence the the adjunction process yields

$$\begin{array}{ccccc} \mathbf{P}^4 & & \mathbf{P}^4 & & \mathbf{P}^2 \\ \cup & & \cup & & \| \\ S^8_5 & \xrightarrow{1} & S^6_3 & \xrightarrow{10} & S^1_0. \end{array}$$

Indeed, we are not in one of the exceptional cases, and the intersection matrix of the first adjoint surface is

$$\begin{pmatrix} H_1^2 & H_1.K_1 \\ H_1.K_1 & K_1^2 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & -1 \end{pmatrix}.$$

Altogether,

$$S \cong \mathbf{P}^4(p_1, \ldots, p_{11}) \quad \text{embedded by} \quad H \equiv 7L - \sum_1^{10} 2E_i - E_{11}.$$

The proof that such a linear system is very ample for general choices of the points is not trivial (see Alexander, 1988). Note that our method actually gives an explicit parametrization for the explicit surface.

EXAMPLE 9.3. The random search in Example 6.5 finds four different families of surfaces with explicitly computed adjunction process as follows:

$$\text{(a)} \quad \begin{array}{ccccc} \mathbf{P}^4 & & \mathbf{P}^9 & & \mathbf{P}^{10} \\ \cup & & \cup & & \cup \\ S^{11}_{10} & \xrightarrow{5} & S^{19}_{11} & \xrightarrow{1} & S^{20}_{11} \end{array}$$

$$\text{(b)} \quad \begin{array}{ccccccccccc} \mathbf{P}^4 & & \mathbf{P}^9 & & \mathbf{P}^{10} & & \mathbf{P}^9 & & \mathbf{P}^7 & & \mathbf{P}^4 \\ \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ S^{11}_{10} & \xrightarrow{4} & S^{19}_{11} & \xrightarrow{1} & S^{19}_{10} & \xrightarrow{0} & S^{16}_{8} & \xrightarrow{0} & S^{11}_{5} & \xrightarrow{5} & S^{4}_{1} \end{array}$$

$$\text{(c)}\quad \begin{array}{ccccccccc} \mathbf{P}^4 & & \mathbf{P}^9 & & \mathbf{P}^{10} & & \mathbf{P}^8 & & \mathbf{P}^4 \\ \cup & & \cup & & \cup & & \cup & & \cup \\ S_{10}^{11} & \xrightarrow{3} & S_{11}^{19} & \xrightarrow{1} & S_9^{18} & \xrightarrow{2} & S_5^{12} & \xrightarrow{4} & S_1^4 \end{array}$$

$$\text{(d)}\quad \begin{array}{ccccccccc} \mathbf{P}^4 & & \mathbf{P}^9 & & \mathbf{P}^{10} & & \mathbf{P}^8 & & \mathbf{P}^5 \\ \cup & & \cup & & \cup & & \cup & & \cup \\ S_{10}^{11} & \xrightarrow{3} & S_{11}^{19} & \xrightarrow{2} & S_9^{18} & \xrightarrow{0} & S_6^{13} & \xrightarrow{3} & S_2^6 \end{array}$$

The intersection matrix of the original surfaces is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ 7 & -6 \end{pmatrix}.$$

Hence in case (a) the second adjoint matrix is

$$\begin{pmatrix} H_2^2 & H_2.K_2 \\ H_2.K_2 & K_2^2 \end{pmatrix} = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $K_2 \equiv 0$ and that the surface is an Enriques surface.

In cases (b) and (c) the adjunction process ends with a Del Pezzo surface of degree 4. In case (d) the final surface is a complete intersection of $\mathbf{P}^1 \times \mathbf{P}^2 \subset \mathbf{P}^5$ with a quadric. So over $\mathbf{C}$ these surfaces are rational. Over the finite field $\mathbf{F}_3$ the examples are sometimes rational and sometimes not, because frequently they simply contain too few $\mathbf{F}_3$-rational points.

It is an open problem to give a construction of all these surfaces without the brute force search.

Acknowledgements

The authors would like to thank the MSRI in Berkeley for its support and for providing a stimulating working atmosphere. We are also grateful to Iustin Coandă for pointing out to us the mistake in Cook's connectedness theorem (see Remark 4.4).

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Originally Received 15 March 1999
Accepted 11 October 1999