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Arc length associated with generalized distance functions

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ABSTRACT

We propose a generalization of the traditional definition of arc length in a manifold. In our definition, the arc length is associated with a distance function d that satisfies the identity property but not necessarily the triangle inequality, non-negativity, definiteness and symmetry. A new class of directed arcs, which we call "d-conservative" arcs, arises in an evident manner from our definition. These arcs satisfy a property of conservation of the d-distance along the arc. Each d-conservative arc has a d-length equal to the d-distance between its endpoints. If d satisfies the triangle inequality, the d-conservative arcs coincide with the arcs of minimum d-length. We prove that the d-length of an arc can be expressed as the integral of a function F along the arc, where F is the one-sided directional derivative of d. This last relation between d and F was proved by Busemann and Mayer (1941) [3] for the Finsler distances, which satisfy, among others, the triangle inequality and non-negativity, requirements that we do not need in our proof. We also prove that if the one-sided directional derivative F of a distance function d is continuous, then d satisfies the triangle inequality if, and only if, F is convex. We analyze an example of a non-positive definite distance function.

1. Introduction

H. Busemann [2] wrote that the goal which Riemann [7] set for himself was the definition and discussion of the most general finite-dimensional space in which every curve has a length $\int F(\mathbf{x}(t), \dot{\mathbf{x}}(t))\,dt$ *derived from an infinitesimal length or line element* $F(\mathbf{x}, d\mathbf{x})$.

We coincide with Riemann in that every smooth curve should have a length. However, in the definition of arc length proposed in this paper, the length is *derived from a given distance function* d (instead of from a line element $F(\mathbf{x}, d\mathbf{x})$). We show that the arc length derived from $F(\mathbf{x}, d\mathbf{x})$ can be obtained from our definition, and in that case $F(\mathbf{x}, d\mathbf{x})$ is the one-sided directional derivative of d at $\mathbf{x}$ in the direction $d\mathbf{x}$.

The most common definition of arc length derived from a distance function d defines length as the least upper bound of the lengths of all possible inscribed polygons. This definition has the disadvantage of requiring that the distance function d satisfies both the triangle inequality and the non-negativity condition.

In Section 2, we give a definition of the length associated with a distance function d (d-length) of a directed arc in a differentiable manifold M in terms of refinements of partitions of the arc. Our concept is based on an alternate definition of the Riemann integral (see, e.g., [8, p. 160]), and claims that the distance function d satisfies the identity property (the distance from one point to itself is zero) but not necessarily the triangle inequality and non-negativity. According to the foregoing, the relevant concept of arc length can be any cumulative attribute of paths (such as energy expended, travel

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time, travel cost or travel distance), and we define a (generalized) distance function on a differentiable manifold M as a binary function satisfying the identity property. The existence of certain arcs, those whose partitions have the same d-length, follows in an evident manner from our definition of arc length. These arcs, which we call d-conservative, satisfy a property of conservation of the distance d (d-distance) along the arc, and they can be thought of as a generalization of the straight line segments in R^n. We show that each directed d-conservative arc $C(\mathbf{a},\mathbf{b})$ has a d-length equal to the d-distance from $\mathbf{a}$ to $\mathbf{b}$, $d(\mathbf{a},\mathbf{b})$, and satisfies the property that for any sequence of points on the arc containing the endpoints $\mathbf{a}$ and $\mathbf{b}$, the sum of the d-distances between all the consecutive points ordered in the direction of the arc is exactly equal to $d(\mathbf{a},\mathbf{b})$. This property is used in a well-known empirical procedure for measuring the distance between two far away points. We also prove that if a distance function d is complete (for every ordered pair of points $\mathbf{a}$, $\mathbf{b}$ in M there is at least one d-conservative arc connecting $\mathbf{a}$ to $\mathbf{b}$), then the triangle inequality is a necessary and sufficient condition for both the d-conservative arcs and the arcs of minimum d-length are the same.

In Section 3, from our definition of arc length we obtain the well-known integral of a continuous function F along an arc $C(\mathbf{a},\mathbf{b})$, $\int F(\mathbf{x}(s),\dot{\mathbf{x}}(s))\,ds$, where $\mathbf{x}:[a,b]\to M$ is a parametric representation of $C(\mathbf{a},\mathbf{b})$; we prove that this integral is the arc length associated with a distance function d if, and only if, the function $F(\mathbf{x}(s),\dot{\mathbf{x}}(s))$ is the one-sided directional derivative of d, i.e.,

$$F(\mathbf{x},\mathbf{v})=\lim_{t\to 0^+}\frac{d(\mathbf{x},\boldsymbol{\sigma}(t))-d(\mathbf{x},\mathbf{x})}{t}=\lim_{t\to 0^+}\frac{d(\mathbf{x},\boldsymbol{\sigma}(t))}{t}\quad\text{for all } \mathbf{x}\in M,\ \mathbf{v}\in T_{\mathbf{x}}M, \tag{$*$}$$

where $\boldsymbol{\sigma}:[0,1]\to M$ is a path in M with $\boldsymbol{\sigma}(0)=\mathbf{x}$ and $(d\boldsymbol{\sigma}/ds)(0)=\mathbf{v}$. The last equality in $(*)$ was proved by Busemann and Mayer [3, p. 186] (see also [1, p. 161]) for Finsler distances, which satisfy the triangle inequality and non-negativity. In this section, we also prove that, if the one-sided directional derivative F of a distance function d is continuous, then d satisfies the triangle inequality if and only if F is convex, and that, in these both cases, the distance function is given by

$$d(\mathbf{a},\mathbf{b})=\min_{\mathbf{x}\in\Omega_{[\mathbf{a},\mathbf{b}]}}\int_{\mathbf{a}}^{\mathbf{b}}F\big(\mathbf{x}(s),\dot{\mathbf{x}}(s)\big)\,ds\quad\text{for all } \mathbf{a},\mathbf{b}\in M,\ \dot{\mathbf{x}}(s)\in T_{\mathbf{x}}M\backslash\{\mathbf{0}\}.$$

In Section 4, we analyze an example of an asymmetric distance function that can take negative values. This distance function is derived from a location problem formulated by Hodgson [5], and it is obtained in the context of an object sliding on an inclined plane. Assuming that the object has no acceleration over the path, and that path turns altogether involve insignificant energy loss, then the major external forces operating on the object are gravity and friction. In this context, we define "length" of an arc as the energy expended to move the abject along the arc on the inclined plane, and the corresponding "distance" is the minimum energy needed to slide the object on the inclined plane from one point to another point. In this case, the distance function satisfies the triangle inequality and the straight line segments are d-conservative arcs.

2. d-Length of an arc and d-conservative arcs

In this section we propose a definition of arc length associated with a distance function in a differentiable manifold. Our definition requires that the distance function satisfies the identity property but not necessarily the triangle inequality, symmetry and non-negativity. Also, in this section we introduce a new class of arcs associated with a distance function d. These arcs (which we call d-conservative) satisfy a property of conservation of the distance d along the arc, and can be thought of as a generalization of the straight line segments in R^n. We prove that, under certain conditions, the triangle inequality is a necessary and sufficient condition for the d-conservative arcs and the arcs of minimum d-length are the same.

Let M be a k-dimensional differentiable manifold in R^n, $k\leqslant n$. Denote by $T_{\mathbf{x}}M$ the tangent space at $\mathbf{x}\in M$, $T_{\mathbf{x}}M:=\{\mathbf{v}\in R^n:\ \mathbf{x}+\mathbf{v}h\in M$ for sufficiently small positive $h\}$, and by $TM:=\bigcup_{\mathbf{x}\in M}T_{\mathbf{x}}M$ the tangent bundle of M. Each element of TM has the form $(\mathbf{x},\mathbf{v})$, where $\mathbf{x}\in M$ and $\mathbf{v}\in T_{\mathbf{x}}M$. The subset of TM whose elements have the form $(\mathbf{x},\mathbf{v})$ where $\mathbf{x}\in M$ and $\mathbf{v}\in T_{\mathbf{x}}M\backslash\{\mathbf{0}\}$ is denoted by $TM\backslash\{\mathbf{0}\}$.

A *path* in M from $\mathbf{a}\in M$ to $\mathbf{b}\in M$ is a continuous function $\mathbf{x}:[a,b]\to M$ where $\mathbf{x}(a)=\mathbf{a}$ and $\mathbf{x}(b)=\mathbf{b}$, and $a<b$ are real numbers. The directed image $C(\mathbf{a},\mathbf{b})\subseteq M$ of the path $\mathbf{x}:[a,b]\to M$ is called a (*directed*) *arc* connecting $\mathbf{a}$ to $\mathbf{b}$. Throughout this paper, an arc will mean a directed arc. An arc $C(\mathbf{a},\mathbf{b})$ is said to be an *arc* C^1 if it has a parametric representation $\mathbf{x}:[a,b]\to M$ with a bounded derivative $\mathbf{x}'$ which is continuous everywhere on $[a,b]$ except (possibly) at a finite number of points. The set of all piecewise C^1 arcs on M is denoted by Ω. For the sake of simplicity, the set of all piecewise C^1 arcs connecting $\mathbf{a}$ to $\mathbf{b}$ and the set of parametric representations of these arcs are denoted by $\Omega_{[\mathbf{a},\mathbf{b}]}$.

Definition 1. We define a (*generalized*) *distance function* d on M as a binary function $d:M\times M\to R$ satisfying the identity property ($d(\mathbf{a},\mathbf{a})=0$ for all $\mathbf{a}\in M$).

If d also satisfies the triangle inequality ($d(\mathbf{a},\mathbf{b})\leqslant d(\mathbf{a},\mathbf{c})+d(\mathbf{c},\mathbf{b})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in M$), then we call it a *premetric*.

A *metric* is a premetric satisfying symmetry ($d(\mathbf{a},\mathbf{b}) = d(\mathbf{b},\mathbf{a})$ for all $\mathbf{a},\mathbf{b} \in M$), non-negativity ($d(\mathbf{a},\mathbf{b}) \geqslant 0$ for all $\mathbf{a},\mathbf{b} \in M$), and definiteness ($d(\mathbf{a},\mathbf{b}) = 0 \Rightarrow \mathbf{a} = \mathbf{b}$ for all $\mathbf{a},\mathbf{b} \in M$).

A sequence of points in M of the form $(\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b})$, where $k \geqslant 0$, is said to be a *sequence in M from* **a** *to* **b**. The points **a** and **b** are called the *endpoints of the sequence*. The set of all sequences in M from **a** to **b** is denoted by $P[\mathbf{a},\mathbf{b}]$.

For each sequence in M given by $P = (\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b})$, the distance function $d : M \times M \to R$ determines a real number $\Lambda(P)$, that we call *d-length of the sequence P*, which is defined to be the sum of d-distances:

$$\Lambda(P) = \sum_{i=0}^{k} d(\mathbf{x}_i, \mathbf{x}_{i+1}) \quad \text{for all } P = (\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b}) \in P[\mathbf{a},\mathbf{b}].$$

Throughout this paper, a partition of an arc will mean a partition of the arc into subarcs. Each partition P of an arc $C(\mathbf{a},\mathbf{b})$ determines a sequence $(\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b})$ of points along the arc $C(\mathbf{a},\mathbf{b})$. Conversely, such sequence $(\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b})$ of points along the arc $C(\mathbf{a},\mathbf{b})$ determines the partition P. In order to simplify, we also denote by P the sequence of points along $C(\mathbf{a},\mathbf{b})$ corresponding to the partition P. The *trivial partition* of an arc $C(\mathbf{a},\mathbf{b})$ is the set $\{C(\mathbf{a},\mathbf{b})\}$, which is determined by the sequence $(\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1 = \mathbf{b})$. A *refinement* of a partition P of the arc $C(\mathbf{a},\mathbf{b})$ is a partition Q of $C(\mathbf{a},\mathbf{b})$ such that each element of Q is contained in an element of P. The set of all partitions of $C(\mathbf{a},\mathbf{b})$ is denoted by $P[C(\mathbf{a},\mathbf{b})]$. We define the *d-length of a partition* $P \in P[C(\mathbf{a},\mathbf{b})]$ of $C(\mathbf{a},\mathbf{b})$ as the length of the sequence P with respect the distance function d, denoted by $\Lambda(P)$.

Definition 2. We define the *length associated with a distance function d* on M of a directed arc C (*d-length* of an arc C) as a real number L such that, for every $\varepsilon > 0$, there is a partition P_ε of C such that $|\Lambda(P) - L| < \varepsilon$ for all refinement P of P_ε. If the d-length of an arc exists, it is unique.

If the d-length of an arc $C(\mathbf{a},\mathbf{b})$ is finite, then $C(\mathbf{a},\mathbf{b})$ is said to be *d-rectifiable*, and $l(C(\mathbf{a},\mathbf{b}))$ denotes the d-length of $C(\mathbf{a},\mathbf{b})$. It is immediate that all the subarcs of any partition P of a d-rectifiable arc $C(\mathbf{a},\mathbf{b})$ are d-rectifiable, and that the sum of their d-lengths is equal to the d-length of $C(\mathbf{a},\mathbf{b})$.

$C(\mathbf{a},\mathbf{b})$ is an arc of *minimum d-length* if it is rectifiable and its d-length is less or equal than the d-length of any other arc from **a** to **b**. Each subarc of an arc of minimum d-length is an arc of minimum d-length.

Definition 2 allowed us to deduce the existence of certain arcs which can be thought of as a generalization of the straight line segments in R^n.

Definition 3. Let $d : M \times M \to R$ be a distance function. An arc $C(\mathbf{a},\mathbf{b})$ is said to be a *d-conservative* arc if all its partitions have the same d-length. Equivalently, $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc if the d-length of each one of its partitions $P = (\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b})$, with $k \geqslant 0$, is equal to the d-distance from **a** to **b**,

$$d(\mathbf{a},\mathbf{b}) = \sum_{i=0}^{k} d(\mathbf{x}_i, \mathbf{x}_{i+1}) \quad \text{for all } P = (\mathbf{a}, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{b}) \in P\big[C(\mathbf{a},\mathbf{b})\big].$$

Then, every d-conservative arc connecting **a** to **b** satisfies that the d-distance from **a** to **b** is equal to the sum of d-distances between all consecutive points of any sequence of points on the arc, where each sequence contains the endpoints **a** and **b**. Therefore, every d-conservative arc $C(\mathbf{a},\mathbf{b})$ is d-rectifiable and its d-length is equal to the d-length of its trivial partition (i.e., $l(C(\mathbf{a},\mathbf{b})) = d(\mathbf{a},\mathbf{b})$). It is immediate that each subarc of a d-conservative arc is a d-conservative arc.

In order to show some properties of the d-conservative arcs, we give the following alternate definitions of a d-conservative arc:

It can be proved by induction that $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc if and only if it satisfies the *triangle equality* with respect to the endpoint **b**,

$$d\big(\mathbf{x}(s),\mathbf{b}\big) = d\big(\mathbf{x}(s),\mathbf{x}(t)\big) + d\big(\mathbf{x}(t),\mathbf{b}\big), \quad \text{where } a \leqslant s \leqslant t \leqslant b, \tag{1}$$

where $\mathbf{x} : [a,b] \to M$ is a parametric representation of $C(\mathbf{a},\mathbf{b})$. It can be proved directly that (1) is equivalent to any of the following two conditions:

$$d\big(\mathbf{x}(s),\mathbf{x}(z)\big) = d\big(\mathbf{x}(s),\mathbf{x}(t)\big) + d\big(\mathbf{x}(t),\mathbf{x}(z)\big), \quad \text{where } a \leqslant s \leqslant t \leqslant z \leqslant b,$$
$$d\big(\mathbf{x}(s),\mathbf{x}(r)\big) - d\big(\mathbf{x}(s),\mathbf{x}(t)\big) = d\big(\mathbf{x}(t),\mathbf{x}(z)\big) - d\big(\mathbf{x}(r),\mathbf{x}(z)\big), \quad a \leqslant s \leqslant t \leqslant r \leqslant z \leqslant b. \tag{2}$$

Clearly, the last two conditions are mutually equivalent. The first one expresses the "triangle equality" for any three ordered points on $C(\mathbf{a},\mathbf{b})$. Eq. (2) expresses a law of conservation of the d-distance: Let $C(\mathbf{a},\mathbf{b})$ be a d-conservative arc with a parametric representation $\mathbf{x} : [a,b] \to M$, where $a \leqslant s \leqslant t \leqslant r \leqslant z \leqslant b$. Let us consider that the arc $C(\mathbf{a},\mathbf{b})$ is traced out by the movement of a particle going from **a** to **b**. The interval $[a,b]$ is thought of as a time interval and the vector $\mathbf{x}(w)$, where $w \in [a,b]$, specifies the position of the particle at time w. For any interval $[t,r] \subseteq [a,b]$ the particle travels along the arc $C(\mathbf{a},\mathbf{b})$ connecting $\mathbf{x}(t)$ to $\mathbf{x}(r)$. If $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc, then when the particle goes from $\mathbf{x}(t)$ to $\mathbf{x}(r)$, the increment of the d-distance from any point on $C(\mathbf{a},\mathbf{b})$ already visited by the particle, $\mathbf{x}(s)$, where $s \leqslant t$, to the particle itself, $d(\mathbf{x}(s), \mathbf{x}(r)) - d(\mathbf{x}(s), \mathbf{x}(t))$, is equal to the decrement of the d-distance from the particle to any point on $C(\mathbf{a},\mathbf{b})$ that will visit the particle, $\mathbf{x}(z)$, where $r \leqslant z$, $d(\mathbf{x}(t),\mathbf{x}(z)) - d(\mathbf{x}(r),\mathbf{x}(z))$. This is why we call the arc $C(\mathbf{a},\mathbf{b})$ "*d-conservative arc*".

A distance function d is *complete* if for every ordered pair of points $(\mathbf{a},\mathbf{b})$ in M there exists at least one piecewise C^1 d-conservative arc connecting $\mathbf{a}$ to $\mathbf{b}$. If a distance function d is complete, then every partition of a d-rectifiable arc $C(\mathbf{a},\mathbf{b})$ has associated a *inscribed d-polygonal* formed by d-conservative arcs, each of which connecting two consecutive points of the partition. Therefore, the d-length of a d-rectifiable arc $C(\mathbf{a},\mathbf{b})$ is equal to the limit of the d-length of the d-rectifiable d-polygonal inscribed in $C(\mathbf{a},\mathbf{b})$.

Theorem 1 *(Properties of premetrics). Let $d : M \times M \to R$ be a distance function satisfying the triangle inequality (i.e., d is a premetric) and let $C(\mathbf{a},\mathbf{b})$ be a piecewise C^1 d-rectifiable arc whose d-length is $l(C(\mathbf{a},\mathbf{b}))$. Then we have:*

(a) *If $P, Q \in P[C(\mathbf{a},\mathbf{b})]$ are two partitions of $C(\mathbf{a},\mathbf{b})$ such that P is a refinement of Q, then $\Lambda(Q) \leqslant \Lambda(P)$;*
(b) *$d(\mathbf{a},\mathbf{b}) \leqslant \Lambda(P) \leqslant l(C(\mathbf{a},\mathbf{b}))$ for all $P \in P[C(\mathbf{a},\mathbf{b})]$;*
(c) *$l(C(\mathbf{a},\mathbf{b})) = \sup\{\Lambda(P):\ P \in P[C(\mathbf{a},\mathbf{b})]\}$;*
(d) *$d(\mathbf{a},\mathbf{b}) = l(C(\mathbf{a},\mathbf{b}))$ if and only if $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc;*
(e) *Every d-conservative arc is an arc of minimum d-length;*
(f) *If d is a complete distance function, then every arc of minimum d-length is a d-conservative arc.*

Proof. (a) It can be proved directly using the triangle inequality.

(b) This conclusion is an immediate consequence of (a): $d(\mathbf{a},\mathbf{b})$, $\Lambda(P)$, and $l(C(\mathbf{a},\mathbf{b}))$ are d-lengths of three partitions of $C(\mathbf{a},\mathbf{b})$. The partition corresponding to $l(C(\mathbf{a},\mathbf{b}))$ is a refinement of P, and P is a refinement of the trivial partition.

(c) Due to (b), $\Lambda(P) \leqslant l(C(\mathbf{a},\mathbf{b}))$ for all $P \in P[C(\mathbf{a},\mathbf{b})]$. Using the definition of d-length of the arc $C(\mathbf{a},\mathbf{b})$ and by the definition of the supremum of a set, we obtain $l(C(\mathbf{a},\mathbf{b})) = \sup\{\Lambda(P):\ P \in P[C(\mathbf{a},\mathbf{b})]\}$.

(d) As we mentioned in Definition 3, if $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc, then $d(\mathbf{a},\mathbf{b}) = l(C(\mathbf{a},\mathbf{b}))$. To prove the converse, assume that $d(\mathbf{a},\mathbf{b}) = l(C(\mathbf{a},\mathbf{b}))$. Using (b), we obtain $d(\mathbf{a},\mathbf{b}) = \Lambda(P)$ for all $P \in P[C(\mathbf{a},\mathbf{b})]$, and therefore $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc.

(e) If $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc which is not an arc of minimum d-length, then $d(\mathbf{a},\mathbf{b}) = l(C(\mathbf{a},\mathbf{b}))$, and $l(C^*(\mathbf{a},\mathbf{b})) < l(C(\mathbf{a},\mathbf{b}))$ is satisfied for some $C^*(\mathbf{a},\mathbf{b}) \in \Omega_{[\mathbf{a},\mathbf{b}]}$. Then $l(C^*(\mathbf{a},\mathbf{b})) < d(\mathbf{a},\mathbf{b})$, which contradicts (b). Therefore every d-conservative arc is an arc of minimum d-length.

(f) Due to (b), $d(\mathbf{a},\mathbf{b}) \leqslant l(C(\mathbf{a},\mathbf{b}))$. In view of the fact that d is a complete distance function, there exists a piecewise C^1 d-conservative arc connecting $\mathbf{a}$ to $\mathbf{b}$ of d-length $d(\mathbf{a},\mathbf{b})$. Since $C(\mathbf{a},\mathbf{b})$ is a piecewise C^1 arc of minimum d-length, then $l(C(\mathbf{a},\mathbf{b})) \leqslant d(\mathbf{a},\mathbf{b})$. Therefore, the equality $d(\mathbf{a},\mathbf{b}) = l(C(\mathbf{a},\mathbf{b}))$ holds. Due to (d), $C(\mathbf{a},\mathbf{b})$ is a d-conservative arc. □

By (c) of Theorem 1, the well-known expression for calculating the length of an arc $C(\mathbf{a},\mathbf{b})$ in an Euclidean space, given by $l(C(\mathbf{a},\mathbf{b})) = \sup\{\Lambda(P):\ P \in P[C(\mathbf{a},\mathbf{b})]\}$ (see, e.g., [6, p. 463]), is applicable to any distance function d that satisfies the triangle inequality. If d does not satisfy the triangle inequality, this expression is not valid, as we show in the following example: Let d_p be a distance function given by $d_p(\mathbf{a},\mathbf{b}) = (\sum_i |b_i - a_i|^p)^{1/p}$, $\forall \mathbf{a},\mathbf{b} \in R^n$, $i = 1,\dots,n$, where $p \in (0,1)$ (a_i and b_i denote vector components). Suppose three points $\mathbf{a} = (0,0)$, $\mathbf{c} = (1,0)$, and $\mathbf{b} = (1,1)$ in R^2. Let $C(\mathbf{a},\mathbf{b})$ be the arc formed by two subarcs, the first one going from $\mathbf{a}$ to $\mathbf{c}$, and the second one going from $\mathbf{c}$ to $\mathbf{b}$. For $p = 0.5$, the $d_{0.5}$-length of $C(\mathbf{a},\mathbf{b})$ is $1 + 1 = 2$. However, the straight line segment going from $\mathbf{a}$ to $\mathbf{b}$ is the supremum of all the partitions of $C(\mathbf{a},\mathbf{b})$, and its $d_{0.5}$-length is $(1+1)^2 = 4$, hence $\sup\{\Lambda(P):\ P \in P[C(\mathbf{a},\mathbf{b})]\} = 4 > 2 = l(C(\mathbf{a},\mathbf{b}))$.

3. Functional *d*-length

In this section, we obtain from our definition of arc length a formula for calculating the arc length associated with a given distance function d. We also prove that, if the one-sided directional derivative F of a distance function d is continuous, then d satisfies the triangle inequality if, and only if, F is convex.

A function $f : M \to R$ is said to have a *one-sided directional derivative at a point* $\mathbf{x} \in M$ with respect to a $\mathbf{v} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\}$ if and only if

$$D^+ f(\mathbf{x};\mathbf{v}) = \lim_{t \to 0^+} \frac{f(\boldsymbol{\sigma}(t)) - f(\mathbf{x})}{t}$$

exists in the set of the extended real numbers, where $\boldsymbol{\sigma} : [0,1] \to M$ is a path in M with $\boldsymbol{\sigma}(0) = \mathbf{x}$ and $(d\boldsymbol{\sigma}(t)/dt)(0) = \mathbf{v}$. It can be shown directly that $D^+ f(\mathbf{x};\mathbf{v})$ is positively homogeneous of degree one in the argument $\mathbf{v}$, i.e.,

$$D^+ f(\mathbf{x};\alpha\mathbf{v}) = \alpha D^+ f(\mathbf{x};\mathbf{v}) \quad \text{for all } \alpha > 0, \text{ for all } \mathbf{x} \in M.$$

Then $D^+ f$ is independent of the parametric representation of the curve $\boldsymbol{\sigma}$. The second argument of $D^+ f(\mathbf{x}; \mathbf{v})$, where $\mathbf{v} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\}$, can be called *a direction at* $\mathbf{x} \in M$.

Let $d : M \times M \to R$ be a distance function on M. The *one-sided directional derivative* of $d(\mathbf{x}, \cdot)$ at $\mathbf{x} \in M$ with respect to a direction $\mathbf{v} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\}$ is defined to be the limit

$$F(\mathbf{x}, \mathbf{v}) = \lim_{t \to 0^+} \frac{d(\mathbf{x}, \boldsymbol{\sigma}(t)) - d(\mathbf{x}, \mathbf{x})}{t},$$

if it exists. By the identity property of d, $F(\mathbf{x}, \mathbf{v})$ becomes

$$F(\mathbf{x}, \mathbf{v}) := \lim_{t \to 0^+} \frac{d(\mathbf{x}, \boldsymbol{\sigma}(t))}{t} \quad \text{for all } \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M \backslash 0, \tag{3}$$

where $\boldsymbol{\sigma} : [0, 1] \to M$ is a path in M such that $\boldsymbol{\sigma}(0) = \mathbf{x}$ and $(d\boldsymbol{\sigma}/ds)(0) = \mathbf{v}$. The function $F : TM \to R$ given by (3) evaluated at the point $\mathbf{x}$ in the direction $\mathbf{v}$ will be denoted by $F(\mathbf{x}, \mathbf{v})$, and the function F along a path $\mathbf{x} : [a, b] \to M$ will be denoted by $F(\mathbf{x}(s), \dot{\mathbf{x}}(s))$ and it is given by

$$F\big(\mathbf{x}(s), \dot{\mathbf{x}}(s)\big) = \lim_{t \to 0^+} \frac{d(\mathbf{x}(s), \mathbf{x}(s + t))}{t}. \tag{4}$$

It is easy to see that $F : TM \to R$ given by (3) is positively homogeneous of degree one in its second argument, i.e., $F(\mathbf{x}, \alpha\mathbf{v}) = \alpha F(\mathbf{x}, \mathbf{v})$ for all $\alpha > 0$, for all $\mathbf{x} \in M$ and $\mathbf{v} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\}$.

A function $F : TM \to R$ is *convex at a point* $\mathbf{x} \in M$ if

$$F\big(\mathbf{x}, \alpha\mathbf{v} + (1 - \alpha)\mathbf{w}\big) \leqslant \alpha F(\mathbf{x}, \mathbf{v}) + (1 - \alpha)F(\mathbf{x}, \mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\},\ \alpha \in [0, 1].$$

Considering that the function $F(\mathbf{x}, \mathbf{v})$ is positively homogeneous of degree one in its second argument, F is convex at $\mathbf{x}$ if and only if $F(\mathbf{x}, \mathbf{v} + \mathbf{w}) \leqslant F(\mathbf{x}, \mathbf{v}) + F(\mathbf{x}, \mathbf{w})$, for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}M \backslash \{\mathbf{0}\}$. We say that F is a *convex function* if F is convex at all $\mathbf{x} \in M$.

Now we will determine the d-length of a piecewise C^1 d-rectifiable arc $C(\mathbf{a}, \mathbf{b})$ in terms of the one-sided directional derivative function F of d. Let $\mathbf{x} : [a, b] \to M$ be a piecewise C^1 parametric representation of $C(\mathbf{a}, \mathbf{b})$. Any set of interior points of the interval $[a, b]$, $s_1, s_2, \ldots, s_k \in (a, b)$, where $k > 0$, determines a non-trivial partition $(a = s_0, s_1, \ldots, s_{k+1} = b)$ of $[a, b]$ and a non-trivial partition $P = (\mathbf{a} = \mathbf{x}(s_0), \mathbf{x}(s_1), \ldots, \mathbf{x}(s_k), \mathbf{x}(s_{k+1}) = \mathbf{b})$ of $C(\mathbf{a}, \mathbf{b})$. Therefore

$$\Lambda(P) = \sum_{i=0}^{k} d\big(\mathbf{x}(s_i), \mathbf{x}(s_{i+1})\big).$$

Suppose that $F(\mathbf{x}, \mathbf{v})$ is a continuous function on its domain. Then $F(\mathbf{x}(s), \dot{\mathbf{x}}(s))$ is a bounded continuous function on $[a, b]$ and therefore integrable. Due to the existence of F at $t \in [a, b)$ for every $\varepsilon > 0$, there is a $\Delta s > 0$ such that

$$\left| F\big(\mathbf{x}(\xi), \dot{\mathbf{x}}(\xi)\big) - \frac{d(\mathbf{x}(\xi), \mathbf{x}(\xi) + \dot{\mathbf{x}}(\xi)\Delta s)}{\Delta s} \right| < \varepsilon \quad \text{for all } \xi \in [t, t + \Delta s).$$

This can be applied to a partition $(a = s_0, s_1, \ldots, s_{k+1} = b)$ of $[a, b]$. Thus, for every $\varepsilon > 0$, there is a partition P_ε of $C(\mathbf{a}, \mathbf{b})$ such that

$$\left| F\big(\mathbf{x}(\xi_i), \dot{\mathbf{x}}(\xi_i)\big) - \frac{d(\mathbf{x}(\xi_i), \mathbf{x}(\xi_i) + \dot{\mathbf{x}}(\xi_i)\Delta s_i)}{\Delta s_i} \right| < \varepsilon \quad \text{for all } \xi_i \in [s_i, s_{i+1}),\ i = 0, \ldots, k,$$

for all refinement P of P_ε.

Therefore,

$$\left| F\big(\mathbf{x}(\xi_i), \dot{\mathbf{x}}(\xi_i)\big)\Delta s_i - d\big(\mathbf{x}(\xi_i), \mathbf{x}(\xi_i) + \dot{\mathbf{x}}(\xi_i)\Delta s_i\big) \right| < \varepsilon\Delta s_i \quad \text{for all } \xi_i \in [s_i, s_{i+1}),\ i = 0, \ldots, k,$$

holds. Since

$$\Lambda(P) = \sum_{i=0}^{k} d\big(\mathbf{x}(\xi_i), \mathbf{x}(\xi_i) + \dot{\mathbf{x}}(\xi_i)\Delta s_i\big),$$

we obtain

$$\left| \sum_{i=0}^{k} F\big(\mathbf{x}(\xi_i), \dot{\mathbf{x}}(\xi_i)\big)\Delta s_i - \Lambda(P) \right| < \varepsilon(b - a) \quad \text{for all } \xi_i \in [s_i, s_{i+1}).$$

The foregoing inequality is also valid if we replace $\varepsilon(b-a)$ by $\varepsilon/2$:

$$\left|\sum_{i=0}^{k} F\big(\mathbf{x}(\xi_i), \dot{\mathbf{x}}(\xi_i)\big)\Delta s_i - \Lambda(P)\right| < \frac{\varepsilon}{2} \quad \text{for all } \xi_i \in [s_i, s_{i+1}),\ i = 0, \ldots, k.$$

Since $C(\mathbf{a}, \mathbf{b})$ is a d-rectifiable arc, for every $\varepsilon > 0$ there is a partition P_ε of $C(\mathbf{a}, \mathbf{b})$ such that

$$\big|L - \Lambda(P)\big| < \frac{\varepsilon}{2},$$

for all refinement P of P_ε, where L is the length of the arc $C(\mathbf{a}, \mathbf{b})$. From the two last inequalities, for every $\varepsilon > 0$, there is a partition P_ε of $C(\mathbf{a}, \mathbf{b})$ such that

$$\left|L - \sum_{i=0}^{k} F\big(\mathbf{x}(\xi_i), \dot{\mathbf{x}}(\xi_i)\big)\Delta s_i\right| < \varepsilon \quad \text{for all } \xi_i \in [s_i, s_{i+1}),\ i = 0, \ldots, k,$$

for all refinement P of P_ε. Therefore (see [8, p. 160]),

$$L = \int_a^b F\big(\mathbf{x}(s), \dot{\mathbf{x}}(s)\big)\, ds.$$

For all arc $C(\mathbf{a}, \mathbf{b})$ of class C^1, $F(\mathbf{x}(s), \dot{\mathbf{x}}(s))$ is a continuous function on $[a, b]$, and therefore L is well defined, and $C(\mathbf{a}, \mathbf{b})$ is a d-rectifiable arc.

We have proved the following theorem:

Theorem 2 *(Functional arc length). The arc length associated with a distance function $d : M \times M \to R$ (d-length of an arc $C(\mathbf{a}, \mathbf{b})$) is given by*

$$l\big(C(\mathbf{a}, \mathbf{b})\big) = \int_a^b F\big(\mathbf{x}(s), \dot{\mathbf{x}}(s)\big)\, ds, \tag{5}$$

if and only if $F : TM \to R$ is continuous and is the one-sided directional derivative of d given by (3)*, where $\mathbf{x} : [a, b] \to M$ is a piecewise C^1 parametric representation of $C(\mathbf{a}, \mathbf{b})$.*

Considering that the function F in (5) does not depend explicitly on the parameter s, and that F is positively homogeneous of degree one in its second argument, it follows that $F(\mathbf{x}(s), \dot{\mathbf{x}}(s))\, ds$ is invariant under any transformation of the parameter s, and therefore the d-length of any arc $C(\mathbf{a}, \mathbf{b})$ is independent of the parametric representation of the curve.

Theorem 3 *(Two characterizations of premetrics). Let d be a distance function on M and let $F : TM \to R$ be the one-sided directional derivative of d. Then we have*:

(a) *If d satisfies the triangle inequality* (*i.e., d is a premetric*)*, then F is a convex function*;
(b) *If F is a continuous convex function, then the distance function d is a premetric and is given by*

$$d(\mathbf{a}, \mathbf{b}) = \min_{\mathbf{x} \in \Omega_{[\mathbf{a},\mathbf{b}]}} \int_a^b F\big(\mathbf{x}(s), \dot{\mathbf{x}}(s)\big)\, ds \quad \textit{for all } \mathbf{a}, \mathbf{b} \in M,\ \dot{\mathbf{x}}(s) \in T_{\mathbf{x}} M \backslash \{\mathbf{0}\}, \tag{6}$$

where $\mathbf{x} : [a, b] \to M$ is a piecewise C^1 parametric representation of $C(\mathbf{a}, \mathbf{b})$.

Proof. (a) Let $\boldsymbol{\sigma} : [0, 1] \to M$ and $\boldsymbol{\rho} : [0, 1] \to M$ be two paths such that $\boldsymbol{\sigma}(0) = \mathbf{x}$, $(d\boldsymbol{\sigma}(t)/dt)(0) = \mathbf{v}$, $\boldsymbol{\rho}(0) = \mathbf{x}$, and $(d\boldsymbol{\rho}(t)/dt)(0) = \mathbf{w}$. By (3) and because of the positive homogeneity of F,

$$\begin{aligned}
F\big(\mathbf{x}, \alpha\mathbf{v} + (1-\alpha)\mathbf{w}\big) &= \lim_{t \to 0^+} \frac{d(\mathbf{x}, \mathbf{x} + \alpha\boldsymbol{\sigma}(t) + (1-\alpha)\boldsymbol{\rho}(t))}{t} \\
&\leqslant \lim_{t \to 0^+} \frac{d(\mathbf{x}, \mathbf{x} + \alpha\boldsymbol{\sigma}(t)) + d(\mathbf{x} + \alpha\boldsymbol{\sigma}(t), (\mathbf{x} + \alpha\boldsymbol{\sigma}(t) + (1-\alpha)\boldsymbol{\rho}(t)))}{t} \\
&= \lim_{t \to 0^+} \frac{d(\mathbf{x}, \mathbf{x} + \alpha\boldsymbol{\sigma}(t))}{t} + \lim_{t \to 0^+} \frac{d(\mathbf{x} + \alpha\boldsymbol{\sigma}(t), (\mathbf{x} + \alpha\boldsymbol{\sigma}(t) + (1-\alpha)\boldsymbol{\rho}(t)))}{t} \\
&= F(\mathbf{x}, \alpha\mathbf{v}) + F\big(\mathbf{x}, (1-\alpha)\mathbf{w}\big) = \alpha F(\mathbf{x}, \mathbf{v}) + (1-\alpha) F(\mathbf{x}, \mathbf{w}).
\end{aligned}$$

(b) By Theorem 2, it is sufficient to prove that if F is convex, then the one-sided directional derivative of the distance function d in (6) is just the integrand of (6). Substituting (6) into (4):

$$\lim_{t\to 0^+}\left(\frac{1}{t}\min_{\mathbf{x}\in\Omega_{[\mathbf{x},\mathbf{x}+t\mathbf{v}]}}\int_{\mathbf{x}}^{\mathbf{x}+t\mathbf{v}} F\big(\mathbf{x}(s),\dot{\mathbf{x}}(s)\big)\,ds\right)=F(\mathbf{x},\mathbf{v}).$$

The last equality can be explained as follows: in the limit as $t\to 0^+$, we can consider $\mathbf{x}(s)$ as a constant, and therefore the integrand $F(\mathbf{x}(s),\dot{\mathbf{x}}(s))$ only depends on $\dot{\mathbf{x}}(s)$; due to the convexity of F, $F(\mathbf{x},\mathbf{v}+\mathbf{w})\leqslant F(\mathbf{x},\mathbf{v})+F(\mathbf{x},\mathbf{w})$ for all $\mathbf{v},\mathbf{w}\in T_{\mathbf{x}}M\backslash\{\mathbf{0}\}$, the integral attains its minimum value if $\dot{\mathbf{x}}(s)$ has the direction $\mathbf{v}$ at each point along the arc joining $\mathbf{x}$ and $\mathbf{x}+\mathbf{v}t$. Therefore, the integrand $F(\mathbf{x}(s),\dot{\mathbf{x}}(s))$ remains constant along the arc joining $\mathbf{x}$ and $\mathbf{x}+\mathbf{v}t$ and it is equal to $F(\mathbf{x},\mathbf{v})$. $\square$

In Finsler geometry, the arc length in a manifold M is given by (5), and the distance function is *defined* by (6), where the function F is positively homogeneous of degree one in its second argument, strictly convex, non-negative, and of class C^∞ on $TM\backslash\{\mathbf{0}\}$ (see, e.g., [4, p. 1] or [9, p. 484]). Busemann and Mayer [3, p. 186] (see also [1, p. 161]) proved that under these conditions, the function F is the one-sided directional derivative of d, given by (3).

The well-known formula for calculating the arc length in R^n,

$$l\big(C(\mathbf{a},\mathbf{b})\big)=\int_{\mathbf{a}}^{\mathbf{b}}\big\|\dot{\mathbf{x}}(s)\big\|\,ds, \tag{7}$$

can be obtained as a particular case of (5): Suppose that d is a distance function on R^n satisfying the condition $d(\mathbf{a},\mathbf{b})=d(0,\mathbf{b}-\mathbf{a})=\|\mathbf{b}-\mathbf{a}\|$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in R^n$, where $\|\cdot\|$ denotes the usual norm in R^n. This condition is satisfied if d is invariant under translations (i.e., $d(\mathbf{a}+\mathbf{c},\mathbf{b}+\mathbf{c})=d(\mathbf{a},\mathbf{b})$, or equivalently, $F(\mathbf{x},\mathbf{v})$ does not depend on $\mathbf{x}$) and is non-negative. It follows from (3) that

$$\lim_{t\to 0^+}\frac{d(\mathbf{x}(s),\mathbf{x}(s+t))}{t}=\lim_{t\to 0^+}\frac{\|\mathbf{x}(s),\mathbf{x}(s+t)\|}{t}=\lim_{t\to 0^+}\left\|\frac{\mathbf{x}(s),\mathbf{x}(s+t)}{t}\right\|=\left\|\lim_{t\to 0^+}\frac{\mathbf{x}(s),\mathbf{x}(s+t)}{t}\right\|=\big\|\dot{\mathbf{x}}(s)\big\|.$$

Substituting the right-hand side into (5) we obtain (7).

4. A non-positive definite and asymmetric distance function

In this section, we analyze an example of an asymmetric distance function that can take negative values. This distance function is obtained from a location problem formulated by Hodgson [5], and is simplified in the context of an object sliding on an inclined plane. Assuming that the object has no acceleration over the path, and that path turns altogether involve insignificant energy loss, then the major external forces operating on the object are gravity and friction. In this context, we define "length" of an arc from one point to another point as the energy expended to move the abject along the arc on the inclined plane, and the corresponding "distance" is the minimum energy needed to slide the object on the inclined plane.

The distance function d we analyze is

$$d(\mathbf{x},\mathbf{y})=(y_2-x_2)\tan\theta+\mu\big((y_1-x_1)^2\cos^2\theta+(y_2-x_2)^2\big)^{1/2}, \tag{8}$$

where $\theta\geqslant 0$ and $\mu>0$ are constants, and (x_1,x_2) and (y_1,y_2) are the coordinates of the points $\mathbf{x}$ and $\mathbf{y}$, respectively. It is immediate that d in (8) satisfies the identity property ($d(\mathbf{x},\mathbf{x})=0$), and therefore, d is a generalized distance function.

The points $\mathbf{x}$ and $\mathbf{y}$ are the projections of the starting point and the ending point, respectively, onto the horizontal plane. The inclined plane has an inclination angle θ with respect to the horizontal plane. The friction coefficient between the object and the inclined plane is denoted by μ. The speed of the object is so small that the kinetic energy can be ignored.

The distance function (8) is uniform, but it is asymmetric due to the antisymmetric term $(y-b)\tan\theta$. Other properties of (8) are determined by the angle of inclination θ: if $\theta=0$, d is symmetric, non-negative and holds definiteness; if $0<\tan\theta<\mu$, d is asymmetric, non-negative and holds definiteness; if $\tan\theta=\mu$, d is asymmetric, non-negative and does not satisfy definiteness; if $\tan\theta>\mu$, d is asymmetric, non-positive definite, and it does not satisfy definiteness.

In the following theorem we show that (8) satisfies the triangle inequality, is complete, and that the straight line segments are both d-conservative arcs and arcs of minimum d-length.

Theorem 4 *(The straight line segments are arcs of minimum d-length and d-conservative). Let d be the distance function given by* (8)*. Then*:

(a) *d satisfies the triangle inequality.*
(b) *The straight line segments in the XY-plane are d-conservative and of minimum d-length.*

(c) *d is a complete premetric (d is not necessarily a metric because it can be asymmetric and non-positive definite).*
(d) *The one-sided directional derivative of d is given by*

$$F\big(\mathbf{x}(s),\dot{\mathbf{x}}(s)\big)=\dot{x}_2(s)\tan\theta+\mu\big(\dot{x}_1^2(s)\cos^2\theta+\dot{x}_2^2(s)\big)^{1/2},$$

and the d-length of a d-rectifiable arc $C(\mathbf{a},\mathbf{b})$ with a parametric representation $\mathbf{x}:[a,b]\to R^n$ is given by

$$l\big(C(\mathbf{a},\mathbf{b})\big)=(b_2-a_2)\tan\theta+\mu\int_{\mathbf{a}}^{\mathbf{b}}\big(\dot{x}_1^2(s)\cos^2\theta+\dot{x}_2^2(s)\big)^{1/2}\,ds. \tag{9}$$

Proof. (a) d is the sum of two distance functions, each of which satisfies the triangle inequality. Therefore, d satisfies the triangle inequality.

(b) The path $\mathbf{x}:[0,1]\to R^n$ given by $\mathbf{x}(t)=\mathbf{a}+(\mathbf{b}-\mathbf{a})t$, where $\mathbf{x}(0)=\mathbf{a}$ and $\mathbf{x}(1)=\mathbf{b}$, is a parametric representation of the straight line segment connecting $\mathbf{a}$ to $\mathbf{b}$. Substituting $\mathbf{x}$ into the right-hand side of (1) and considering $0\leqslant s\leqslant 1$ and $s\leqslant t\leqslant 1$:

$$\begin{aligned}d\big(\mathbf{x}(s),\mathbf{x}(t)\big)+d\big(\mathbf{x}(t),\mathbf{b}\big)&=(b_2-a_2)(t-s)\tan\theta+(b_2-a_2)(1-t)\tan\theta\\&\quad+\mu\big((b_1-a_1)^2(t-s)^2\cos^2\theta+(b_2-a_2)^2(t-s)^2\big)^{1/2}\\&\quad+\mu\big((b_1-a_1)^2(1-t)^2\cos^2\theta+(b_2-a_2)^2(1-t)^2\big)^{1/2}\\&=(b_2-a_2)(1-s)\tan\theta+\mu\big((b_1-a_1)^2\cos^2\theta+(b_2-a_2)^2\big)^{1/2}[t-s+1-t]\\&=d\big(\mathbf{x}(s),\mathbf{b}\big).\end{aligned}$$

Thereby, $\mathbf{x}$ satisfies (1). Therefore, each straight line segment is a d-conservative arc. By (a), d satisfies the triangle inequality, and by (e) of Theorem 1, each straight line segment is an arc of minimum d-length.

(c) By (b), the distance function d is complete, and by (a), d is a premetric.

(d) Substituting d given by (8) into (4), we obtain

$$\begin{aligned}F\big(\mathbf{x}(s),\dot{\mathbf{x}}(s)\big)&=\lim_{h\to 0^+}\frac{d(\mathbf{x}(s),\mathbf{x}(s)+\dot{\mathbf{x}}(s)h)}{h}=\lim_{h\to 0^+}\frac{(\dot{x}_2h)\tan\theta+\mu[(\dot{x}_1h)^2\cos^2\theta+(\dot{x}_2h)^2]^{1/2}}{h}\\&=\dot{x}_2\tan\theta+\mu\big(\dot{x}_1^2\cos^2\theta+\dot{x}_2^2\big)^{1/2}.\end{aligned}$$

Substituting $F(\mathbf{x}(s),\dot{\mathbf{x}}(s))$ into (5), we obtain (9). □

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