ELSEVIER

Bull. Sci. math. 132 (2008) 486–499

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Fundamental solutions for a class of non-elliptic homogeneous differential operators

Brice Camus

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstr. 150, D-44780 Bochum, Germany

Received 5 June 2007

Available online 22 June 2007

Abstract

We compute temperate fundamental solutions of homogeneous differential operators with real-principal type symbols. Via analytic continuation of meromorphic distributions, fundamental solutions for these non-elliptic operators can be constructed in terms of radial averages and invariant distributions on the unit sphere.

Keywords: Fundamental solutions; PDE; Singularities

1. Introduction and main results

If $P := P(D_x)$ is a differential operator on $\mathbb{R}^n$ a temperate fundamental solution to P is a distribution $\mathfrak{s} \in \mathcal{S}'(\mathbb{R}^n)$ such that $P(D_x)\mathfrak{s} = \delta$, where δ is the delta-Dirac distribution at the origin. Fundamental solutions play a major role in the theory of PDE. For a large overview on this subject, and applications, we refer to [5] vols. 1 and 2. It is well known, see e.g. [1,4,5], that differential operators with constant coefficients have temperate fundamental solutions. But, apart in very trivial cases like the Laplacian, it is difficult to produce explicitly a solution. The case of order 3 homogeneous operators, in dimension 3, was treated in [6]. Always in dimension 3, the case of elliptic quartic operators was considered in [7] and our contribution in [2] was to obtain temperate fundamental solutions for homogeneous elliptic operators of any degree and in any dimension. Also, we mention that the book of J.E. Björk [1] contains a very nice study of the algebraic and analytic properties of fundamental solutions for operators with polynomial or

E-mail address: brice.camus@uni-duisburg-essen.de.

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2007.06.004

analytic symbols and constant coefficients. In particular the presence of logarithmic distributions, as occurring in the present contribution, is predicted in a very general setting.

1.1. Hypotheses and definitions

We are here interested in the case of a non-definite homogeneous polynomial p on $\mathbb{R}^n$, i.e., $p(\lambda\xi) = \lambda^k p(\xi)$. In all this article k is the degree of p. To simplify, we restrict our study to a real principal type singularity, i.e. we assume that:

$$(\mathcal{H}):\quad \begin{cases} p \text{ is real valued,} \\ p(x) = 0 \text{ and } \nabla p(x) = 0 \Leftrightarrow x = 0. \end{cases}$$

But p complex valued is admissible, see Section 2. In what follows, we write:

$$\mathfrak{C}(p) = \{\theta \in \mathbb{S}^{n-1} \mid p(\theta) = 0\},$$

the trace of the characteristic set of p on the unit-sphere. In terms of polar coordinates, $(\mathcal{H})$ implies that the restriction of p to $\mathbb{S}^{n-1}$ satisfies:

$$\nabla_\theta p(\theta) \neq 0 \quad \text{near } \mathfrak{C}(p).$$

By a standard result of differential geometry, see e.g. [4] Chapter 3, condition $(\mathcal{H})$ insures the existence of a canonical $(n-2)$-dimensional measure $d\mathfrak{L}$ smooth on the level sets $p(\theta) = \varepsilon$, for $\varepsilon > 0$ small enough. This measure, traditionally called Liouville or Guelfand–Leray measure, satisfies the coarea formula:

$$\int_{\mathbb{S}^{n-1}} h(\theta)\, d\theta = \int_{\mathbb{R}} \Bigg(\int_{p(\theta)=u} h\, d\mathfrak{L} \Bigg) du,$$

for all h with support in $K_\varepsilon = \{\theta \in \mathbb{S}^{n-1} \mid |p(\theta)| \leqslant \varepsilon\}$. This relation defines a new function: $u \mapsto \mathfrak{L}(h)(u)$, obtained by integration of f in the fibers $p^{-1}(u)$ w.r.t. $d\mathfrak{L}$. By Sard's Theorem this function is finite almost everywhere and for any $h \in C^\infty(\mathbb{S}^{n-1})$ it is easy to check that $\mathfrak{L}(h)$ can be extended as an integrable function with compact support $\operatorname{supp}(\mathfrak{L}(h)) \subset [\inf_{\mathbb{S}^{n-1}} p(\theta), \sup_{\mathbb{S}^{n-1}} p(\theta)]$. With these elementary facts in mind we introduce:

Definition 1. For a general function $g \in \mathcal{S}(\mathbb{R}^n)$ we define the polar Guelfand–Leray transform of g as:

$$\mathfrak{L}\big(g(r\theta)\big)(u) := \mathfrak{L}(g)(r, u) = \int_{p(\theta)=u} g(r\theta)\, d\mathfrak{L}(\theta),$$

simply by viewing the radius r as a parameter.

In all what follows the map $\mathfrak{L}$ is defined w.r.t. the restriction of p to $\mathbb{S}^{n-1}$ and:

$$\mathfrak{L}^{(l)}\big(g(r\theta)\big)(u) = \frac{d^l}{du^l}\Bigg(\int_{p(\theta)=u} g(r\theta)\, d\mathfrak{L}(\theta) \Bigg),$$

is the exterior derivative of degree l w.r.t. the argument u. Finally:

$$\hat{h}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} h(x)\, dx,$$

stands for the Fourier transform.

1.2. Main results

Theorem 2. *Assume that $n \geqslant 2$ and that the symbol p satisfies condition $(\mathcal{H})$. A fundamental solution $\mathfrak{s} \in \mathcal{S}'(\mathbb{R}^n)$ to P is respectively given by*:

(A) *If $k < n$ (locally integrable singularity)*:

$$\langle \mathfrak{s}, f\rangle = \frac{1}{(2\pi)^n} \int_0^\infty \big\langle \log\big(|u|\big);\ \mathfrak{L}^{(1)}\big(\hat{f}\big(r\theta\big)\big)(u)\big\rangle r^{n-k-1}\, dr.$$

(B) *If $k \geqslant n$ (non-integrable case) then we have*:

$$\begin{aligned}\langle \mathfrak{s}, f\rangle = {}& \frac{1}{(2\pi)^n} \frac{\gamma + \Psi(k)}{\Gamma(2k)} \frac{\partial^{2k-1}}{\partial r^{2k-1}} \big(r^{k+n-1} \big\langle \log\big(|u|\big);\ \mathfrak{L}^{(1)}\big(\hat{f}(r\theta)\big)(u)\big\rangle\big)_{|r=0} \\ &+ \frac{1}{(2\pi)^n \Gamma(1+2k)} \frac{\partial^{2k-1}}{\partial r^{2k-1}} \big(r^{k+n-1} \big\langle \log\big(|u|\big)^2;\ \mathfrak{L}^{(1)}\big(\hat{f}(r\theta)\big)(u)\big\rangle\big)_{|r=0} \\ &+ \frac{1}{(2\pi)^n \Gamma(k)} \int_{\mathbb{R}^+} \log(r) \frac{\partial^{2k}}{\partial r^{2k}} \big(r^{k+n-1} \big\langle \log\big(|u|\big);\ \mathfrak{L}^{(1)}\big(\hat{f}(r\theta)\big)(u)\big\rangle\big)\, dr.\end{aligned}$$

Here γ is Euler's constant and $\Psi(z) = \Gamma'(z)/\Gamma(z)$.

The trivial case $n = 1$, i.e. a monomial symbol, can be treated directly and for $n = 2$ the map $\mathfrak{L}$ is simply related to Dirac masses at $\mathbb{S}^1 \cap \{p = 0\}$. Note that the results are very different from the case of an elliptic operator. In particular observe the presence of singularities supported in the lacuna set of p since distributions $\log(|u|)^j$, $j = 1, 2$, are not smooth in $u = 0$.

For non-integrable singularities we can say more and the method we use allows to produce a one-parameter family of solutions:

Corollary 3. *Under the conditions of Theorem 2 and if $k \geqslant n$ a temperate solution of $P(D)\mathfrak{s}_\mathrm{o} = 0$ is given by*:

$$\langle \mathfrak{s}_\mathrm{o}, f\rangle = \frac{\partial^{2k-1}}{\partial r^{2k-1}} \big(r^{k+n-1} \big\langle \log\big(|u|\big);\ \mathfrak{L}^{(1)}\big(\hat{f}(r\theta)\big)(u)\big\rangle\big)_{|r=0}.$$

Hence each $\mathfrak{s} + \lambda\mathfrak{s}_\mathrm{o}$, $\lambda \in \mathbb{C}$, is a temperate fundamental solution to $P(D)$.

2. Proof of the main result

The strategy is as follows. If p is positive for all $f \in \mathcal{S}(\mathbb{R}^n)$ we have:

$$\lim_{\zeta \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(\xi)^\zeta \hat{f}(\xi) = f(0) = \langle \delta, f\rangle. \tag{1}$$

If p is a polynomial, or more generally an analytic function, the integral in Eq. (1) defines a meromorphic distribution $\mathcal{P}(\zeta)$. See [1] for this point. The Laurent development around $\zeta = -1$ can be written:

$$\mathcal{P}(\zeta - 1) = \sum_{j=-1}^{-d} \mu_j \zeta^j + \mu_0 + \sum_{j=1}^{\infty} \mu_j \zeta^j. \tag{2}$$

But, according to Eq. (1), we have:

$$\lim_{\zeta\to 0}\langle P(D)f, \mathcal{P}(\zeta-1)\rangle = \langle \delta, f\rangle,$$

and it follows that μ_0 is a temperate fundamental solution to $P(D)$.

Remark 4. Eq. (1), combined with Eq. (2), provides the set of relations:

$$P(D)\mu_j = 0, \quad \forall j < 0,$$

in the sense of distributions of $\mathcal{S}'(\mathbb{R}^n)$. If such non-zero terms exist, any affine combination $\mu_0 + \sum_{j=1}^{d}\alpha_j\mu_{-j}$, $(\alpha_1,\dots,\alpha_d)\in\mathbb{C}^d$, is a temperate fundamental solution. This remark provides the basic strategy to establish Corollary 3.

When p is no more positive, or complex valued, the trick is to compute the fundamental solution ρ_0 attached to $|p|^2$. With $|p|^2 = p(\xi)\bar{p}(\xi)$, it is easy to check that:

$$\mu_0 = \bar{P}(D)\rho_0,$$

is a fundamental solution to P. Hence, to attain our objective we have to construct meromorphic extensions of the family of distributions:

$$\zeta \mapsto \int_{\mathbb{R}^n} \left(\left|p(\xi)\right|^2\right)^{\zeta} g(\xi)\,d\xi, \quad g\in\mathcal{S}(\mathbb{R}^n).$$

To solve a non-elliptic equation we transform the problem into a positive, and hence simpler, problem. The expense is that $|p(\xi)|^{-2}$ is more singular than $|p(\xi)|^{-1}$ and this induces extra computations in the proof. We start by solving, locally, the singularities of p. We have:

Lemma 5. *If p satisfies $(\mathcal{H})$ there exists local coordinates ω (strictly speaking outside of the origin), such that we have the local diffeomorphism*:

$$p(\xi)\simeq\begin{cases} -\omega_1^k \text{ or } \omega_1^k, & \text{outside of } \mathfrak{C}(p)\times\,]0,\infty[,\\ \omega_1^k\omega_2 \text{ in a neighborhood of } \mathfrak{C}(p)\times\,]0,\infty[. \end{cases}$$

Proof. To blow up the singularity, we use polar coordinates $\xi = (r,\theta)$. By homogeneity we have $p(r\theta) = r^k p(\theta)$. First if $\theta_0\notin\mathfrak{C}(p)$ we choose:

$$(\omega_1,\omega_2,\dots,\omega_n)(r,\theta) = \left(r\left|p(\theta)\right|^{1/k},\theta\right). \tag{3}$$

We have $p(\xi)\simeq\pm\omega_1(r,\theta)^k$ in a conical neighborhood of θ_0. The sign is obviously given by the sign of $p(\theta_0)$ and the Jacobian is $|J\omega|(r,\theta) = |p(\theta)|^{1/k}\neq 0$. Next, if $\theta_0\in\mathfrak{C}(p)$ by condition $(\mathcal{H})$ and by homogeneity we have $\nabla_\theta p(\theta_0)\neq 0$. We can assume that $\partial_{\theta_1}p(\theta)\neq 0$ and we chose:

$$(\omega_1,\omega_2,\omega_3,\dots,\omega_n)(r,\theta) = (r, p(\theta), \theta_2,\dots,\theta_{n_1}).$$

We have:

$$|J\omega|(r,\theta_0) = \left|\frac{\partial p}{\partial\theta_1}(\theta_0)\right| dr\,d\theta \neq 0.$$

By continuity, this result holds in a sufficiently small neighborhood of θ_0. Since $\mathfrak{C}(p)$ is a compact subset of $\mathbb{S}^{n-1}$ we can easily globalize the construction. □

To use these normal forms, we construct an adapted partition of unity on $\mathbb{S}^{n-1}$. We pick a family of positive function Ω_j on $\mathbb{S}^{n-1}$ such that:

$$\sum_{j=1}^{N} \Omega_j(\theta) = 1 \quad \text{near } \mathfrak{C}(p),$$

with the existence of a normal form $\omega_1^k \omega_2$ inside each $\operatorname{supp}(\Omega_j)$. Next, since the previous construction depends only on the set $\mathfrak{C}(p)$, we can assume that $\operatorname{supp}(\Omega_j) \subset K_\varepsilon$ for $\varepsilon > 0$ chosen small enough so that the measures $d\mathfrak{L}$ are well defined on each $\operatorname{supp}(\Omega_j)$. Finally we can complete this finite set as partition of unity on $\mathbb{S}^{n-1}$ with $\Omega_0 = 1 - \sum_j \Omega_j$. The support of Ω_0 is generally not connected, as shows the case $n = 3$. With this partition of unity we have:

$$\int_{\mathbb{R}^n} \left(\left|p(\xi)\right|^2\right)^\zeta g(\xi)\, d\xi = \sum_{j=0}^{N} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^+} \Omega_j(\theta) \left|p(r,\theta)\right|^{-2\zeta} g(r,\theta) r^{n-1}\, dr\, d\theta.$$

With this localization argument we use Lemma 5 to trivialize locally the problem and we have to study the elementary quantities:

$$\mu^{\mathrm{ell}}(\zeta) = \int_{\mathbb{R}^+} \omega_1^{2k\zeta} G(\omega_1)\, d\omega_1,$$

$$\mu_j^{\mathrm{sing}}(\zeta) = \int_{\mathbb{R}^+\times\mathbb{R}} \omega_1^{2k\zeta} \left(\omega_2^2\right)^\zeta G_j(\omega_1,\omega_2)\, d\omega_1\, d\omega_2.$$

These new functions are obtained by pullback and integration:

$$G(\omega_1) = \int \omega^*\left(\Omega_0(\theta) g(r,\theta) r^{n-1}\right)(\omega_1,\ldots,\omega_n)\, d\omega_2 \cdots d\omega_n,$$

$$G_j(\omega_1,\omega_2) = \int \omega^*\left(\Omega_j(\theta) g(r,\theta) r^{n-1}\right)(\omega_1,\ldots,\omega_n)\, d\omega_3 \cdots d\omega_n,$$

where ω^* stands for the pullback including the multiplication by the Jacobian.

2.1. Trivial contribution

We start by the analytic continuation of the elliptic part $\mu^{\mathrm{ell}}(\zeta)$. We have:

$$\frac{\partial^{2k}}{\partial\omega_1^{2k}} \omega_1^{2k\zeta} = \omega_1^{2k(\zeta-1)} \prod_{j=0}^{2k-1} (2k\zeta - j),$$

and after $2k$ integrations by parts we obtain:

$$\mu^{\mathrm{ell}}(\zeta - 1) = \int_{\mathbb{R}^+} \omega_1^{2k(\zeta-1)} G(\omega_1)\, d\omega_1 = \left(\prod_{j=0}^{2k-1} \frac{1}{2k\zeta - j}\right) \int_{\mathbb{R}^+} \omega_1^{2k\zeta} \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1.$$

The integral in the r.h.s. defines an holomorphic function near $\zeta = 0$. The constant term of the Laurent series at the origin, determined by the rational function, is given by:

$$\mu_0^{\mathrm{ell}} = \lim_{\zeta\to 0} \frac{\partial}{\partial\zeta}\left(\zeta \mu^{\mathrm{ell}}(\zeta - 1)\right).$$

With the holomorphic function near $\zeta = 0$:

$$h(\zeta) = \zeta \prod_{j=0}^{2k-1} \frac{1}{2k\zeta - j} = \frac{1}{2k} \prod_{j=1}^{2k-1} \frac{1}{2k\zeta - j},$$

we obtain:

$$\mu_0^{\mathrm{ell}} = h'(0) \int_{\mathbb{R}^+} \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1 + 2kh(0) \int_{\mathbb{R}^+} \log(\omega_1) \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1.$$

Clearly $2kh(0) = -1/\Gamma(2k)$ and a direct computation yields:

$$h'(0) = -\frac{\gamma + \Psi(2k)}{\Gamma(2k)}.$$

Here $\Psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$ is the usual polygamma function of order 0 and

$$\gamma = \lim_{L\to\infty} \left(\sum_{j=1}^{L} \frac{1}{j} - \log(L) \right),$$

is Euler's constant.

2.2. Non-trivial contribution

Now, we study the singular term $\mu^{\mathrm{sing}}(\zeta) = \sum_{j=1}^{N} \mu_j^{\mathrm{sing}}(\zeta)$. We have:

$$\frac{\partial^{2k+2}}{\partial\omega_1^{2k}\partial\omega_2^2} \omega_1^{2k\zeta} (\omega_2^2)^{\zeta} = \mathfrak{b}(\zeta) \omega_1^{2k(\zeta-1)} (\omega_2^2)^{\zeta-1},$$

$$\mathfrak{b}(\zeta) = 2\zeta(2\zeta - 1) \prod_{j=0}^{2k-1} (2k\zeta - j).$$

Accordingly, $\zeta = 0$ is a pole of order 2 of the meromorphic extension:

$$\mu_j^{\mathrm{sing}}(\zeta - 1) = \frac{1}{\mathfrak{b}(\zeta)} \int_{\mathbb{R}^+\times\mathbb{R}} \omega_1^{2k(\zeta-1)} (\omega_2^2)^{\zeta-1} \frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2.$$

The constant term of the Laurent expansion is given by:

$$\mu_{0,j}^{\mathrm{sing}} = \frac{1}{2} \lim_{\zeta\to 0} \frac{\partial^2}{\partial\zeta^2} \bigl(\zeta^2 \mu_j^{\mathrm{sing}}(\zeta - 1)\bigr).$$

Hence with the auxiliary functions:

$$m(\zeta) = \frac{\zeta^2}{\mathfrak{b}(\zeta)} = \frac{1}{4k(2\zeta - 1)} \prod_{j=1}^{2k-1} \frac{1}{2k\zeta - j},$$

$$M_j(\zeta) = \int_{\mathbb{R}^+\times\mathbb{R}} \omega_1^{2k(\zeta-1)} (\omega_2^2)^{\zeta-1} \frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2,$$

we obtain that the term of interest is given by:

$$\mu_{0,j}^{\text{sing}} = \frac{1}{2}\big(m(0)M_j''(0) + 2m'(0)M_j'(0) + m''(0)M_j(0)\big). \tag{4}$$

By some elementary calculations we obtain respectively:

$$m(0) = \frac{1}{2\Gamma(1+2k)},$$
$$m'(0) = \frac{1 + k(\gamma + \Psi(2k))}{\Gamma(1+2k)}.$$

The coefficient $m''(0)$ plays no rôle here, see Eq. (5) below. The next step is to evaluate $\mu_{0,j}^{\text{sing}}$ in the coordinates ω. After integration by parts w.r.t. ω_2, we have:

$$M_j(0) = \int\limits_{\mathbb{R}^+\times\mathbb{R}} \frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 = 0. \tag{5}$$

For the next distributional coefficient we find that:

$$\begin{aligned} M_j'(0) &= \int\limits_{\mathbb{R}^+\times\mathbb{R}} \big(2k\log(\omega_1) + 2\log\big(|\omega_2|\big)\big)\frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 \\ &= \int\limits_{\mathbb{R}} 2\log\big(|\omega_2|\big)\frac{\partial^{2k+1} G_j}{\partial\omega_1^{2k-1}\partial\omega_2^2}(0,\omega_2)\, d\omega_2. \end{aligned} \tag{6}$$

Finally, we obtain similarly:

$$\begin{aligned} M_j''(0) &= \int\limits_{\mathbb{R}^+\times\mathbb{R}} \big(2k\log(\omega_1) + 2\log\big(|\omega_2|\big)\big)^2\frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 \\ &= 4\int\limits_{\mathbb{R}} \log\big(|\omega_2|\big)^2\frac{\partial^{2k+1} G_j}{\partial\omega_1^{2k-1}\partial\omega_2^2}(0,\omega_2)\, d\omega_2 \\ &\quad + 8k\int\limits_{\mathbb{R}^+\times\mathbb{R}} \log(\omega_1)\log\big(|\omega_2|\big)\frac{\partial^{2k+2} G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2. \end{aligned} \tag{7}$$

After expanding the square in the integral we have, once more, discarded the term attached to $\log(\omega_1)^2$, vanishing after integration w.r.t. ω_2.

2.3. Invariant formulation

To achieve the proof we must formulate our distributions in a geometrical way, also independent of the partition of unity attached to the coordinates ω. First, by construction, we have to evaluate our distribution on $P(D)f$ so that after Fourier transformation $g(\xi) = p(\xi)\hat{f}(\xi)$. Since p is of degree k, we have $G(\omega_1) = \mathcal{O}(\omega_1^{k+n-1})$ near $\omega_1 = 0$. Same remark for $G_j(\omega_1,\omega_2) = \mathcal{O}(\omega_1^{k+n-1})$ near $\omega_1 = 0$. These properties are important since several coefficient expressed below are related to Dirac-delta distributions supported in $\omega_1 = 0$. According to Eqs. (6) and (7) at worst 3 different terms occur which we treat separately distinguishing out the case of $|p|^{-1}$ locally integrable or not.

2.3.1. 1-contribution of the elliptic directions

We have:

$$\int_0^\infty \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1 = -\partial_{\omega_1}^{2k-1} G(0).$$

If $2k-1 < k+n-1$ this term vanishes and for $k \geqslant n$ we have:

$$\partial_{\omega_1}^{2k-1} G(0) = \frac{\partial^{2k-1}}{\partial r^{2k-1}} \left(r^{n+k-1} \int_{\mathbb{S}^{n-1}} \hat{f}(r\theta) \Omega_0(\theta) \frac{d\theta}{p(\theta)} \right)_{|r=0}.$$

This identity holds after inversion of our diffeomorphism and the substitution $g(\xi) = p(\xi)\hat{f}(\xi)$. When $k < n$, we can integrate by parts the logarithmic contribution to obtain:

$$\begin{aligned}\int_{\mathbb{R}^+} \log(\omega_1) \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1 &= (2k-1)! \int_{\mathbb{R}^+} G(\omega_1) \frac{d\omega_1}{\omega_1^{2k}} \\ &= (2k-1)! \int_{\mathbb{R}^+ \times \mathbb{S}^{n-1}} \Omega_0(\theta) \hat{f}(r\theta) r^{n-k-1}\, dr \frac{d\theta}{p(\theta)}.\end{aligned}$$

Observe that the integral w.r.t. r is precisely convergent, for any $f \in \mathcal{S}(\mathbb{R}^n)$, if and only if $k < n$. If $k \geqslant n$ this argument does not holds, but we can write:

$$\partial_{\omega_1}^{2k} G(\omega_1) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ip\omega_1} (ip)^{2k} \hat{G}(p)\, dp.$$

After inversion of our diffeomorphism and scaling out the spherical term $p(\theta)$ in the phase, we obtain the contribution:

$$\int_{\mathbb{R}^+} \log(\omega_1) \partial_{\omega_1}^{2k} G(\omega_1)\, d\omega_1 = \int_{\mathbb{R}^+ \times \mathbb{S}^{n-1}} \log\left(r \left|p(\theta)\right|^{1/k}\right) \frac{\partial^{2k}}{\partial r^{2k}} \left(\hat{f}(r\theta) r^{n+k-1}\right) \Omega_0(\theta)\, dr \frac{d\theta}{p(\theta)}.$$

2.3.2. 2-contribution of the non-elliptic directions

To express our amplitudes, we use the Schwartz kernel technique. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2}$, then:

$$\begin{aligned}D^\alpha G_j(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} \int e^{i(y_1\omega_1 + y_2\omega_2)} y^\alpha \hat{G}_j(y_1, y_2)\, dy \\ &= \frac{1}{(2\pi)^2} \int e^{i(y_1(\omega_1 - x_1) + y_2(\omega_2 - x_2))} y^\alpha G_j(x_1, x_2)\, dy\, dx.\end{aligned}$$

For this integral we can inverse our diffeomorphism via $x_1(r,\theta) = r$ and $x_2(r,\theta) = p(\theta)$, locally on $\operatorname{supp}(\Omega_j)$. For the r-integration we can extend the integrand by 0 for $r < 0$ and we obtain first:

$$\begin{aligned}D^\alpha G_j(\omega_1, \omega_2) &= \frac{1}{(2\pi)} \int e^{i\langle y_2, \omega_2 - p(\theta)\rangle} y_2^{\alpha_2} \frac{\partial}{\partial \omega_1^{\alpha_1}} \int \Omega_j(\theta) g(\omega_1\theta) \omega_1^{n-1}\, d\theta\, dy_2 \\ &= \frac{1}{(2\pi)} \int e^{i\langle y_2, \omega_2 - p(\theta)\rangle} y_2^{\alpha_2} \frac{\partial}{\partial \omega_1^{\alpha_1}} \int \omega_2 \Omega_j(\theta) \hat{f}(\omega_1\theta) \omega_1^{k+n-1}\, d\theta\, dy_2.\end{aligned}$$

The remaining integral is simply the exterior derivative, of order α_2, of the Liouville measure on the surface $p(\theta) = \omega_2$. For $\alpha_2 = 2$, observe that:

$$\mathfrak{L}^{(2)}\big(p(\theta)\Omega_j(\theta)\hat{f}(\omega_1\theta)\big)(\omega_2) = \frac{\partial^2}{\partial\omega_2^2}\big(\omega_2\mathfrak{L}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)\big)(\omega_2)\big)$$
$$= \omega_2\mathfrak{L}^{(2)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)\big)(\omega_2) + 2\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)\big)(\omega_2),$$

and that by construction the functions $\mathfrak{L}(\Omega_j\hat{f})$ are smooth. Choosing $\alpha_1 = 2k$, we have obtained:

$$\begin{aligned}\frac{\partial^{2k+2}G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2) &= \frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(2)}\big(p(\theta)\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\\ &= \frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\big(\omega_2\mathfrak{L}^{(2)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\\ &\quad + 2\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\big).\end{aligned} \tag{8}$$

By degree considerations w.r.t. ω_1 we have respectively:

$$\int_{\mathbb{R}}\log\big(|\omega_2|\big)\frac{\partial^{2k+1}G_j}{\partial\omega_1^{2k-1}\partial\omega_2^2}(0,\omega_2)\,d\omega_2 = \begin{cases}0 & \text{if } k < n,\\ C(f) \neq 0 & \text{if } k \geqslant n.\end{cases}$$

$$\int_{\mathbb{R}}\log\big(|\omega_2|\big)^2\frac{\partial^{2k+1}G_j}{\partial\omega_1^{2k-1}\partial\omega_2^2}(0,\omega_2)\,d\omega_2 = \begin{cases}0 & \text{if } k < n,\\ D(f) \neq 0 & \text{if } k \geqslant n.\end{cases}$$

Where C and D are obtained by inserting Eq. (8) in the integrals. Finally, in Eq. (7) the term attached to the product of logarithms is given by:

$$\int_{\mathbb{R}^+\times\mathbb{R}}\log(\omega_1)\log\big(|\omega_2|\big)\frac{\partial^{2k+2}G_j}{\partial\omega_1^{2k}\partial\omega_2^2}(\omega_1,\omega_2)\,d\omega_1\,d\omega_2, \quad \text{if } k \geqslant n,$$

$$(2k-1)!\int_{\mathbb{R}^+\times\mathbb{R}}\log\big(|\omega_2|\big)\frac{\partial^2G_j}{\partial\omega_2^2}(\omega_1,\omega_2)\frac{d\omega_1}{\omega_1^{2k}}\,d\omega_2, \quad \text{if } k < n.$$

For $k \geqslant n$ integrations by parts are not allowed but we can anyhow conclude with Eq. (8). We treat now separately parts (A) and (B) of Theorem 2.

Proof of part (A). To obtain the final result we sum over the partition of unity. According to the considerations of homogeneity above, for $k < n$ the full contribution is generated by μ_0^{ell} and $M_j''(0)$. With the explicit values of $h(0)$ and $m(0)$, we obtain that $(2\pi)^n\mu_0(f)$ equals:

$$\int_{\mathbb{R}^+\times\mathbb{S}^{n-1}}\Omega_0(\theta)\hat{f}(r\theta)r^{n-k-1}\frac{d\theta}{p(\theta)}\,dr + \sum_j\int_{\mathbb{R}^+\times\mathbb{R}}\log\big(|\omega_2|\big)\frac{\partial^2G_j}{\partial\omega_2^2}(\omega_1,\omega_2)\frac{d\omega_1}{\omega_1^{2k}}\,d\omega_2.$$

With $\Omega_0 = 0$ near $\mathfrak{C}(p)\cap\mathbb{S}^{n-1}$, we have $\mathfrak{L}(\Omega_0(\theta)\hat{f}(r\theta))(u) = 0$ in a neighborhood of $u = 0$. Hence, in the first term, the integral w.r.t. θ equals:

$$\int_{u\in\mathbb{R}}\mathfrak{L}\big(\Omega_0(\theta)\hat{f}(r\theta)\big)(u)\frac{du}{u} = \big\langle\log\big(|u|\big);\,\mathfrak{L}^{(1)}\big(\Omega_0(\theta)\hat{f}(r\theta)\big)(u)\big\rangle.$$

The derivation is in sense of distributions. For the coefficients attached to $M_j''(0)$ we obtain:

$$\int_{\mathbb{R}} u\log\bigl(|u|\bigr)\mathfrak{L}^{(2)}\bigl(\Omega_j(\theta)\hat{f}(r\theta)\bigr)(u)\,du + 2\int_{\mathbb{R}} \log\bigl(|u|\bigr)\mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(r\theta)\bigr)(u)\,du.$$

Since $(u\log(|u|))' = \log(|u|) + 1$, via one integration by parts:

$$\int_{\mathbb{R}} u\log\bigl(|u|\bigr)\mathfrak{L}^{(2)}\bigl(\Omega_j(\theta)\hat{f}(r\theta)\bigr)(u)\,du = -\int_{\mathbb{R}} \mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(r\theta)\bigr)(u)\bigl(\log\bigl(|u|\bigr)+1\bigr)\,du.$$

Observe the minus sign which fits with the weak derivation above. Since for each r and $j > 0$, $u \mapsto \mathfrak{L}(\Omega_j(\theta)\hat{f}(r\theta))(u) \in C_0^\infty(\mathbb{R})$, we get:

$$\int_{\mathbb{R}} \mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(r\theta)\bigr)(u)\,du = 0.$$

By integration w.r.t. r and summation over the partition of unity we obtain:

$$\mu_0(f) = \frac{1}{(2\pi)^n}\int_0^\infty \bigl\langle \log\bigl(|\omega_2|\bigr);\, \mathfrak{L}^{(1)}\bigl(\hat{f}(r\theta)\bigr)(\omega_2)\bigr\rangle r^{n-k-1}\,dr,$$

which is the desired result when $k < n$.

Proof of part (B). Now, we consider $k \geqslant n$. All coefficients contribute via:

$$\begin{aligned}(2\pi)^n\mu_0(f) = {}& -h'(0)\frac{\partial^{2k-1}}{\partial r^{2k-1}}\left(r^{n+k-1}\int_{\mathbb{S}^{n-1}} \hat{f}(r\theta)\Omega_0(\theta)\frac{d\theta}{p(\theta)}\right)_{|r=0}\\ &+ 2kh(0)\int_{\mathbb{R}^+\times\mathbb{S}^{n-1}} \log\bigl(r\bigl|p(\theta)\bigr|^{1/k}\bigr)\frac{\partial^{2k}}{\partial r^{2k}}\bigl(\hat{f}(r\theta)r^{n+k-1}\bigr)\Omega_0(\theta)\frac{d\theta}{p(\theta)}\,dr\\ &+ \frac{1}{2}\sum_j\bigl(m(0)M_j''(0) + 2m'(0)M_j'(0)\bigr).\end{aligned}$$

If we split the integral with the logarithm we obtain two terms:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{S}^{n-1}} \log(r)\frac{\partial^{2k}}{\partial r^{2k}}\bigl(\hat{f}(r\theta)r^{n+k-1}\bigr)\Omega_0(\theta)\frac{d\theta}{p(\theta)}\,dr\\ &\quad - \frac{1}{k}\frac{\partial^{2k-1}}{\partial r^{2k-1}}\left(\int_{\mathbb{S}^{n-1}} \hat{f}(r\theta)\log\bigl(\bigl|p(\theta)\bigr|\bigr)\Omega_0(\theta)\frac{d\theta}{p(\theta)}\right)_{|r=0}.\end{aligned}$$

Observe that, by construction, all integrals are well defined. First, we express the contributions near $\mathfrak{C}(p)$. Combining Eqs. (6) and (8), we find that:

$$\begin{aligned}M_j'(0) = {}& 2\int_{\mathbb{R}} \log\bigl(|\omega_2|\bigr)\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\bigl(\omega_2\mathfrak{L}^{(2)}\bigl(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\bigr)\bigr)\bigr)_{|\omega_1=0}\,d\omega_2\\ &+ 4\int_{\mathbb{R}} \log\bigl(|\omega_2|\bigr)\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\bigr)\bigr)_{|\omega_1=0}\,d\omega_2.\end{aligned}$$

This term can be treated as in part (A) and we obtain:

$$M_j'(0) = 2\int_{\mathbb{R}} \log(|\omega_2|)\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)_{|\omega_1=0}\,d\omega_2.$$

Next, combining Eqs. (7) and (8) we have:

$$\begin{aligned}M_j''(0) &= 4\int_{\mathbb{R}} \log(|\omega_2|)^2\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\big(\omega_1^{k+n-1}\big(\omega_2\mathfrak{L}^{(2)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\big)_{|\omega_1=0}\,d\omega_2\\
&\quad + 8\int_{\mathbb{R}} \log(|\omega_2|)^2\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)_{|\omega_1=0}\,d\omega_2\\
&\quad + 8k\int_{\mathbb{R}^+\times\mathbb{R}} \log(\omega_1)\log(|\omega_2|)\frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\big(\omega_2\mathfrak{L}^{(2)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\big)\,d\omega_1\,d\omega_2\\
&\quad + 16k\int_{\mathbb{R}^+\times\mathbb{R}} \log(\omega_1)\log(|\omega_2|)\frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\,d\omega_1\,d\omega_2.\end{aligned}$$

The last two integrals can be combined as above. For the others, we use:

$$\big(u\log(|u|)^2\big)' = \log(|u|)^2 + 2\log(|u|),\quad \forall u\neq 0,$$

and proceed to integrations by parts, which is legal since the factors $\mathfrak{L}^{(k)}(\cdot)(\omega_2)$ vanish for ω_2 large and $\omega_2\log(|\omega_2|)$ also vanishes at the origin. We obtain:

$$\begin{aligned}M_j''(0) &= 4\int_{\mathbb{R}} \log(|\omega_2|)^2\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)_{|\omega_1=0}\,d\omega_2\\
&\quad - 8\int_{\mathbb{R}} \log(|\omega_2|)\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)_{|\omega_1=0}\,d\omega_2\\
&\quad + 8k\int_{\mathbb{R}^+\times\mathbb{R}} \log(\omega_1)\log(|\omega_2|)\frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\big(\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)(\omega_2)\big)\big)\big)\,d\omega_1\,d\omega_2.\end{aligned}$$

Observe that we have 3 different coefficients, like for the coefficients attached to the set Ω_0. We combine each of these contributions by nature and by gathering carefully the constants. First, we consider the term involving two logarithms:

$$\begin{aligned}&2kh(0)\int_{\mathbb{R}^+\times\mathbb{S}^{n-1}} \log(r)\frac{\partial^{2k}}{\partial r^{2k}}\big(\hat{f}(r\theta)r^{n+k-1}\big)\Omega_0(\theta)\frac{d\theta}{p(\theta)}\,dr\\
&\quad + 4km(0)\sum_j\int_{\mathbb{R}^+\times\mathbb{R}} \log(\omega_1)\log(|\omega_2|)\\
&\quad \times\frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\big(\Omega_j(\theta)\hat{f}(\omega_1\theta)\big)(\omega_2)\big)\,d\omega_1\,d\omega_2\\
&= \frac{1}{(k-1)!}\int_{\mathbb{R}^+} \log(\omega_1)\frac{\partial^{2k}}{\partial\omega_1^{2k}}\big(\omega_1^{k+n-1}\big\langle\log(|\omega_2|);\ \mathfrak{L}^{(1)}\big(\hat{f}(\omega_1\theta)\big)(\omega_2)\big\rangle\big)\,d\omega_1.\end{aligned}$$

The change of sign for comes from a derivation in the sense of distributions, a similar comment applies below. Next, we have:

$$-\frac{2kh(0)}{k}\int_{\mathbb{S}^{n-1}}\log\bigl(\bigl|p(\theta)\bigr|\bigr)\frac{\partial^{2k-1}}{\partial r^{2k-1}}\bigl(\hat{f}(r\theta)r^{n+k-1}\bigr)_{|r=0}\Omega_0(\theta)\frac{d\theta}{p(\theta)}$$

$$+2m(0)\sum_j\int_{\mathbb{R}}\log\bigl(|\omega_2|\bigr)^2\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(\omega_1\theta)\bigr)(\omega_2)\bigr)_{|\omega_1=0}\,d\omega_2$$

$$=\frac{1}{(2k)!}\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\bigl\langle\log\bigl(|\omega_2|\bigr)^2;\ \mathfrak{L}^{(1)}\bigl(\hat{f}(\omega_1\theta)\bigr)(\omega_2)\bigr\rangle\bigr)_{|\omega_1=0}.$$

Finally, we combine the remaining terms to obtain:

$$-h'(0)\int_{\mathbb{S}^{n-1}}\frac{\partial^{2k-1}}{\partial r^{2k-1}}\bigl(r^{n+k-1}\hat{f}(r\theta)\bigr)_{|r=0}\Omega_0(\theta)\frac{d\theta}{p(\theta)}+\bigl(2m'(0)-4m(0)\bigr)$$

$$\times\sum_j\int_{\mathbb{R}}\log\bigl(|\omega_2|\bigr)\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\mathfrak{L}^{(1)}\bigl(\Omega_j(\theta)\hat{f}(\omega_1\theta)\bigr)(\omega_2)\bigr)_{|\omega_1=0}\,d\omega_2$$

$$=\frac{\gamma+\Psi(k)}{\Gamma(2k)}\frac{\partial^{2k-1}}{\partial\omega_1^{2k-1}}\bigl(\omega_1^{k+n-1}\bigl\langle\log\bigl(|\omega_2|\bigr);\ \mathfrak{L}^{(1)}\bigl(\hat{f}(\omega_1\theta)\bigr)(\omega_2)\bigr\rangle\bigr)_{|\omega_1=0}.$$

This proves parts (B) of Theorem 2.

Proof of Corollary 3. We start by the analytic continuation of the elliptic part $\mu^{\mathrm{ell}}(\zeta)$. The pole $\zeta=-1$ is simple and the term of interest is given by:

$$\mu^{\mathrm{ell}}_{0,-1}=\lim_{\zeta\to0}\bigl(\zeta\mu^{\mathrm{ell}}(\zeta-1)\bigr)=\frac{1}{\Gamma(2k+1)}\partial^{2k-1}_{\omega_1}G(0).$$

The value of this coefficient was determined in the proof of Theorem 2.

As concerns the singular term $\mu^{\mathrm{sing}}(\zeta)=\sum_{j=1}^N\mu_j^{\mathrm{sing}}(\zeta)$, $\zeta=-1$ is a pole of order 2. Accordingly, the coefficients of degree -2 and -1 are respectively given by:

$$a^{\mathrm{sing}}_{-2,j}=\lim_{\zeta\to0}\bigl(\zeta^2\mu_j^{\mathrm{sing}}(\zeta-1)\bigr),$$

$$a^{\mathrm{sing}}_{-1,j}=\lim_{\zeta\to0}\frac{\partial}{\partial\zeta}\bigl(\zeta^2\mu_j^{\mathrm{sing}}(\zeta-1)\bigr).$$

Since $M_j(0)=0$, we have $a^{\mathrm{sing}}_{-2,j}=0$ and $a^{\mathrm{sing}}_{-1,j}=m(0)M_j'(0)$. To evaluate this distributional coefficient we proceed exactly as above and obtain:

$$a^{\mathrm{sing}}_{-1,j}=\frac{1}{\Gamma(1+2k)}\int_{\mathbb{R}}\log\bigl(|\omega_2|\bigr)\frac{\partial^{2k+1}G_j}{\partial\omega_1^{2k-1}\partial\omega_2^2}(0,\omega_2)\,d\omega_2.$$

The discussion concerning the value of this term, established in the proof of Theorem 2, gives the announced result. □

2.4. Duality brackets

Condition $(\mathcal{H})$ only insures that the Liouville measure is smooth in a neighborhood of the origin. But the distributions $\log(|y|)^\alpha$, $\alpha > 0$, are smooth away from the origin. With a smooth cut-off χ, supported in a neighborhood of the origin, we write $\langle \log(|y|)^\alpha; \mathfrak{L}^{(p)}(f)(y)\rangle$ as:

$$\left\langle \log\left(|y|\right)^\alpha;\ \chi(y)\mathfrak{L}^{(p)}(f)(y)\right\rangle + \left\langle \log\left(|y|\right)^\alpha;\ \left(1-\chi(y)\right)\mathfrak{L}^{(p)}(f)(y)\right\rangle.$$

Away from the origin, we can integrate by part the logarithmic distribution. On the other side, we use that $y \mapsto \mathfrak{L}(f)(y)$ is smooth on $\operatorname{supp}(\chi)$ if this support is chosen small enough. This duality bracket is well defined since both distribution have disjoint singular support.

Finally, this construction is independent from the cut-off χ if $\operatorname{supp}(\chi)$ is small enough with respect to the covering of $\mathfrak{C}(p)$ introduced before. Conversely, for any covering of $\mathfrak{C}(p)$ chosen such that $|p(\theta)| \leqslant \varepsilon$ on each $\operatorname{supp}(\Omega_j)$, $j \geqslant 1$, there exists a cut-off χ with the previous properties. Hence the final value is independent from the choice of our partition of unity on $\mathbb{S}^{n-1}$.

2.5. Comments

- The relation between special functions, in particular Γ and hypergeometric, and fundamental solutions has attracted much attention by the past. That's why we have greatly detailed the coefficients appearing in our setting.
- Residuum, and poles, of meromorphic distributions play also an important rôle in asymptotic expansion of oscillatory and fiber integrals. For example, the value of $m''(0)$ is exactly:

$$\frac{12 + k(6\gamma(2+k\gamma) + k\pi^2) + 6k(\Psi(2k)(2+2k\gamma + k\Psi(2k)) - k\Psi^{(1)}(2k))}{3\Gamma(1+2k)},$$

  where $\Psi^{(1)}(\zeta) = \partial_\zeta \Psi(\zeta)$ is the polygamma-function of order 1. Such a coefficient is useful to compute the second term of the asymptotic expansion of oscillatory integrals with phase $p(\xi)$ or $p(\xi)^2$. See [8] for this point.
- The determination of Liouville measures, and a fortiori of their exterior differentials, is generally not possible. In the case of homogeneous singularity, the determination of these measures is sometimes possible in terms of generalized elliptic integrals. See [6] or [3] for different examples.
- The condition that $k \in \mathbb{N}$ can be relaxed. We can consider operators with a singularity at the origin providing that their symbols are regular enough. If $\alpha > 1$ is the degree, a similar proof holds by using the integer part $k = [\alpha] + 1$. All constants are well defined as analytic functions of α and one has to replace the radial derivations by the action of some pseudo-differential operators with homogeneous symbol. If $\alpha \leqslant 1$ the symbol p is generally not C^1 and our approach fails.

References

[1] J.-E. Björk, Rings of Differential Operators, Math. Library, vol. 21, North-Holland, 1979.

[2] B. Camus, Fundamental solutions of homogeneous elliptic differential operators, Bulletin des Sciences Mathématiques 130 (3) (2006) 264–268.

[3] B. Camus, Asymptotic approximation of degenerate fiber integrals, Journal of Mathematical Analysis and Applications 320 (2) (2006) 30–44.

[4] I.M. Guelfand, G.E. Chilov, Les Distributions, Collection Universitaire de Mathématiques, vol. VIII, Dunod, Paris, 1962.

[5] L. Hörmander, The Analysis of Linear Partial Differential Operators 1, 2, 3, 4, Springer-Verlag, 1985.
[6] P. Wagner, Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions, Acta Mathematica 182 (1999) 283–300.
[7] P. Wagner, On the fundamental solutions of a class of elliptic quartic operators in dimension 3, Journal de Mathématiques Pures Appliquées (9) 81 (11) (2002) 1191–1206.
[8] R. Wong, Asymptotic Approximations of Integrals, Academic Press Inc., 1989.