Journal of Pure and Applied Algebra 90 (1993) 55–67
North-Holland

On groups of type $(\mathrm{FP})_\infty$

Peter H. Kropholler

School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, United Kingdom

Communicated by K.W. Gruenberg
Received 19 November 1992

Abstract

Kropholler, P.H., On groups of type $(\mathrm{FP})_\infty$, Journal of Pure and Applied Algebra 90 (1993) 55–67.

Let G be a group. A $\mathbb{Z}G$-module M is said to be of type $(\mathrm{FP})_\infty$ over $\mathbb{Z}G$ if and only if there is a projective resolution $P_* \twoheadrightarrow M$ in which every P_i is finitely generated. We show that if G belongs to a large class of torsion-free groups, which includes torsion-free linear and soluble-by-finite groups, then every $\mathbb{Z}G$-module of type $(\mathrm{FP})_\infty$ has finite projective dimension. We also prove that every soluble or linear group of type $(\mathrm{FP})_\infty$ is virtually of type (FP). The arguments apply to groups which admit hierarchical decompositions. We also make crucial use of a generalized theory of Tate cohomology recently developed by Mislin.

1. Introduction

Throughout this paper, $\mathfrak{X}$ denotes a class of groups and k denotes a non-zero commutative ring. In applications $\mathfrak{X}$ will usually be a class of finite groups and k will usually be a subring of $\mathbb{Q}$ or a finite field. For a functor F between two module categories, we shall say that F is continuous if and only if the natural map $\varinjlim_\lambda F(M_\lambda) \to F(\varinjlim_\lambda M_\lambda)$ is an isomorphism for all direct limit systems (M_λ) of modules.

It is shown in [12] that every soluble group of type $(\mathrm{FP})_\infty$ has finite torsion-free rank. In this paper we combine the new methods of [12] with recent work of Mislin [13] to obtain results of far greater generality. In Section 2 we introduce a new closure operation H for classes of groups. The main theorem

Correspondence to: Dr P.H. Kropholler, School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, United Kingdom. Email: p.h.kropholler@qmw.ac.uk.

concerns certain cohomological functors for groups which belong to $\mathrm{LH}\mathfrak{X}$, the class of locally $\mathrm{H}\mathfrak{X}$-groups.

Theorem A. *Let G be an $\mathrm{LH}\mathfrak{X}$-group and let M be a kG-module. Suppose that the functors $\mathrm{Ext}^l_{kG}(M,-)$ are continuous for infinitely many non-negative l. Then the following are equivalent.*

(i) *M has finite projective dimension over kG.*

(ii) *M has finite projective dimension over kH for all $\mathfrak{X}$-subgroups H of G.*

The usefulness of this result depends on the fact that $\mathrm{LH}\mathfrak{X}$ is often a very much larger class of groups than $\mathfrak{X}$. For example, taking $\mathfrak{X}$ to be the class $\mathfrak{F}$ of all finite groups, we shall show that

- every soluble-by-finite group belongs to $\mathrm{LH}\mathfrak{F}$,
- every linear group belongs to $\mathrm{LH}\mathfrak{F}$,
- $\mathrm{LH}\mathfrak{F}$ is extension closed, subgroup closed and closed under directed unions,
- if G is the fundamental group of a graph of $\mathrm{LH}\mathfrak{F}$-groups then G belongs to $\mathrm{LH}\mathfrak{F}$.

Recall that a kG-module M is said to be of type $(\mathrm{FP})_\infty$ over kG if and only if there is a projective resolution $P_* \twoheadrightarrow M$ in which every P_i is finitely generated as a kG-module. Theorem A applies naturally to this class of modules because the $(\mathrm{FP})_\infty$ property is equivalent to the assertion that all the functors $\mathrm{Ext}^l_{kG}(M,-)$ are continuous. The group G is said to be of type $(\mathrm{FP})_\infty$ over k if and only if the trivial module k is of type $(\mathrm{FP})_\infty$ over kG. Bieri's book [3] provides an excellent introduction to the $(\mathrm{FP})_\infty$ property for groups. We give a sample application which can be proved by applying Theorem A with $k = \mathbb{Z}$ and $\mathfrak{X} = \mathfrak{F}$.

Corollary. *Let G be a torsion-free linear group. Then every $\mathbb{Z}G$-module of type $(\mathrm{FP})_\infty$ has finite projective dimension. In particular, if G is of type $(\mathrm{FP})_\infty$ then it has finite cohomological dimension.*

We leave the reader to devise further applications. One can, for example, obtain results for groups in $\mathrm{LH}\mathfrak{F}$ which are not torsion-free by applying Theorem A with $k = \mathbb{Q}$, although the conclusions are less strong than one might wish. However, one can obtain rather better results by combining Theorem A with the following general result:

Proposition. *Let G be a group of type $(\mathrm{FP})_\infty$ over $\mathbb{Z}$. Suppose that G has finite cohomological dimension over $\mathbb{Q}$. Then there is a finitely generated subring S of $\mathbb{Q}$ such that G has finite cohomological dimension over S, and there is a bound on the orders of the finite subgroups of G.*

The consequence for $(\mathrm{FP})_\infty$-groups is as follows.

Theorem B. *If G belongs to $\mathrm{LH}\mathfrak{F}$ and is of type* $(\mathrm{FP})_\infty$ *over $\mathbb{Z}$ then there is a finitely generated subring S of $\mathbb{Q}$ such that G has finite cohomological dimension over S, and there is a bound on the orders of the finite subgroups of G.*

One corollary of Theorem B may be worth stating explicitly. This solves a question raised by Bieri in 1978, which appears as Problem 6.5 in the Kourovka Notebook [10].

Corollary. *Every soluble group of type* $(\mathrm{FP})_\infty$ *is constructible.*

The first step in this direction was taken by Bieri and Groves [4], who proved that every metabelian group of type $(\mathrm{FP})_\infty$ is virtually of type (FP). In [12], the author proves that soluble groups of type $(\mathrm{FP})_\infty$ have finite torsion-free rank. In this paper, we can use Theorems A and B to show that every soluble group of type $(\mathrm{FP})_\infty$ is virtually of type (FP), and then the Corollary follows from results in the literature. The Proposition, Theorem B and its Corollary are discussed in detail in Section 5.

In view of the great generality of these results it is natural to ask whether there exist groups for which Theorem A fails. Such examples are known and have been studied by Brown and Geoghegan [5,6]. The simplest is the group Γ with presentation

$$\langle x_0, x_1, x_2, \dots \mid x_n^{x_i} = x_{n+1} \text{ for all } i < n\rangle.$$

Brown and Geoghegan show that this group is torsion-free of infinite cohomological dimension and of type $(\mathrm{FP})_\infty$ over $\mathbb{Z}$. One can deduce that Γ does not belong to $\mathrm{LH}\mathfrak{F}$.

2. Closure operations

We shall adopt the usual convention that every class of groups contains the trivial group and contains a group G whenever it contains an isomorphic copy of G. Following Philip Hall, a closure operation A is defined to be an operation which associates a new class of groups $\mathrm{A}\mathfrak{X}$ to any class $\mathfrak{X}$ in such a way that

$$\mathfrak{X} \leq \mathrm{A}\mathfrak{X} = \mathrm{A}^2\mathfrak{X},$$

and for classes $\mathfrak{X} \leq \mathfrak{Y}$,

$$\mathrm{A}\mathfrak{X} \leq \mathrm{A}\mathfrak{Y}.$$

Amongst the most common closure operations are P, Q, S for extension, quotient and subgroup closure respectively. We also use $\mathrm{L}\mathfrak{X}$ to denote the class of locally $\mathfrak{X}$-groups: these are the groups for which every finite subset is contained in an $\mathfrak{X}$-subgroup. Here we introduce a new closure operation H based on group actions on contractible spaces. We shall always consider cellular actions on cell complexes with the property that the setwise stabilizer of every cell coincides with the pointwise stabilizer.

Definition 2.1. Let $\mathfrak{X}$ be a class of groups. Then H$\mathfrak{X}$ denotes the smallest class of groups containing $\mathfrak{X}$ with the property that whenever a group G acts cellularly on a finite-dimensional contractible complex with all isotropy groups in H$\mathfrak{X}$ then G itself belongs to H$\mathfrak{X}$.

Classes of groups closely related to H$\mathfrak{F}$ have been considered by Ikenaga [9] and by Gedrich and Gruenberg [8].

It is immediate from the definition that H is a closure operation, but to understand H further we need the following hierarchical description. We define operations (which are not in general closure operations) H_α for each ordinal α inductively:

- $\text{H}_0\mathfrak{X} = \mathfrak{X}$,

and for ordinals $\alpha > 0$,

- $\text{H}_\alpha\mathfrak{X}$ is the class of groups G which admit a cellular action on a finite-dimensional contractible cell complex in such a way that each isotropy group belongs to $\text{H}_\beta\mathfrak{X}$ for some $\beta < \alpha$ (where β may depend on the isotropy group).

With these definitions it is automatic that a group G belongs to H$\mathfrak{X}$ if and only if there is an ordinal α such that G belongs to $\text{H}_\alpha\mathfrak{X}$, and many elementary facts can be established by transfinite induction on the least such α.

2.2. *For any class* $\mathfrak{X}$,

(i) $\text{SH}\mathfrak{X} \leq \text{HS}\mathfrak{X}$,

(ii) H$\mathfrak{X}$ *is closed under countable directed unions,*

(iii) H$\mathfrak{X}$ *is closed under both free products with amalgamation and HNN-extensions.*

Proof. (i) can be proved by using transfinite induction to show that $\text{SH}_\beta\mathfrak{X} \leq \text{H}_\beta\text{S}\mathfrak{X}$ for each ordinal β.

(ii) and (iii) follow from the more general fact that if G is the fundamental group of a graph of groups in which the vertex and edge groups belong to H$\mathfrak{X}$ then G itself belongs to H$\mathfrak{X}$. The point is that such a G acts on a tree (a 1-dimensional contractible complex) with isotropy groups in H$\mathfrak{X}$. We refer the reader to [7] for further information about fundamental groups of graphs of groups and group actions on trees. □

2.3. H$\mathfrak{F}$ *is extension closed.*

Proof. First we show that for any class $\mathfrak{X}$,

$$(\mathrm{H}\mathfrak{X})\mathfrak{F} \leq \mathrm{HS}(\mathfrak{X}\mathfrak{F}).$$

Let G be a group with a normal $\mathrm{H}\mathfrak{X}$-subgroup H of finite index. There is an ordinal α such that H belongs to $\mathrm{H}_\alpha\mathfrak{X}$ and we work by induction on α. Let X be a finite-dimensional contractible H-complex for which each isotropy group belongs to $\mathrm{H}_\beta\mathfrak{X}$ for some $\beta < \alpha$. There is a natural action of the wreath product $H \wr G/H$ on the cartesian product $Y = X \times \cdots \times X$ of $|G/H|$ copies of X, and using the canonical embedding of G into $H \wr G/H$, we obtain an action of G on Y. For this action of G, all the isotropy groups are finite extensions of subgroups of isotropy groups for the action of H on X, and so each one belongs to $\mathrm{H}_\beta\mathrm{S}(\mathfrak{X}\mathfrak{F})$ for some $\beta < \alpha$. Thus G belongs to $\mathrm{H}_\alpha\mathrm{S}(\mathfrak{X}\mathfrak{F})$, and this completes the induction.

It follows that $(\mathrm{H}\mathfrak{F})\mathfrak{F} = \mathrm{H}\mathfrak{F}$ and now an easy induction, of which this is the initial case, shows that $(\mathrm{H}\mathfrak{F})(\mathrm{H}_\alpha\mathfrak{F}) = \mathrm{H}\mathfrak{F}$ for each α. This completes the proof. □

2.4. $\mathrm{LH}\mathfrak{F}$ *is subgroup closed, extension closed, and closed under arbitrary directed unions.*

Proof. The subgroup closure follows from the subgroup closure of $\mathrm{H}\mathfrak{F}$ (2.2(i)). Closure under directed unions is automatic. Now suppose that $N \rightarrowtail G \twoheadrightarrow Q$ is a group extension in which N and Q belong to $\mathrm{LH}\mathfrak{F}$. It suffices to prove that every finitely generated subgroup of G belongs to $\mathrm{H}\mathfrak{F}$, and so we may assume that G is finitely generated. This implies that Q belongs to $\mathrm{H}\mathfrak{F}$ and N is a countable $\mathrm{LH}\mathfrak{F}$-group. The finitely generated subgroups of N belong to $\mathrm{H}\mathfrak{F}$, and because $\mathrm{H}\mathfrak{F}$ is closed under countable directed unions, by 2.2(ii), it follows that N actually belongs to $\mathrm{H}\mathfrak{F}$. Thus the extension closure of $\mathrm{LH}\mathfrak{F}$ follows from that of $\mathrm{H}\mathfrak{F}$. □

2.5. (i) *Every group of finite cohomological dimension over* $\mathbb{Z}$ *belongs to* $\mathrm{LH}\mathfrak{F}$.
(ii) *Every soluble-by-finite group belongs to* $\mathrm{LH}\mathfrak{F}$.
(iii) *Every linear group belongs to* $\mathrm{LH}\mathfrak{F}$.

Proof. (i) Let $\mathcal{I}$ denote the class of the trivial group. Then $\mathrm{H}_1\mathcal{I}$ is the class of all groups which admit free actions on a finite-dimensional contractible cell complex, or equivalently, the class of all groups of finite cohomological dimension over $\mathbb{Z}$. The assertion follows because $\mathrm{H}_1\mathcal{I} \leq \mathrm{LH}\mathfrak{F}$.

(ii) By (i), $\mathrm{LH}\mathfrak{F}$ contains every free abelian group of finite rank. Since $\mathrm{LH}\mathfrak{F}$ also contains all finite groups and is extension and locally closed it follows that every soluble-by-finite group belongs to $\mathrm{LH}\mathfrak{F}$.

(iii) We first show that if A is a finitely generated subring of $\mathbb{C}$ then $\mathrm{SL}_n(A)$ belongs to $\mathrm{LH}\mathfrak{F}$. This follows from the hierarchical description of finitely generated linear groups established by Alperin and Shalen [1]. Alperin and Shalen distinguish the special class of linear groups of *integral characteristic*. These are the subgroups G of $\mathrm{GL}_n(\mathbb{C})$ with the property that the coefficients of the characteristic polynomial of every element of G are algebraic integers. According to Proposition 3.2 of [1], every subgroup G of $\mathrm{SL}_n(A)$ has a permutation-free $(n-1)$-dimensional hierarchy supported on a family of subgroups of G having integral characteristic. Translating into the language of this paper it is immediate that $\mathrm{SL}_n(A)$ and all its subgroups belong to $\mathrm{H}\mathfrak{X}$, where $\mathfrak{X}$ denotes the class of subgroups of $\mathrm{SL}_n(A)$ having integral characteristic.

Now it suffices to show that $\mathfrak{X}$ is contained in $\mathrm{LH}\mathfrak{F}$. Let G be a subgroup of $\mathrm{SL}_n(A)$ of integral characteristic, and let F be the field of fractions of A. As in the proof of Theorem 3.3 of [1], we can consider the natural linear action of G on $V = F^n$. Let G_0 be a normal subgroup of finite index in G whose action on V has a composition series of maximal length. Since $\mathrm{LH}\mathfrak{F}$ is closed under finite extension, it is enough to prove that G_0 belongs to $\mathrm{LH}\mathfrak{F}$. Let $0 = V_0 < V_1 < \cdots < V_m = V$ be a composition series for the action of G_0 on V, and let G_i denote the image of G_0 in its representation on V_i/V_{i-1}. Then there is a homomorphism $G_0 \to G_1 \times \cdots \times G_m$ with unipotent kernel. All unipotent subgroups belong to $\mathrm{LH}\mathfrak{F}$, because they are soluble, and since $\mathrm{LH}\mathfrak{F}$ is extension and subgroup closed, it now suffices to prove that each G_i belongs to $\mathrm{LH}\mathfrak{F}$. This follows from Proposition 2.3 of [1], which shows that each G_i has finite virtual cohomological dimension.

Thus every subgroup of $\mathrm{SL}_n(A)$ belongs to $\mathrm{LH}\mathfrak{F}$. The closure properties of $\mathrm{LH}\mathfrak{F}$ now guarantee that every characteristic-zero linear group belongs to $\mathrm{LH}\mathfrak{F}$. □

3. Cohomological functors

Let R and S be rings. Following Gedrich and Gruenberg [8], we define a $(-\infty,\infty)$-cohomological functor from R-modules to S-modules to be a sequence of additive functors $(T^i \mid i \in \mathbb{Z})$ from R-modules to S-modules together with natural connecting homomorphisms so that for each short exact sequence $0 \to M'' \to M \to M' \to 0$ of R-modules, one obtains a corresponding long exact sequence of S-modules:

$$\cdots \to T^{n-1}(M') \to T^n(M'') \to T^n(M) \to T^n(M') \to \cdots.$$

The most common examples arising in group theory are

(i) the cohomology functors

$$(H^i(G,-) \mid i \in \mathbb{Z}),$$

where $H^i(G,-)$ is defined to be zero for $i < 0$,

(ii) the homology functors

$$(H_{-i}(G,-) \mid i \in \mathbb{Z}),$$

where $H_j(G,-)$ is defined to be zero for $j < 0$, and

(iii) in case G is a finite group, the Tate cohomology functors

$$(\widehat{H}^i(G,-) \mid i \in \mathbb{Z}).$$

These are all examples of $(-\infty,\infty)$-cohomological functors from $\mathbb{Z}G$-modules to $\mathbb{Z}$-modules.

Notation. We fix the following notation for the remainder of this section: k denotes a non-zero commutative ring, G is a group, and T^* is a $(-\infty,\infty)$-cohomological functor from kG-modules to k-modules such that T^l is continuous for all l.

We shall need the following variation on Lemma 1 of [12]. This is a simple general fact which does not depend on the assumption that the T^l are continuous.

3.1. *Let*

$$0 \to M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 \to L \to 0$$

be an exact sequence of kG-modules. If i is an integer such that $T^i(L)$ is non-zero then there exists a j with $0 \le j \le r$ such that $T^{i+j}(M_j)$ is non-zero.

Proof. We give a direct proof by dimension shifting: Let K_j denote the image of the map $M_{j+1} \to M_j$ for $0 \le j \le r-1$. Thus we can express the exact sequence of modules as a series of r short exact sequences:

$$\begin{aligned} &0 \to K_0 \to M_0 \to L \to 0, \\ &0 \to K_1 \to M_1 \to K_0 \to 0, \\ &\quad\vdots \\ &0 \to K_{r-1} \to M_{r-1} \to K_{r-2} \to 0. \end{aligned}$$

Now suppose that $T^{i+j}(M_j) = 0$ for $j \le r-1$. The long exact sequence of cohomology applied to the first short exact sequence shows that $T^i(L)$ imbeds in $T^{i+1}(K_0)$. Using the same argument with the second sequence shows that $T^{i+1}(K_0)$ imbeds in $T^{i+2}(K_1)$. Continuing in this way we obtain a succession of embeddings

$$T^i(L) \subseteq T^{i+1}(K_0) \subseteq \cdots \subseteq T^{i+r}(K_{r-1}).$$

Since $T^i(L)$ is non-zero, it follows that $T^{i+r}(K_{r-1})$ is non-zero. This proves the assertion, because $K_{r-1} = M_r$. $\square$

We now come to the main result of this section. Here, the continuity of the T^l plays a key role. As a matter of fact this is the only point at which continuity enters directly into the argument.

3.2. *Let G be an* LH$\mathfrak{X}$*-group. If i is an integer and V is a kG-module such that $T^i(V)$ is non-zero, then there exist $j \geq i$ and $H \leq G$ such that H is an $\mathfrak{X}$-group and $T^j(\mathrm{Ind}_H^G V)$ is non-zero.*

Proof. Consider the collection $\mathcal{O}$ of ordinals β for which there exist $j \geq i$ and $H \leq G$ such that H is an $\mathrm{H}_\beta\mathfrak{X}$-subgroup and $T^j(\mathrm{Ind}_H^G V)$ is non-zero. It suffices to prove that 0 belongs to $\mathcal{O}$.

Step 1: *$\mathcal{O}$ is non-empty.* Let $(G_\lambda \mid \lambda \in \Lambda)$ be the family of all finitely generated subgroups of G. Then, just as G can be viewed as the direct limit of the G_λ, so V can be viewed as the direct limit of the induced modules $\mathrm{Ind}_{G_\lambda}^G V$,

$$V = \varinjlim_{\lambda} \mathrm{Ind}_{G_\lambda}^G V.$$

Since $T^i(-)$ is a continuous functor and $T^i(V)$ is non-zero, it follows that for some λ, the induced map $T^i(\mathrm{Ind}_{G_\lambda}^G V) \to T^i(V)$ is non-zero. Fix this λ. Since G_λ is finitely generated and G is locally H$\mathfrak{X}$, there is an H$\mathfrak{X}$-subgroup H of G containing G_λ. The map $T^i(\mathrm{Ind}_{G_\lambda}^G V) \to T^i(V)$ factors through $T^i(\mathrm{Ind}_H^G V)$ and hence $T^i(\mathrm{Ind}_H^G V)$ is non-zero. Let β be an ordinal such that H belongs to $\mathrm{H}_\beta\mathfrak{X}$. Then β belongs to $\mathcal{O}$.

Step 2: *If β is a non-zero ordinal in $\mathcal{O}$ then there is an ordinal $\gamma < \beta$ which also belongs to $\mathcal{O}$.* Suppose that β is a non-zero ordinal which belongs to $\mathcal{O}$. Then there exists $j \geq i$ and $H \in \mathrm{H}_\beta\mathfrak{X}$ such that $T^j(\mathrm{Ind}_H^G V)$ is non-zero. By definition, there is a cellular action of H on a finite-dimensional contractible space with each isotropy group belonging to H_γ for some $\gamma < \beta$. The cellular chain complex of this space is an exact sequence of H-modules:

$$0 \to C_r \to C_{r-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0,$$

where each C_l is the permutation module coming from the action of H as a group of permutations of the l-dimensional cells. Tensoring with V and then applying the induction functor $\mathrm{Ind}_H^G -$ yields the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ind}_H^G(V \otimes C_r) \to \cdots \to \mathrm{Ind}_H^G(V \otimes C_1) \\ &\to \mathrm{Ind}_H^G(V \otimes C_0) \to \mathrm{Ind}_H^G V \to 0, \end{aligned}$$

of G-modules. Now we may apply 3.1 with $L = \mathrm{Ind}_H^G V$ and $M_l = \mathrm{Ind}_H^G V \otimes C_l$. The conclusion is that for some $l \geq 0$,

$$T^{j+l}(\mathrm{Ind}_H^G(V \otimes C_l)) \quad \text{is non-zero.}$$

Let Δ be a set of H-orbit representatives for the l-dimensional cells. Then $C_l = \bigoplus_{\sigma\in\Delta} \mathrm{Ind}_{H_\sigma}^H \mathbb{Z}$, and hence

$$\mathrm{Ind}_H^G(V \otimes C_l) = \bigoplus_{\sigma\in\Delta} \mathrm{Ind}_H^G(V \otimes \mathrm{Ind}_{H_\sigma}^H \mathbb{Z}) = \bigoplus_{\sigma\in\Delta} \mathrm{Ind}_{H_\sigma}^G V.$$

Since the functor $T^{j+l}(-)$ is continuous, it commutes with direct sums and we conclude that there is a $\sigma \in \Delta$ such that

$$T^{j+l}(\mathrm{Ind}_{H_\sigma}^G V) \quad \text{is non-zero.}$$

Since H_σ belongs to $\mathrm{H}_\gamma\mathfrak{X}$ for some $\gamma < \beta$ we have established Step 2.

By transfinite induction, 0 belongs to $\mathcal{O}$. This means that there exist $j \geq i$ and $H \in \mathfrak{X}$ such that $T^j(\mathrm{Ind}_H^G V)$ is non-zero, as required. □

The principal application we need is as follows: Here, the continuity of the T^l has to be assumed solely because the argument uses 3.2.

3.3. *Let G be an* $\mathrm{LH}\mathfrak{X}$*-group. Let M be a kG-module which has finite projective dimension over kH for all $\mathfrak{X}$-subgroups H of G. If the T^l vanish on projectives for all $l \geq 0$ then $T^0(M) = 0$.*

Proof. If $T^0(M)$ is non-zero then 3.2 shows that there exist $i \geq 0$ and $H \leq G$ such that H belongs to $\mathfrak{X}$ and $T^i(\mathrm{Ind}_H^G M)$ is non-zero. Now M has finite projective dimension as a kH-module, and hence $\mathrm{Ind}_H^G M$ has finite projective dimension as a kG-module. Let

$$0 \to P_r \to \cdots \to P_1 \to P_0 \to \mathrm{Ind}_H^G M \to 0$$

be a projective resolution. Since $T^i(\mathrm{Ind}_H^G M)$ is non-zero, an application of 3.1 shows that for some $j \geq 0$, $T^{i+j}(P_j)$ is non-zero. This contradiction completes the proof. □

4. Mislin's generalization of Tate cohomology

In [2] Benson and Carlson formulate Tate cohomology of finite groups in a way that applies to any group, finite or infinite. Mislin [13] has now laid down an axiomatic approach to this generalized Tate cohomology theory. For groups of finite virtual cohomological dimension this coincides with the usual Tate–Farrell cohomology.

Throughout this section, let R and S be rings and let T^* be a $(-\infty,\infty)$-cohomological functor from R-modules to S-modules. Mislin defines a new cohomological functor $\widehat{T}^*$, in terms of left satellite functors, by the formula

$$\widehat{T}^n(M) = \varinjlim_{i\geq 0} S^{-i}T^{n+i}(M).$$

This new functor has the property that it vanishes on all projective R-modules. Moreover, there is a natural map $T^* \to \widehat{T}^*$, and T^* satisfies the universal property that given any other cohomological functor U^* which vanishes on projectives then every natural map $T^* \to U^*$ factors uniquely through the map $T^* \to \widehat{T}^*$. We refer the reader to Mislin's paper [13] for further details.

4.1. (i) *If T^i is zero for infinitely many non-negative i then $\widehat{T}^i$ is zero for all i.*

(ii) *If T^i is continuous for infinitely many non-negative i then $\widehat{T}^i$ is continuous for all i.*

(iii) *If T^i commutes with direct sums for infinitely many non-negative i then $\widehat{T}^i$ commutes with direct sums for all i.*

Proof. (i) This is clear.

(ii) We first show that if T is any half-exact continuous additive functor then its left satellite $S^{-1}T$ is continuous. Following Mislin [13], we define $S^{-1}T(M)$ to be the kernel of the induced map $T(\Omega M) \to T(FM)$, where FM denotes the free module on the underlying set of M and ΩM denotes the kernel of the natural surjection $FM \twoheadrightarrow M$. It is easy to see that the functor $F : M \mapsto FM$ is continuous, and therefore so is Ω. Let $(M_\lambda \mid \lambda \in \Lambda)$ be a direct limit system of modules. The continuity of T, F and Ω ensures that the natural maps $\varinjlim_\lambda T(FM_\lambda) \to T(F(\varinjlim_\lambda M_\lambda))$ and $\varinjlim_\lambda T(\Omega M_\lambda) \to T(\Omega(\varinjlim_\lambda M_\lambda))$ are isomorphisms, and consequently the natural map

$$\varinjlim_{\lambda} S^{-1}T(M_\lambda) \to S^{-1}T(\varinjlim_{\lambda} M_\lambda)$$

is an isomorphism. Thus $S^{-1}T$ is continuous. It also half exact and additive. The higher left satellites $S^{-i}T$ are defined inductively by $S^{-i}T = S^{-1}S^{-i+1}T$ and so the continuity of all these follows by induction.

Combining this argument with the fact that the T^l are continuous for infinitely many non-negative i, we see that for any fixed n, the functors $S^{-i}T^{n+i}$ are continuous for infinitely many non-negative i. Continuity now follows: if I is the set of i for which $S^{-i}T^{n+i}$ is continuous then we have

$$\begin{aligned}\varinjlim_{\lambda} \widehat{T}^n(M_\lambda) &= \varinjlim_{\lambda} \varinjlim_{i} S^{-i}T^{n+i}(M_\lambda) = \varinjlim_{i} \varinjlim_{\lambda} S^{-i}T^{n+i}(M_\lambda) \\ &= \varinjlim_{i\in I} \varinjlim_{\lambda} S^{-i}T^{n+i}(M_\lambda) = \varinjlim_{i\in I} S^{-i}T^{n+i}(\varinjlim_{\lambda} M_\lambda) \\ &= \widehat{T}^n(\varinjlim_{\lambda} M_\lambda),\end{aligned}$$

as required.

(iii) can be proved in a similar way to (ii). □

4.2. *Let M be an R-module. Then the following are equivalent.*

(i) *M has finite projective dimension over R.*
(ii) $\mathrm{Ext}^i_R(M,-)$ *is zero for all sufficiently large i.*
(iii) $\widehat{\mathrm{Ext}}^i_R(M,-)$ *is zero for all i.*
(iv) $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are elementary, and (ii) $\Rightarrow$ (iii) is an application of 4.1(i). We shall prove (iv) $\Rightarrow$ (i).

For R-modules L and N, $[L,N]$ denotes the quotient of $\mathrm{Hom}_R(L,N)$ by the additive subgroup of homomorphisms which factor through a projective module. The functor Ω induces maps $[L,N] \to [\Omega L, \Omega N]$. It can be shown that $\widehat{\mathrm{Ext}}^n_R(L,N)$ is naturally isomorphic to the direct limit $\varinjlim_i [\Omega^{n+i}L, \Omega^i N]$. Mislin proves this in the case when R is a group ring and L is the trivial module, in Section 4 of [13], but the argument applies essentially without change to the general situation. Thus the vanishing of $\widehat{\mathrm{Ext}}^0_R(M,M)$ is equivalent to the assertion $\varinjlim_i [\Omega^i M, \Omega^i M] = 0$. This implies that there exists $i \geq 0$ such that the identity endomorphism of M becomes zero under the natural map $\mathrm{Hom}_R(M,M) \twoheadrightarrow [M,M] \to [\Omega^i M, \Omega^i M]$. For such an i, the identity endomorphism of $\Omega^i M$ factors through a projective module and hence $\Omega^i M$ is projective and M has projective dimension at most i. □

Proof of Theorem A. Let G and M be as in the statement of Theorem A. If M has finite projective dimension over kG then it automatically has finite projective dimension over kH for all subgroups H of G. Conversely, assume that M has finite projective dimension over kH for all $\mathfrak{X}$-subgroups H of G. The functors $\widehat{\mathrm{Ext}}^l_{kG}(M,-)$ are continuous for all l, by 4.1(ii). Since the Tate–Mislin cohomology always vanishes on projective modules, 3.3 can be applied, and the conclusion is that $\widehat{\mathrm{Ext}}^0_{kG}(M,M) = 0$. The theorem now follows from 4.2, (iv) $\Rightarrow$ (i). □

It is perhaps worth remarking that one has the following variation on Theorem A:

Theorem A′. *Let G be an* H$\mathfrak{X}$*-group and let M be a kG-module. Suppose that the functors* $\mathrm{Ext}^l_{kG}(M,-)$ *commute with direct sums for infinitely many non-negative l. Then the following are equivalent.*
(i) *M has finite projective dimension over kG.*
(ii) *M has finite projective dimension over kH for all $\mathfrak{X}$-subgroups H of G.*

Proof (*Outline*). The reason is that in the crucial Step 2 of the proof of 3.2, the argument uses only the fact that the functors commute with direct sums, and not the stronger assumption of continuity. Continuity is used in Step 1 of that proof, but if we assume that G belongs to H$\mathfrak{X}$ rather than LH$\mathfrak{X}$, then this step becomes unnecessary. One must, of course, use 4.1 (iii) in proving Theorem A′, instead of 4.1 (ii). □

5. Some general properties of $(\mathrm{FP})_\infty$-groups

We begin by proving the proposition mentioned in the Introduction:

Proposition. *Let G be a group of type $(\mathrm{FP})_\infty$ over $\mathbb{Z}$. Suppose that G has finite cohomological dimension over $\mathbb{Q}$. Then there is a finitely generated subring S of $\mathbb{Q}$ such that G has finite cohomological dimension over S, and there is a bound on the orders of the finite subgroups of G.*

Proof. Since G is of type $(\mathrm{FP})_\infty$, the functors $H^i(G,-)$ are continuous for all i, and hence, by 4.1 (ii), it follows that $\widehat{H}^0(G,-)$ is continuous. This shows that $\widehat{H}^0(G,\mathbb{Q}) = \widehat{H}^0(G,\mathbb{Z}) \otimes \mathbb{Q}$. But $\mathrm{cd}_{\mathbb{Q}}(G) < \infty$ implies that $\widehat{H}^0(G,\mathbb{Q}) = 0$, by 4.2. Hence $\widehat{H}^0(G,\mathbb{Z})$ is a torsion group.

Now $\widehat{H}^0(G,\mathbb{Z})$ is also a ring. Being torsion, the subring generated by 1 is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n. Let S be the subring of $\mathbb{Q}$ generated by $\frac{1}{n}$. Then $\widehat{H}^0(G,S) = \widehat{H}^0(G,\mathbb{Z}) \otimes S = 0$, and it follows from 4.2 that G has finite cohomological dimension over S. Finally, if H is a finite subgroup of G then the restriction map $\widehat{H}^0(G,\mathbb{Z}) \to \widehat{H}^0(H,\mathbb{Z})$ is a ring homomorphism and the Tate–Mislin cohomology of H coincides with the classical Tate cohomology. Thus there is a ring homomorphism $\mathbb{Z}/n\mathbb{Z} \to \widehat{H}^0(H,\mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}$ and hence $|H|$ divides n. □

Proof of Theorem B. If G belongs to $\mathrm{LH}\mathfrak{F}$ and has type $(\mathrm{FP})_\infty$ then Theorem A, applied with $k = \mathbb{Q}$ and $M = \mathbb{Q}$, shows that $\mathrm{cd}_{\mathbb{Q}}(G) < \infty$. Now the Proposition can be applied, and the result follows. □

Proof of the Corollary. Let G be a soluble group of type $(\mathrm{FP})_\infty$ over $\mathbb{Z}$. By Theorem B, G has finite cohomological dimension over $\mathbb{Q}$ and there is a bound on the orders of the finite subgroups of G. We deduce that G is a soluble group of finite rank, and using Theorem 10.33 of [14] that G has a torsion-free subgroup H of finite index. Now H is again of type $(\mathrm{FP})_\infty$. Applying Theorem A to H with $k = \mathbb{Z}$ and $M = \mathbb{Z}$ shows that H has finite cohomological dimension over $\mathbb{Z}$. Therefore, H is of type (FP) and G is virtually of type (FP). The main theorem of [11] now shows that G is constructible in the sense that it can be built up from the trivial group by a finite sequence of finite extensions and ascending HNN-extensions. □

Acknowledgement

I am deeply indebted to Karl Gruenberg for many stimulating discussions which have had a strong influence on this paper.

References

[1] R.C. Alperin and P.B. Shalen, Linear groups of finite cohomological dimension, Invent. Math. 66 (1982) 89–98.
[2] D.J. Benson and J. Carlson, Products in negative cohomology, J. Pure Appl. Algebra 82 (2) (1992) 107–130.
[3] R. Bieri, Homological Dimension of Discrete Groups (Queen Mary College, London, UK, 1981).
[4] R. Bieri and J.R.J. Groves, Metabelian groups of type $(FP)_\infty$ are virtually of type (FP), Proc. London Math. Soc. (3) 45 (1982) 365–384.
[5] K.S. Brown, Finiteness properties of groups, J. Pure Appl. Algebra 44 (1987) 45–75.
[6] K.S. Brown and R. Geoghegan, An infinite dimensional FP_∞-group, Invent. Math. 77 (1984) 367–381.
[7] W. Dicks and M.J. Dunwoody, Groups Acting on Graphs (Cambridge University Press, Cambridge, 1989).
[8] T.V. Gedrich and K.W. Gruenberg, Complete cohomological functors on groups, Topology Appl. 25 (1987) 203–223.
[9] B.M. Ikenaga, Homological dimension and Farrell cohomology, J. Algebra 87 (1984) 422–457.
[10] Kourovka Notebook, B.M. Kopytov et al., eds., Unsolved Problems of the Theory of Groups (Novosibirsk, 1990).
[11] P.H. Kropholler, Cohomological dimension of soluble groups, J. Pure Appl. Algebra 43 (1986) 281–287.
[12] P.H. Kropholler, Soluble groups of type $(FP)_\infty$ have finite torsion-free rank, Bull. London Math. Soc., to appear.
[13] G. Mislin, Tate cohomology for arbitrary groups via satellites, to appear.
[14] D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Part 2, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 63 (Springer, Berlin), to appear.