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Enumeration formulas for generalized q-Euler numbers

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ABSTRACT

We find an enumeration formula for a (t,q)-Euler number which is a generalization of the q-Euler number introduced by Han, Randrianarivony, and Zeng. We also give a combinatorial expression for the (t,q)-Euler number and find another formula when $t = \pm q^r$ for any integer r. Special cases of our latter formula include the formula of the q-Euler number recently found by Josuat-Vergès and Touchard–Riordan's formula.

1. Introduction

The *Euler number* E_n is defined by

$$\sum_{n\geqslant 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Thus E_{2n} and E_{2n+1} are also called the *secant number* and the *tangent number* respectively. In 1879, André [1] showed that E_n is equal to the number of *alternating permutations* of $\{1,2,\dots,n\}$, i.e., the permutations $\pi = \pi_1 \dots \pi_n$ such that $\pi_1 < \pi_2 > \pi_3 < \cdots$.

There are several q-Euler numbers studied in the literature, for instance, see [5,7–9,15]. In this paper we consider the following q-Euler number $E_n(q)$ introduced by Han et al. [7]:

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$$\sum_{n\geqslant 0} E_{2n}(q)x^n = \cfrac{1}{1-\cfrac{[1]_q^2 x}{1-\cfrac{[2]_q^2 x}{\cdots}}}, \qquad \sum_{n\geqslant 0} E_{2n+1}(q)x^n = \cfrac{1}{1-\cfrac{[1]_q[2]_q x}{1-\cfrac{[2]_q[3]_q x}{\cdots}}}, \tag{1}$$

where $[n]_q = (1-q^n)/(1-q)$. We will use the standard notations:

$$(a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

This q-Euler number also has a nice combinatorial expression found by Chebikin [2]:

$$E_n(q) = \sum_{\pi\in\mathfrak{A}_n} q^{\text{31-2}(\pi)},$$

where $\mathfrak{A}_n$ denotes the set of alternating permutations of $\{1,2,\dots,n\}$ and 31-2(π) denotes the number of 31-2 patterns in π.

Recently, Josuat-Vergès [9] found a formula for $E_n(q)$. In Section 6 we show that, by elementary manipulations, his formula can be rewritten as follows:

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{k(k+1)} \sum_{i=-k}^{k} (-q)^{-i^2}, \tag{2}$$

$$E_{2n+1}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{k(k+2)} A_k\left(q^{-1}\right), \tag{3}$$

where $A_0(q) = 1$ and for $k \geqslant 1$,

$$A_k(q) = \frac{1}{1-q}\sum_{i=-k}^{k} (-q)^{i^2} + \frac{q^{2k+1}}{1-q}\sum_{i=-(k-1)}^{k-1} (-q)^{i^2}.$$

Shin and Zeng [15, Theorem 12] found a parity-independent formula for $E_n(q)$.

We note that (2) is similar to the following formula of Touchard [17] and Riordan [14]:

$$d_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) (-1)^k q^{\frac{k(k+1)}{2}}, \tag{4}$$

where d_n is defined by

$$\sum_{n\geqslant 0} d_n x^n = \cfrac{1}{1-\cfrac{[1]_q x}{1-\cfrac{[2]_q x}{\cdots}}}. \tag{5}$$

In this paper we introduce the (t,q)-*Euler numbers* $E_n(t,q)$ defined by

$$\sum_{n\geqslant 0} E_n(t,q)x^n = \cfrac{1}{1-\cfrac{[1]_q[1]_{t,q}x}{1-\cfrac{[2]_q[2]_{t,q}x}{\cdots}}}, \tag{6}$$

where $[n]_{t,q} = (1-tq^n)/(1-q)$. Note that $(1-q)^{2n}E_n(0,q) = (1-\sqrt{q})^{2n}E_n(-1,\sqrt{q}) = (1-q)^n d_n$, $E_n(1,q) = E_{2n}(q)$, and $E_n(q,q) = E_{2n+1}(q)$. In fact $E_n(t,q)$ is a special case of the $2n$th moment $\mu_{2n}(a,b;q)$ of Al-Salam–Chihara polynomials $Q_n(x)$ defined by the recurrence

$$2xQ_n(x) = Q_{n+1} + (a+b)q^n Q_n(x) + \left(1-q^n\right)\left(1-abq^{n-1}\right)Q_{n-1}(x),$$

and the initial conditions $Q_{-1}(x)=0$ and $Q_0(x)=1$. If $a=\sqrt{-qt}$ and $b=-\sqrt{-qt}$, then the $2n$th moment $\mu_{2n}(a,b;q)$ satisfies $(1-q)^{2n}E_n(t,q) = 2^{2n}\mu_{2n}(\sqrt{-qt},-\sqrt{-qt};q)$. Josuat-Vergès [10, Theorem 6.1.1 or Eq. (46)] found a formula for $\mu_n(a,b;q)$, which implies that

$$\begin{aligned} E_n(t,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \\ \times \sum_{i,j\geqslant 0} (-1)^{k+i} q^{\binom{j+1}{2}} (qt)^{k-j} \begin{bmatrix} 2k-j \\ j \end{bmatrix}_q \begin{bmatrix} 2k-2j \\ i \end{bmatrix}_q. \end{aligned} \tag{7}$$

In the same paper, Josuat-Vergès showed that (2) and (3) can be obtained from (7) using certain summation formulas.

The original motivation of this paper is to find a formula from which one can easily obtain (2), (3), and (4). The main results in this paper are Theorems 1.1 and 1.3 below.

Theorem 1.1. *We have*

$$E_n(t,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) t^k q^{k(k+1)} T_k\left(t^{-1},q^{-1}\right),$$

where $\{T_k(t,q)\}_{k\geqslant 0}$ is a family of polynomials in t and q determined uniquely by the recurrence relation: $T_0(t,q)=1$ and for $k\geqslant 1$,

$$T_k(t,q) = T_{k-1}(t,q) + (1+t)(-q)^{k^2} + \left(1-t^2\right)\sum_{i=1}^{k-1}(-q)^{k^2-i^2}T_{i-1}(t,q). \tag{8}$$

From the recurrence of $T_k(t,q)$, we immediately get $T_k(-1,q)=1$ and $T_k(1,q)=\sum_{i=-k}^{k}(-q)^{i^2}$, which imply (4) and (2) respectively. Using certain weighted lattice paths satisfying the same recurrence relation we obtain the following formula for $T_k(t,q)$.

Corollary 1.2. *We have*

$$T_k(t,q) = \sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j+i}t^{2i}q^{j^2+i^2+i}\begin{bmatrix} k-j \\ i \end{bmatrix}_{q^2}\left(\begin{bmatrix} k-i \\ j-i \end{bmatrix}_{q^2} + t\begin{bmatrix} k-i-1 \\ j-i-1 \end{bmatrix}_{q^2}\right).$$

As a consequence of the proof of Corollary 1.2 we can express $T_k(t,q)$ using what we call self-conjugate overpartitions, see Theorem 4.1. This combinatorial expression allows us to find a functional equation for $T_k(t,q)$ which gives a recurrence relation for $T_k(\pm q^r,q)$, see Corollary 4.2. Solving the recurrence relation, we get the following formulas for $T_n(\pm q^r,q)$ for any integer r.

Theorem 1.3. *For $b \geqslant 0$ and $k \geqslant 1$, we have*

$$T_k\big(q^b,q\big) = \sum_{i=0}^{k-1} \frac{q^{i(2k+1)}}{(q;q)_b} \begin{bmatrix} b \\ i \end{bmatrix}_{q^2} \sum_{j=-(k-i)}^{k-i} (-q)^{j^2} + \sum_{i=0}^{b-1} \frac{(q;q)_i}{(q;q)_b} q^{k(2k+2i+1)} \begin{bmatrix} b-i-1 \\ k-1 \end{bmatrix}_{q^2}, \tag{9}$$

$$T_k\big(-q^b,q\big) = \sum_{i=0}^{k-1} \frac{q^{i(2k+1)}}{(-q;q)_b} \begin{bmatrix} b \\ i \end{bmatrix}_{q^2} + \sum_{i=0}^{b-1} \frac{(-q;q)_i}{(-q;q)_b} q^{k(2k+2i+1)} \begin{bmatrix} b-i-1 \\ k-1 \end{bmatrix}_{q^2}, \tag{10}$$

and for $b \geqslant 1$ and $k \geqslant 0$, we have

$$T_k\big(q^{-b},q\big) = \sum_{i=0}^{b-1} \big(q^{1-b};q\big)_i (-q)^{k(k-2b+2)+2i} \begin{bmatrix} k+i-1 \\ i \end{bmatrix}_{q^2}, \tag{11}$$

$$T_k\big(-q^{-b},q\big) = \sum_{i=0}^{k-1} \big(-q^{1-b};q\big)_b (-q)^{i(2k-2b-i+2)} \begin{bmatrix} b+i-1 \\ i \end{bmatrix}_{q^2} + (-q)^{k^2+2k-2kb} \sum_{i=0}^{b-1} \big(-q^{1-b};q\big)_i q^{2i} \begin{bmatrix} k+i-1 \\ i \end{bmatrix}_{q^2}. \tag{12}$$

Note that (2) and (3) follows immediately from (9) when $b=0$ and $b=1$, and (4) from (10) when $b=0$. When $t=-q$ and $t=-1/q$, we get simple formulas, see Propositions 5.9 and 5.16.

We note that it is possible to obtain another formula for $T_k(q^b,q)$ for a positive integer b from a result in [13, Section 6], see Section 7.

The rest of this paper is organized as follows. In Section 2 we interpret $E_n(t,q)$ using δ_k-configurations introduced in [11]. In Section 3 we prove Theorem 1.1 and Corollary 1.2. In Section 4 we show that $T_k(t,q)$ can be expressed as the sum of certain weights of symmetric overpartitions. Using this expression we also find a functional equation for $T_k(t,q)$. In Section 5 using the functional equation obtained in the previous section we prove Theorem 1.3 which is divided into Corollaries 5.7, 5.8, 5.14, and 5.15. In Section 6 we show that the original formula of $E_n(q)$ in [9] is equivalent to (2) and (3). In Section 7 we propose some open problems.

2. Interpretation of $E_n(t,q)$ using δ_k-configurations

In this section we interpret $E_n(t,q)$ using δ_k-configurations introduced in [11]. The idea is basically the same as in [11].

2.1. *S-fractions and weighted lattice paths*

An *S-fraction* is a continued fraction of the following form:

$$\cfrac{1}{1-\cfrac{c_1x}{1-\cfrac{c_2x}{\cdots}}}.$$

Thus all continued fractions appeared in the introduction are S-fractions. There is a simple combinatorial interpretation for S-fractions using weighted Dyck paths. In this subsection we will find formulas equivalent to Theorem 1.1 using this combinatorial interpretation.

Definition 1. A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ in $\mathbb{N}^2$ consisting of up steps $(1,1)$ and down steps $(1,-1)$. We denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$. A *marked Dyck path* is a Dyck path in which each up step and down step may be marked. We denote by $\overline{\mathcal{D}}_n$ the set of marked Dyck paths of length $2n$. We also denote by $\overline{\mathcal{D}}_n^*$ the subset of $\overline{\mathcal{D}}_n$ consisting of the marked Dyck paths without marked peaks. Here, a *marked peak* means a marked up step immediately followed by a marked down step. Given two sequences $\mathcal{A}=(a_1,a_2,\dots)$, $\mathcal{B}=(b_1,b_2,\dots)$ and $p\in\overline{\mathcal{D}}_n$, we define the weight $\mathrm{wt}(p;\mathcal{A},\mathcal{B})$ to be the product of a_h (resp. b_h) for each non-marked up step (resp. non-marked down step) between height h and $h-1$.

Observe that every marked step can be considered as a step of weight 1. We will consider a Dyck path as a marked Dyck path without marked steps. In this identification we have $\mathcal{D}_n\subset\overline{\mathcal{D}}_n$.

The following combinatorial interpretation of S-fractions is well-known, see [4].

Lemma 2.1. *For two sequences $\mathcal{A}=(a_1,a_2,\dots)$, $\mathcal{B}=(b_1,b_2,\dots)$, we have*

$$\cfrac{1}{1-\cfrac{a_1b_1x}{1-\cfrac{a_2b_2x}{\cdots}}}=\sum_{n\geqslant 0}x^n\sum_{p\in\mathcal{D}_n}\mathrm{wt}(p;\mathcal{A},\mathcal{B}).$$

The reader may have noticed that every formula in the introduction has the factor $\binom{2n}{n-k}-\binom{2n}{n-k-1}$ in its summand. This can be explained by the following lemma.

Lemma 2.2. *(See [11, Lemma 1.2].) For two sequences $\mathcal{A}$ and $\mathcal{B}$ we have*

$$\sum_{p\in\mathcal{D}_n}\mathrm{wt}(p;\mathcal{A},\mathcal{B})=\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\sum_{p\in\overline{\mathcal{D}}_k^*}\mathrm{wt}(p;\mathcal{A}-\mathbf{1},\mathcal{B}-\mathbf{1}),$$

where, if $\mathcal{A}=(a_1,a_2,\dots)$, the sequence $\mathcal{A}-\mathbf{1}$ means $(a_1-1,a_2-1,\dots)$.

From now on we fix the following sequences:

$$\mathcal{U}=\left(-q,-q^2,\dots\right),\qquad \mathcal{V}_t=\left(-tq,-tq^2,\dots\right).$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}E_n(t,q)&=\sum_{p\in\mathcal{D}_n}\mathrm{wt}\big(p;\big([1]_q,[2]_q,\dots\big),\big([1]_{t,q},[2]_{t,q},\dots\big)\big)\\&=\frac{1}{(1-q)^{2n}}\sum_{p\in\mathcal{D}_n}\mathrm{wt}\big(p;\big(1-q,1-q^2,\dots\big),\big(1-tq,1-tq^2,\dots\big)\big)\\&=\frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\sum_{p\in\overline{\mathcal{D}}_k^*}\mathrm{wt}(p;\mathcal{U},\mathcal{V}_t).\end{aligned}\tag{13}$$

2.2. δ_k^+-configurations

We now recall δ_k-configurations. We first need some terminologies on integer partitions.

Definition 2. A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integer. Sometime we will consider that infinitely many zeros are attached at the end of λ so that $\lambda_i = 0$ for all $i > \ell$. Each integer λ_i is called a *part* of λ and the size of λ, denoted $|\lambda|$, is the sum of all parts. The *Ferrers diagram* of λ is the arrangement of left-justified square cells in which the ith topmost row has λ_i cells. We will identify a partition with its Ferrers diagram. *Row* i (resp. *Column* i) means the ith topmost row (resp. leftmost column). The (i, j)-*cell* means the cell in Row i and Column j. An *inner corner* (resp. *outer corner*) of λ is a cell $c \in \lambda$ (resp. $c \in \delta_k/\lambda$) such that $\lambda \setminus \{c\}$ (resp. $\lambda \cup \{c\}$) is a partition. For a partition λ, the *transpose* (or *conjugate*) of λ is the partition, denoted λ^{tr}, such that λ^{tr} has the (i, j)-cell if and only if λ has the (j, i)-cell. For two partition λ and μ we write $\mu \subset \lambda$ if the Ferrers diagram of μ is contained in that of λ. In this case we denote their difference as sets by λ/μ.

Let δ_k denote the staircase partition $(k, k-1, \dots, 1)$. Let $B(m,n)$ denote the box with m rows and n columns, that is, $B(m,n) = (\overbrace{n, n, \dots, n}^{m})$. It is well-known, for instance see [16], that

$$\sum_{\lambda \subset B(m,n)} q^{|\lambda|} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q . \tag{14}$$

Definition 3. A δ_k-*configuration* is a pair (λ, A) of a partition $\lambda \subset \delta_{k-1}$ and a set A of arrows each of which occupies a whole row or a whole column of δ_k/λ or δ_{k-1}/λ. If an arrow occupies a whole row or a whole column of δ_k/λ (resp. δ_{k-1}/λ), we call the arrow a k-arrow (resp. $(k-1)$-arrow). The *length* of an arrow is the number of cells occupied by the arrow. A *fillable corner* is an outer corner which is occupied by one k-arrow and one $(k-1)$-arrow. A *forbidden corner* is an outer corner which is occupied by two k-arrows. A δ_k^+-*configuration* is a δ_k-configuration without forbidden corners nor $(k-1)$-arrows.

We note that an arrow in a δ_k-configuration can have length 0. We will represent an arrow of length 0 as a half dot as shown in Fig. 1.

There is a natural bijection between $\overline{\mathcal{D}}_k^*$ and Δ_k^+ as follows. For $(\lambda, A) \in \Delta_k^+$, the north-west border of δ_k/λ defines a marked Dyck path of length $2k$ where the marked steps correspond to the segments on the border with arrows, see Fig. 2.

For a δ_k-configuration $C = (\lambda, A)$, we define the weight $\mathrm{wt}_{t,q}(C)$ by

$$\mathrm{wt}_{t,q}(C) = (-1)^{|A|} t^{\mathrm{h}(A)} q^{2|\lambda| + \|A\|},$$

where $\|A\|$ is the sum of the arrow lengths and $\mathrm{h}(A)$ is the number of horizontal arrows. For example, if C is the δ_k-configuration in Fig. 2, we have $\mathrm{wt}_{t,q}(C) = (-1)^7 t^4 q^{2\cdot 8+1+3+4+3+3+3+2}$.

Lemma 2.3. *Suppose that $C \in \Delta_k^+$ corresponds to $p \in \overline{\mathcal{D}}_k^*$ in the bijection described above. Then we have*

$$\mathrm{wt}(p; \mathcal{U}, \mathcal{V}_t) = t^k q^{k(k+1)} \mathrm{wt}_{t^{-1}, q^{-1}}(C), \tag{15}$$

which implies that

$$\sum_{p \in \overline{\mathcal{D}}_k^*} \mathrm{wt}(p; \mathcal{U}, \mathcal{V}_t) = t^k q^{k(k+1)} \sum_{C \in \Delta_k^+} \mathrm{wt}_{t^{-1}, q^{-1}}(C). \tag{16}$$

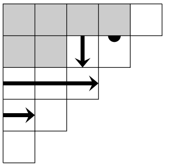


Fig. 1. An example of δ_k-configuration.

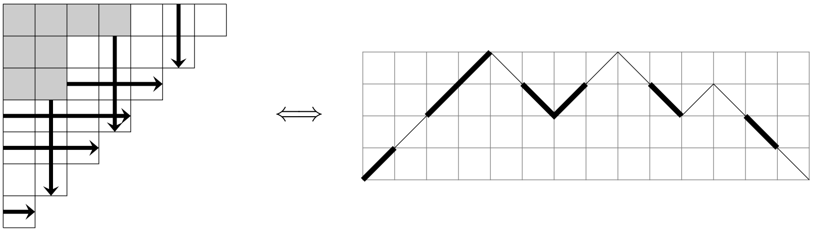


Fig. 2. A δ_k^+-configuration and the corresponding marked Dyck path, where the marked steps are the thicker steps.

Proof. Let $C = (\lambda, A)$. By the construction of the bijection sending C to p, it is easy to see that

$$\mathrm{wt}(p; \mathcal{U}, \mathcal{V}_t) = \prod_{i=1}^{k} r(i)c(i),$$

where $r(i) = -tq^{(k+1-i)-\lambda_i}$ if there is no horizontal arrow in Row i and $r(i) = 1$ otherwise, and $c(i) = -tq^{(k+1-i)-\lambda_i^{\mathrm{tr}}}$ if there is no vertical arrow in Column i and $c(i) = 1$ otherwise.

Now consider $t^k q^{k(k+1)} \mathrm{wt}_{t^{-1},q^{-1}}(C)$. By the identities

$$t^k q^{k(k+1)} = \prod_{i=1}^{k} \bigl(-tq^{k+1-i}\bigr)\bigl(-q^{k+1-i}\bigr),$$

$$2|\lambda| = \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \lambda_i^{\mathrm{tr}},$$

and the fact that if there is an arrow in Row i (resp. Column i) then its length is $(k+1-i) - \lambda_i$ (resp. $(k+1-i) - \lambda_i^{\mathrm{tr}}$), it is easy to check

$$t^k q^{k(k+1)} \mathrm{wt}_{t^{-1},q^{-1}}(C) = \prod_{i=1}^{k} r(i)c(i),$$

which finishes the proof. □

3. Proofs of Theorem 1.1 and Corollary 1.2

From now on we denote

$$T_k(t,q) = \sum_{C \in \Delta_k^+} \mathrm{wt}_{t,q}(C).$$

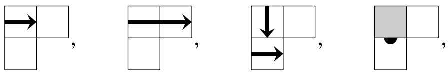
For brevity we will also write T_k instead of $T_k(t,q)$.

By (13) and (16), we have

$$E_n(t,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) t^k q^{k(k+1)} T_k\left(t^{-1}, q^{-1}\right).$$

Thus in order to prove Theorem 1.1, it remains to prove the recurrence relation (8). We need some results in [11]. We begin by defining a set which is in bijection with Δ_k^+.

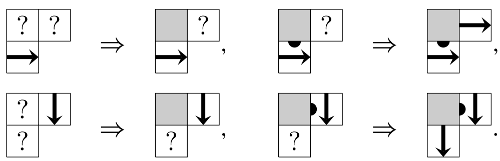
A *miniature* of a δ_k-configuration is the restriction of it to the $(k-i,i)$-cell, the $(k-i,i+1)$-cell, and the $(k-i+1,i)$-cell for some $1 \leqslant i \leqslant k-1$, where any $(k-1)$-arrows in Column $i+1$ or Row $k-i+1$ are ignored. For example, the miniatures of the δ_k-configuration in Fig. 1 are



where the bottommost miniature appears first.

Definition 4. A δ_k^--*configuration* is a δ_k-configuration (λ, A) satisfying the following conditions.

1. There is neither fillable corner nor forbidden corner.
2. Every k-arrow has length 1.
3. For any miniature, if there is a horizontal (resp. vertical) k-arrow in the bottom (resp. right) cell, then the middle cell is contained in λ. Moreover, if the bottom (resp. right) cell has a horizontal (resp. vertical) k-arrow and a vertical (resp. horizontal) $(k-1)$-arrow, then the right (resp. bottom) cell has a horizontal (resp. vertical) k-arrow. Pictorially, these mean the following:



The set of δ_k^--configurations is denoted by Δ_k^-.

Josuat-Vergès and the author [11, Proposition 4.1] found a bijection $\psi : \Delta_k^+ \to \Delta_k^-$ preserving $\mathrm{wt}_{1,q}$, i.e. $\mathrm{wt}_{1,q}(\psi(C)) = \mathrm{wt}_{1,q}(C)$ for all $C \in \Delta_k^+$. From the construction of ψ in their paper, it is clear that ψ also preserves the number of horizontal arrows. Thus we also have $\mathrm{wt}_{t,q}(\psi(C)) = \mathrm{wt}_{t,q}(C)$ for all $C \in \Delta_k^+$, which implies

$$T_k = \sum_{C \in \Delta_k^-} \mathrm{wt}_{t,q}(C). \tag{17}$$

Since $\Delta_{k-1}^+ \subset \Delta_k^-$, we can rewrite (17) as

$$T_k = T_{k-1} + \sum_{C \in \Delta_k^- \setminus \Delta_{k-1}^+} \mathrm{wt}_{t,q}(C). \tag{18}$$

In order to compute the sum in (18), we need a property of the elements in $\Delta_k^- \setminus \Delta_{k-1}^+$.

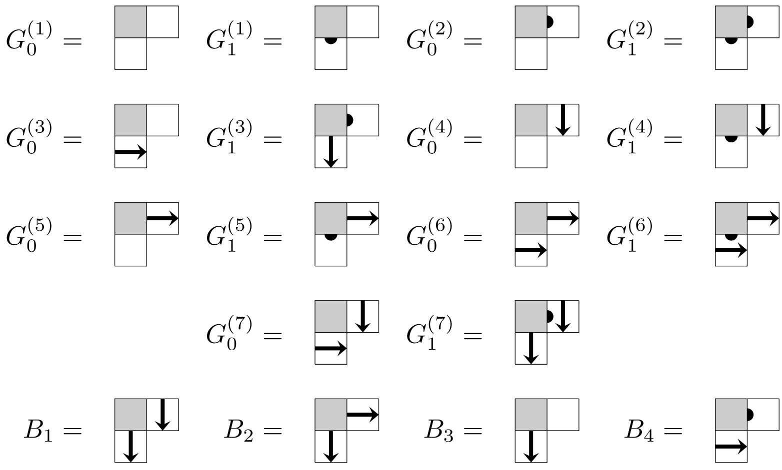


Fig. 3. List of exceptions.

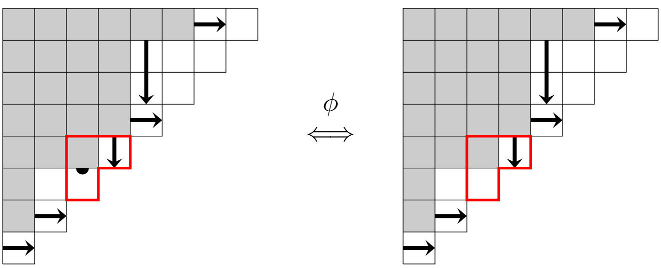


Fig. 4. The sign-reversing involution ϕ on $\Delta_k^- \setminus \Delta_{k-1}^+$. The topmost good exception is colored red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 3.1. *(See [11, Lemma 4.2].) Let $C \in \Delta_k^-$. Then $C \in \Delta_k^- \setminus \Delta_{k-1}^+$ if and only if C has a miniature listed in Fig. 3.*

We call the miniatures in Fig. 3 *exceptions*. The exceptions B_1, B_2, B_3, and B_4 are called *bad exceptions*, and the others are called *good exceptions*.

Now we can compute the sum in (18).

Lemma 3.2. *We have*

$$\sum_{C \in \Delta_k^- \setminus \Delta_{k-1}^+} \mathrm{wt}_{t,q}(C) = (1+t)(-q)^{k^2} + \left(1-t^2\right) \sum_{i=1}^{k-1} (-q)^{k^2-i^2} T_{i-1}.$$

Proof. We will construct a sign-reversing involution ϕ on $\Delta_k^- \setminus \Delta_{k-1}^+$, i.e. an involution satisfying $\mathrm{wt}_{t,q}(\phi(C)) = -\mathrm{wt}_{t,q}(C)$ if $\phi(C) \neq C$. If $\phi(C) = C$, we call C a *fixed point* of ϕ.

Suppose $C \in \Delta_k^- \setminus \Delta_{k-1}^+$. By Lemma 3.1, C has an exception. If C has a good exception, find the topmost good exception. If the topmost good exception is $G_0^{(i)}$ (resp. $G_1^{(i)}$) for some $i = 1, 2, \dots, 7$, we define $\phi(C)$ to be the configuration obtained from C by replacing the topmost good exception with $G_1^{(i)}$ (resp. $G_0^{(i)}$), see Fig. 4. If C has no good exceptions, we define $\phi(C) = C$. The map ϕ is certainly a sign-reversing involution whose fixed points are those containing only bad exceptions.

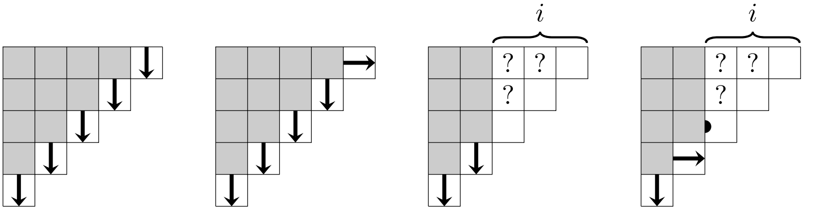


Fig. 5. The elements in $\Delta_k^- \setminus \Delta_{k-1}^+$ containing only bad exceptions.

Now suppose that C has only bad exceptions. Note that the topmost bad exception determined the miniatures below it because the miniature below B_1, B_2, B_3 must be B_1 and the miniature below B_4 must be B_2. Furthermore, B_1 or B_2 can be the topmost exception only if it intersects with the first row. Thus C looks like one of the configurations in Fig. 5. Since there is no exception above the topmost bad exception, the sub-configurations consisting of ?'s in the last two configurations in Fig. 5 are contained in Δ_{i-1}^+, where i can be any integer in $\{1, 2, \ldots, k-1\}$. Thus the weight sum of the configurations in Fig. 5 are, from left to right,

$$(-q)^{k^2}, \quad t(-q)^{k^2}, \quad \sum_{i=1}^{k-1} (-q)^{k^2-i^2} T_{i-1}, \quad \sum_{i=1}^{k-1} -t^2(-q)^{k^2-i^2} T_{i-1}.$$

Since the left hand side of the equation of the lemma is the weight sum of fixed points of ϕ, we are done. $\square$

From (18) and Lemma 3.2 we get the recurrence relation (8) for T_k, thus completing the proof of Theorem 1.1.

In order to find a formula for T_k from the above recurrence relation, we introduce a lattice path model for T_k. We consider the integer lattice $\mathbb{Z} \times \mathbb{Z}$ in which the unit length is defined to be $\sqrt{2}$ so that the area of a unit square is 2. In this lattice the area of the right triangle with three vertices $(0,0)$, $(1,0)$, and $(0,-1)$ is 1.

For nonnegative integers k and j, let $M[(k,0) \to (0,-j)]$ denote the set of paths from $(k,0)$ to $(0,-j)$ consisting of west steps $(-1,0)$ and southwest steps $(-1,-1)$. We define the weight $w(p)$ of $p \in M[(k,0) \to (0,-j)]$ to be

$$w(p) = (-1)^j q^{A(R)} \left(1 - t^2\right)^s V, \tag{19}$$

where $A(R)$ is the area of the region R bounded by the x-axis, the y-axis, and p, s is the number of southwest steps immediately followed by a west step, and $V = 1 + t$ if the last step is southwest, and $V = 1$ otherwise.

Lemma 3.3. *For $k \geqslant 0$, we have*

$$T_k = \sum_{j \geqslant 0} \sum_{p \in M[(k,0) \to (0,-j)]} w(p).$$

Proof. Let T'_k denote the right hand side of the equation. We will show that T'_k satisfies the same recurrence relation in (8).

Observe that T'_k is the sum of $w(p)$ for all paths p from $(k,0)$ to a point on the y-axis consisting of west steps and southwest steps. The weight sum of such paths p starting with a west step is T'_{k-1}. Suppose now that p starts with a southwest step. If p has only southwest steps, then p must be a path from $(k,0)$ to $(0,-k)$ and $w(p) = (-1)^k q^{k^2}(1+t)$. Otherwise we may assume that the first

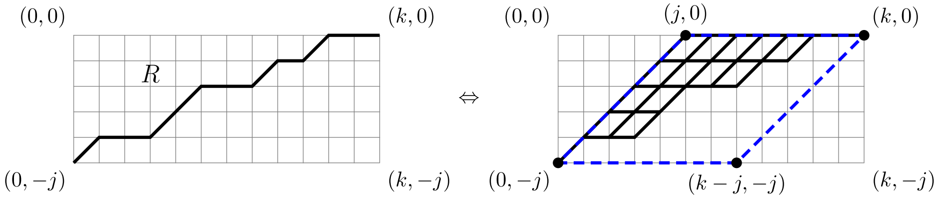


Fig. 6. An example of $p \in M[(b,k) \to (0,-j)]$. The region S obtained from R by removing the right triangle with three vertices $(0,0)$, $(0,-j)$, and $(j,0)$ can be identified with the partition $\lambda = (5,4,2,2) \subset B(j,k-j)$.

west step of p is the $(i+1)$st step for some $1 \leqslant i \leqslant k-1$. Let p' be the path obtained from p by removing the first $i+1$ steps and shifting the remaining path upwards by i units. Then p' is a path from $(k-i-1,0)$ to a point on the y-axis and $w(p) = (-1)^i q^{k^2-(k-i)^2}(1-t^2)w(p')$. Summarizing these, we get

$$T'_k = T'_{k-1} + (1+t)(-q)^{k^2} + \left(1-t^2\right)\sum_{i=1}^{k-1}(-1)^i q^{k^2-(k-i)^2} T'_{k-i-1}.$$

Changing the index i to $k-i$ in the above sum, we obtain that T_k and T'_k satisfy the same recurrence relation. Since $T_0 = T'_0 = 1$, we have $T_k = T'_k$. □

Suppose $p \in M[(k,0) \to (0,-j)]$. Then the region R in (19) contains the right triangle with three vertices $(0,0)$, $(j,0)$, and $(0,-j)$ whose area is j^2. If we remove this right triangle from R, the remaining region S can be identified with a partition $\lambda \subset B(j,k-j)$ as shown in Fig. 6. Then we have $A(S) = 2|\lambda|$. Moreover, s equals the number of inner corners of λ, which is the number $\mathrm{dist}(\lambda)$ of distinct parts, and $V = 1+t$ if $\lambda_j = 0$, and $V = 1$ if $\lambda_j > 0$. Therefore, we have

$$w(p) = (-q)^{j^2} q^{2|\lambda|}\left(1-t^2\right)^{\mathrm{dist}(\lambda)} V, \tag{20}$$

where $V = 1+t$ if $\lambda_j = 0$, and $V = 1$ if $\lambda_j > 0$. Since $M[(b,k) \to (0,-j)] = \emptyset$ if $j > k$, we get

$$T_k = \sum_{j=0}^{k}(-q)^{j^2}\left(\sum_{\lambda\subset B(j,k-j)} q^{2|\lambda|}\left(1-t^2\right)^{\mathrm{dist}(\lambda)} + \sum_{\lambda\subset B(j-1,k-j)} tq^{2|\lambda|}\left(1-t^2\right)^{\mathrm{dist}(\lambda)}\right). \tag{21}$$

Lemma 3.4. *For nonnegative integers m and n, we have*

$$\sum_{\lambda\subset B(m,n)} x^{\mathrm{dist}(\lambda)} q^{|\lambda|} = \sum_{i=0}^{m} q^{\binom{i+1}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} n+m-i \\ m-i \end{bmatrix}_q (x-1)^i.$$

Proof. Let P_n denote the set of partitions such that the largest part is at most n and every part is nonzero. It is not hard to see that

$$\sum_{\lambda\in P_n} y^{\ell(\lambda)} x^{\mathrm{dist}(\lambda)} q^{|\lambda|} = \prod_{i=1}^{n}\left(1+\frac{yxq^i}{1-yq^i}\right) = \prod_{i=1}^{n}\left(1+y(x-1)q^i\right)\prod_{j=1}^{n}\frac{1}{1-yq^j},$$

where $\ell(\lambda)$ is the number of parts of λ. Then by the q-binomial theorem [6, Exercise 1.2(vi)], we have

$$\prod_{i=1}^{n}\left(1+y(x-1)q^i\right)=\sum_{i=0}^{n}q^{\binom{i+1}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q y^i(x-1)^i.$$

Since the condition $\lambda \subset B(m,n)$ is equivalent to $\lambda \in P_n$ with $\ell(\lambda) \leqslant m$, we have

$$\begin{aligned}\sum_{\lambda\subset B(m,n)} x^{\mathrm{dist}(\lambda)}q^{|\lambda|} &= \left[y^{\leqslant m}\right]\left(\sum_{\lambda\in P_n} y^{\ell(\lambda)}x^{\mathrm{dist}(\lambda)}q^{|\lambda|}\right)\\ &= \left[y^{\leqslant m}\right]\left(\sum_{i=0}^{n}q^{\binom{i+1}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q y^i(x-1)^i\prod_{j=1}^{n}\frac{1}{1-yq^j}\right)\\ &= \sum_{i=0}^{\min(m,n)} q^{\binom{i+1}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q (x-1)^i\cdot\left[y^{\leqslant m-i}\right]\left(\prod_{j=1}^{n}\frac{1}{1-yq^j}\right),\end{aligned}$$

where $[y^{\leqslant m}]f(y)$ means the sum of the coefficients of y^j in $f(y)$ for $j \leqslant m$. Note that it is no harm to replace $\min(m,n)$ with m in the last sum of the above equation. Since

$$\left[y^{\leqslant m-i}\right]\left(\prod_{j=1}^{n}\frac{1}{1-yq^j}\right)=\sum_{\lambda\subset B(m-i,n)}q^{|\lambda|}=\begin{bmatrix}n+m-i\\m-i\end{bmatrix}_q,$$

we are done. □

Now we can complete the proof of Corollary 1.2.

Proof of Corollary 1.2. Applying Lemma 3.4 to (21), we obtain that T_k is equal to

$$\sum_{j=0}^{k}(-q)^{j^2}\left(\sum_{i=0}^{j}q^{i^2+i}\begin{bmatrix}k-j\\i\end{bmatrix}_{q^2}\begin{bmatrix}k-i\\j-i\end{bmatrix}_{q^2}\left(-t^2\right)^i+\sum_{i=0}^{j-1}tq^{i^2+i}\begin{bmatrix}k-j\\i\end{bmatrix}_{q^2}\begin{bmatrix}k-i-1\\j-i-1\end{bmatrix}_{q^2}\left(-t^2\right)^i\right),$$

which gives the desired formula. □

4. Self-conjugate overpartitions

In this section we will express the sum $T_k(t,q)$ in the previous section using overpartitions. Overpartitions were first introduced by Corteel and Lovejoy [3]. We define overpartitions in a slightly different way, but it should be clear that the two definitions are equivalent.

Definition 5. An *overpartition* is a partition in which each inner corner may be marked. For an overpartition λ, we define the *conjugate* of λ in the natural way: the partition is transposed and the cell (i,j) is marked if and only if the cell (j,i) is marked in λ, see Fig. 7. A *self-conjugate overpartition* is an overpartition whose conjugate is equal to itself. We denote by $\mathcal{SOP}(k)$ the set of self-conjugate overpartitions whose underlying partitions are contained in $B(k,k)$. A *diagonal cell* is the (i,i)-cell for some i. For an overpartition λ, the number of diagonal cells is denoted by $\mathrm{diag}(\lambda)$, and the number of marked cells is denoted by $\mathrm{mark}(\lambda)$. The *main diagonal* is the infinite set of (i,i)-cells (not necessarily contained in λ) for all i.

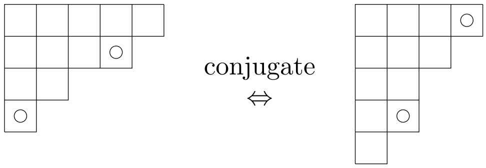


Fig. 7. An overpartition and its conjugate.

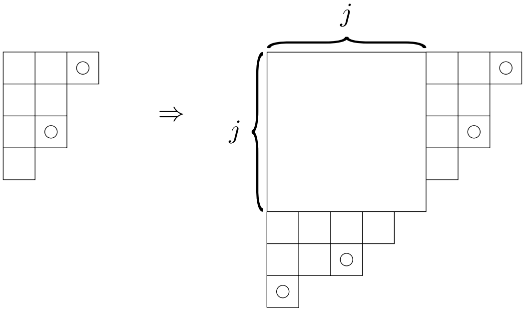


Fig. 8. The construction of $\nu \in \mathcal{SOP}(k)$ from an overpartition λ whose underlying partition is contained in $B(j, k-j)$.

Recall that by Lemma 3.3 and (20) we have

$$T_k(t,q) = \sum_{j=0}^{k} \sum_{\lambda \subset B(j,k-j)} (-q)^{j^2} q^{2|\lambda|} \left(1-t^2\right)^{\mathrm{dist}(\lambda)} V, \tag{22}$$

where $\mathrm{dist}(\lambda)$ is the number of distinct parts of λ, and $V = 1+t$ if $\lambda_j = 0$, and $V = 1$ if $\lambda_j > 0$. Since $\mathrm{dist}(\lambda)$ is equal to the number of inner corners of λ, the factor $(1-t^2)^{\mathrm{dist}(\lambda)}$ in (22) can be understood as marking each inner corner or not. Thus (22) can be rewritten as

$$T_k(t,q) = \sum_{j=0}^{k} \sum_{\lambda} (-1)^{j+\mathrm{mark}(\lambda)} t^{2\,\mathrm{mark}(\lambda)} q^{j^2} q^{2|\lambda|} V, \tag{23}$$

where the latter sum is over all overpartitions λ whose underlying partitions are contained in $B(j, k-j)$. For such an overpartition λ, we construct $\nu \in \mathcal{SOP}(k)$ which is obtained from the box $B(j, j)$ by attaching λ to the right of the box and its conjugate to the bottom of the box as shown in Fig. 8. Then ν always has even number of marked cells and

$$(-1)^{j+\mathrm{mark}(\lambda)} t^{2\,\mathrm{mark}(\lambda)} q^{j^2} q^{2|\lambda|} = (-1)^{\mathrm{diag}(\nu)+\frac{\mathrm{mark}(\nu)}{2}} t^{\mathrm{mark}(\nu)} q^{|\nu|}.$$

On the other hand, in (23) $V = 1+t$ if $\lambda_j = 0$, and $V = 1$ if $\lambda_j > 0$, equivalently, $V = 1+t$ if ν has an inner corner on the main diagonal, and $V = 1$ otherwise. Considering $V = 1+t$ as marking the diagonal inner corner or not, we can express $T_k(t,q)$ as follows.

Theorem 4.1. *We have*

$$T_k(t,q) = \sum_{\nu \in \mathcal{SOP}(k)} (-1)^{\mathrm{diag}(\nu)+\lfloor \frac{\mathrm{mark}(\nu)}{2} \rfloor} t^{\mathrm{mark}(\nu)} q^{|\nu|}.$$

We close this section by finding a functional equation for $T_k(t,q)$ which will serve as a recurrence relation in the next section.

Corollary 4.2. *For $k \geqslant 1$, we have*

$$(1-tq)T_k(tq,q) = T_k(t,q) + t^2q^{2k+1}T_{k-1}(t,q).$$

Proof. For $\nu \in \mathcal{SOP}(k)$, let $\omega(\nu) = (-1)^{\mathrm{diag}(\nu)+\lfloor \frac{\mathrm{mark}(\nu)}{2} \rfloor} t^{\mathrm{mark}(\nu)} q^{|\nu|}$. Then

$$T_k(t,q) = \sum_{\nu \in \mathcal{SOP}(k)} \omega(\nu).$$

We can think of $\omega(\nu)$ as the product of the weight of the cells and marks in ν, which are defined as follows:

(1) every non-diagonal cell has weight q,
(2) every diagonal cell has weight $-q$,
(3) every mark above the main diagonal has weight $-t$, and
(4) every mark below or on the main diagonal has weight t.

In order to express the left hand side of the equation we define $\mathcal{SOP}'(k)$ to be the set of $\nu \in \mathcal{SOP}(k)$ in which the unique corner on the main diagonal may have a special mark. Note that the corner of the main diagonal can be an inner corner or an outer corner depending on ν, and if it is an inner corner, then this corner may have two marks, one is non-special and the other is special. For $\nu \in \mathcal{SOP}'(k)$, we define $\omega'(\nu)$ to be the product of weights of the cells and marks, which are defined as follows:

(1) every non-diagonal cell has weight q,
(2) every diagonal cell has weight $-q$,
(3) every mark above the main diagonal has weight $-tq$,
(4) every mark below or on the main diagonal has weight tq, and
(5) if there is a special mark, it has weight $-tq$.

It is easy to see that

$$(1-tq)T_k(tq,q) = \sum_{\nu \in \mathcal{SOP}'(k)} \omega'(\nu).$$

Let X be the set of $\nu \in \mathcal{SOP}'(k)$ which has an inner corner on the main diagonal with only one mark. For $\nu \in X$, we define ν' to be the element in X that is obtained by switching the mark in the inner corner on the main diagonal to special one or non-special one. It is clear that $\omega'(\nu') = -\omega'(\nu)$. Thus the sum of $\omega'(\nu)$ for all $\nu \in X$ is zero and we get

$$(1-tq)T_k(tq,q) = \sum_{\nu \in \mathcal{SOP}'(k)\setminus X} \omega'(\nu).$$

Now suppose $\nu \in \mathcal{SOP}'(k) \setminus X$. For each mark above (resp. below) the main diagonal, if it is in Row i (resp. Column i), delete the mark and add a cell in Row $i+1$ (resp. Column $i+1$) and mark the new cell. If there is a special mark in the outer corner on the diagonal, then add a cell to ν to fill this outer corner and change the special mark to a non-special mark, see Fig. 9. If there are one non-special mark and one special mark in the inner corner on the main diagonal, which is in Row i and Column i, then delete the two marks, add one cell to Row $i+1$ and one cell to Column $i+1$, and

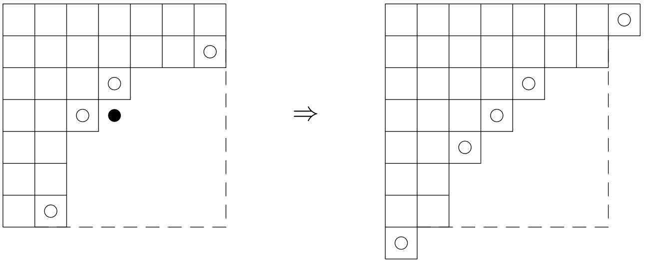


Fig. 9. Moving the marks in $\nu \in \mathcal{SOP}'(k)$ when there is a special mark in the outer corner on the main diagonal.

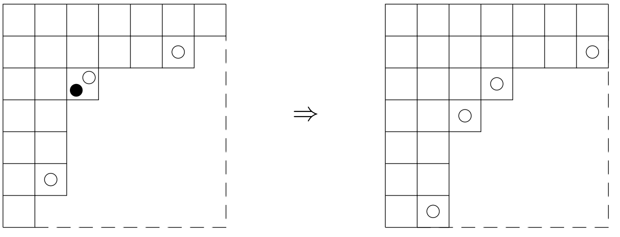


Fig. 10. Moving the marks in $\nu \in \mathcal{SOP}'(k)$ when there is a special mark in the inner corner on the main diagonal.

mark the two new cells, see Fig. 10. Let μ be the resulting overpartition. From the construction it is clear that $\omega'(\nu) = \omega(\mu)$. Also, it is not hard to see that μ is an element in $\mathcal{SOP}(k)$ or an element in $\mathcal{SOP}(k+1)$. Moreover, if $\mu \in \mathcal{SOP}(k+1)$, the $(1, k+1)$-cell and the $(k+1, 1)$-cell of μ are marked inner corners, and the overpartition μ' obtained from μ by deleting Row 1 and Column 1 satisfies $\mu' \in \mathcal{SOP}(k-1)$ and $\omega(\mu) = t^2 q^{2k+1} \omega(\mu')$. Note that the sign does not change because μ' has one less diagonal cells and two less marks than μ. Thus we have

$$\sum_{\nu \in \mathcal{SOP}'(k) \setminus X} \omega'(\nu) = \sum_{\nu \in \mathcal{SOP}(k)} \omega(\nu) + t^2 q^{2k+1} \sum_{\nu \in \mathcal{SOP}(k-1)} \omega(\nu),$$

which finishes the proof. □

5. Another formula for $T_k(\pm q^r, q)$

In this section we will find another formula for $T_k(t, q)$ when $t = \pm q^r$ for any integer r. To this end we need to divide the cases when $r \geqslant 0$ and $r \leqslant 0$. For a sign $\epsilon \in \{+, -\}$, and nonnegative integers b and k, we define

$$\alpha_\epsilon(b, k) = T_k\big(\epsilon q^b, q\big), \qquad \beta_\epsilon(b, k) = T_k\big(\epsilon q^{-b}, q\big).$$

Note that for $b \geqslant 0$, we have

$$\alpha_\epsilon(b, 0) = \beta_\epsilon(b, 0) = 1. \tag{24}$$

Recall that from the recurrence (8) of $T_k(t, q)$, we immediately get $T_k(-1, q) = 1$ and $T_k(1, q) = \sum_{i=-k}^{k} (-q)^{i^2}$. Thus we have

$$\alpha_-(0,k) = \beta_-(0,k) = T_k(-1,q) = 1, \tag{25}$$

$$\alpha_+(0,k) = \beta_+(0,k) = T_k(1,q) = \sum_{i=-k}^{k} (-q)^{i^2}. \tag{26}$$

Substituting $t = \epsilon q^{b-1}$ in Corollary 4.2, we obtain

$$\left(1-\epsilon q^b\right) T_k\left(\epsilon q^b, q\right) = T_k\left(\epsilon q^{b-1}, q\right) + q^{2k+2b-1} T_{k-1}\left(\epsilon q^{b-1}, q\right).$$

If $b \geqslant 1$, we can divide the both sides of the above equation by $1-\epsilon q^b$ to get the following lemma.

Lemma 5.1. *For integers $b, k \geqslant 1$, we have*

$$\alpha_\epsilon(b,k) = \frac{1}{1-\epsilon q^b} \alpha_\epsilon(b-1,k) + \frac{q^{2k+2b-1}}{1-\epsilon q^b} \alpha_\epsilon(b-1,k-1).$$

Substituting $t = \epsilon q^{-b}$ in Corollary 4.2, we obtain

$$\left(1-\epsilon q^{1-b}\right) T_k\left(\epsilon q^{1-b}, q\right) = T_k\left(\epsilon q^{-b}, q\right) + q^{2k-2b+1} T_{k-1}\left(\epsilon q^{-b}, q\right),$$

which implies the following lemma.

Lemma 5.2. *For integers $b, k \geqslant 1$, we have*

$$\beta_\epsilon(b,k) = \left(1-\epsilon q^{1-b}\right) \beta_\epsilon(b-1,k) - q^{2k-2b+1} \beta_\epsilon(b,k-1).$$

Now we have recurrence relations and initial conditions for $\alpha_\epsilon(b,k)$ and $\beta_\epsilon(b,k)$. Thus we can use the idea in Section 4 to compute $\alpha_\epsilon(b,k)$ and $\beta_\epsilon(b,k)$. As we did in Section 4 we define the unit length in the lattice $\mathbb{Z} \times \mathbb{Z}$ to be $\sqrt{2}$.

5.1. *Formula for $T_k(\pm q^r, q)$ when $r \geqslant 0$*

Suppose m and n are nonnegative integers with $m = 0$ or $n = 0$. We define $L[(b,k) \to (m,n)]$ to be the set of lattice paths from (b,k) to (m,n) consisting of west steps $(-1,0)$ and southwest steps $(-1,-1)$ without any west steps on the x-axis. The condition that there is no west step on the x-axis guarantees that the lattice path ends when it first touches the x-axis or the y-axis.

For $p \in L[(b,k) \to (m,n)]$ we define the weight $w(p)$ by

$$w(p) = q^{A(R)} \prod_{i=m+1}^{b} \frac{1}{1-\epsilon q^i} \prod_{i=n+1}^{k} q^{2i}, \tag{27}$$

where $A(R)$ is the area of the upper region R of the rectangle with four vertices $(0,0)$, $(b,0)$, $(0,k)$, and (b,k) divided by the path p.

Lemma 5.3. *For $b, k \geqslant 0$, we have*

$$\alpha_\epsilon(b,k) = \sum_{\substack{m,n\geqslant 0\\ mn=0}} \alpha_\epsilon(m,n) \sum_{p\in L[(b,k)\to(m,n)]} w(p).$$

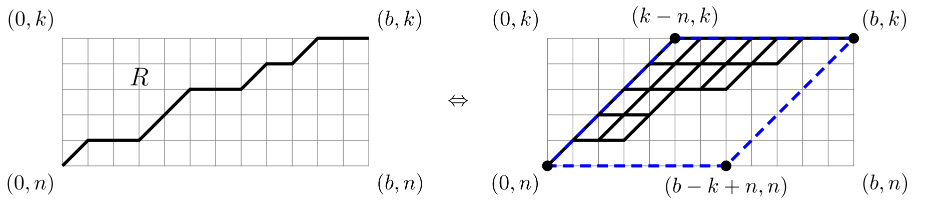


Fig. 11. An example of $p \in L[(b,k) \to (0,n)]$.

Proof. Let $F(b,k)$ denote the right hand side and let $f_{m,n}(b,k)$ denote the latter sum there. Using a similar argument as in the proof of Lemma 3.3, one can easily check that for $b,k \geqslant 1$,

$$f_{m,n}(b,k) = \frac{1}{1-\epsilon q^b} f_{m,n}(b-1,k) + \frac{q^{2k+2b-1}}{1-\epsilon q^b} f_{m,n}(b-1,k-1).$$

Thus $F(b,k)$ and $\alpha_\epsilon(b,k)$ satisfy the same recurrence relation. Since $F(b,k) = \alpha_\epsilon(b,k)$ when $b=0$ or $k=0$, we get $F(b,k) = \alpha_\epsilon(b,k)$ for all $b,k \geqslant 0$. $\square$

Since $\alpha_\epsilon(m,0) = 1$, the formula in the previous lemma can be written as

$$\alpha_\epsilon(b,k) = \sum_{n\geqslant 1} \alpha_\epsilon(0,n) \sum_{p\in L[(b,k)\to(0,n)]} w(p) + \sum_{m\geqslant 0} \sum_{p\in L[(b,k)\to(m,0)]} w(p). \tag{28}$$

Now we compute the weight sums in (28).

Lemma 5.4. *For $b,k \geqslant 0$ and $n \geqslant 1$, we have*

$$\sum_{p\in L[(b,k)\to(0,n)]} w(p) = \frac{q^{(k-n)(2k+1)}}{(\epsilon q;q)_b} \begin{bmatrix} b \\ k-n \end{bmatrix}_{q^2}.$$

Proof. Let $p \in L[(b,k) \to (0,n)]$. From the definition of $w(p)$ in (27), we have

$$w(p) = \frac{q^{k(k+1)-n(n+1)}}{(\epsilon q;q)_b} \cdot q^{A(R)}.$$

Since p consists of west steps and southwest steps, the region contains the right triangle with three vertices $(0,n)$, $(0,k)$, and $(k-n,k)$, whose area is $(k-n)^2$, see Fig. 11. Let S be the region obtained from R by removing this right triangle. Then S is contained in the quadrilateral with four vertices $(0,n)$, $(k-n,k)$, (b,k), and $(b-k+n,n)$. Again by the fact that p consists of west steps and southwest steps, one can identify S with a partition λ contained in $B(k-n, b-k+n)$. In this identification we have $A(S) = 2|\lambda|$. Thus, we get

$$\sum_{p\in L[(b,k)\to(0,n)]} w(p) = \frac{q^{k(k+1)-n(n+1)}}{(\epsilon q;q)_b} \cdot q^{(k-n)^2} \sum_{\lambda\subset B(k-n,b-k+n)} q^{2|\lambda|}$$
$$= \frac{q^{(k-n)(2k+1)}}{(\epsilon q;q)_b} \begin{bmatrix} b \\ k-n \end{bmatrix}_{q^2}. \qquad \square$$

Lemma 5.5. *For $b \geqslant 0$, $k \geqslant 1$ and $m \geqslant 0$, we have*

$$\sum_{p \in L[(b,k)\to(m,0)]} w(p) = \frac{(\epsilon q;q)_m}{(\epsilon q;q)_b} q^{k(2k+2m+1)} \begin{bmatrix} b-m-1 \\ k-1 \end{bmatrix}_{q^2}.$$

Proof. This is similar to the proof of the previous lemma. The only difference is that since the last step of p is always a southwest step, p visits $(m+1,1)$ right before its end point. Then the same argument works, so we omit the details. □

Finally, we obtain a formula for $\alpha_\epsilon(b,k)$.

Theorem 5.6. *For $b \geqslant 0$ and $k \geqslant 1$, we have*

$$\alpha_\epsilon(b,k) = \sum_{i=0}^{k-1} \frac{q^{i(2k+1)}}{(\epsilon q;q)_b} \begin{bmatrix} b \\ i \end{bmatrix}_{q^2} \alpha_\epsilon(0,k-i) + \sum_{i=0}^{b-1} \frac{(\epsilon q;q)_i}{(\epsilon q;q)_b} q^{k(2k+2i+1)} \begin{bmatrix} b-i-1 \\ k-1 \end{bmatrix}_{q^2}.$$

Proof. By (28) and Lemmas 5.4 and 5.5, we have

$$\alpha_\epsilon(b,k) = \sum_{n\geqslant 1} \frac{q^{(k-n)(2k+1)}}{(\epsilon q;q)_b} \begin{bmatrix} b \\ k-n \end{bmatrix}_{q^2} \alpha_\epsilon(0,n) + \sum_{m\geqslant 0} \frac{(\epsilon q;q)_m}{(\epsilon q;q)_b} q^{k(2k+2m+1)} \begin{bmatrix} b-m-1 \\ k-1 \end{bmatrix}_{q^2}.$$

In the first sum the summand is zero unless $k-n \geqslant 0$, and in the second sum the summand is zero unless $m \leqslant b-1$. Replacing $k-n$ with i in the first sum and m with i in the second sum we get the desired formula. □

By Theorem 5.6 with $\epsilon = +$ and (26), we get a formula for $T_k(q^b,q)$.

Corollary 5.7. *For $b \geqslant 0$ and $k \geqslant 1$, we have*

$$T_k\big(q^b,q\big) = \sum_{i=0}^{k-1} \frac{q^{i(2k+1)}}{(q;q)_b} \begin{bmatrix} b \\ i \end{bmatrix}_{q^2} \sum_{j=-(k-i)}^{k-i} (-q)^{j^2} + \sum_{i=0}^{b-1} \frac{(q;q)_i}{(q;q)_b} q^{k(2k+2i+1)} \begin{bmatrix} b-i-1 \\ k-1 \end{bmatrix}_{q^2}.$$

If $b=1$ in Corollary 5.7, we have that for $k \geqslant 1$,

$$T_k(q,q) = \frac{1}{1-q} \sum_{i=-k}^{k} (-q)^{i^2} + \frac{q^{2k+1}}{1-q} \sum_{i=-(k-1)}^{k-1} (-q)^{i^2},$$

which together with Theorem 1.1 implies (3).

By Theorem 5.6 with $\epsilon = -$ and (25), we get a formula for $T_k(-q^b,q)$.

Corollary 5.8. *For $b \geqslant 0$ and $k \geqslant 1$, we have*

$$T_k\big(-q^b,q\big) = \sum_{i=0}^{k-1} \frac{q^{i(2k+1)}}{(-q;q)_b} \begin{bmatrix} b \\ i \end{bmatrix}_{q^2} + \sum_{i=0}^{b-1} \frac{(-q;q)_i}{(-q;q)_b} q^{k(2k+2i+1)} \begin{bmatrix} b-i-1 \\ k-1 \end{bmatrix}_{q^2}.$$

If $b=1$ in Corollary 5.8, we have that for $k \geqslant 1$,

$$T_k(-q,q) = \frac{1+q^{2k+1}}{1+q}. \tag{29}$$

Note that the above identity is also true for $k=0$. This gives the following formula for $E_n(-q,q)$.

Proposition 5.9. *We have*

$$E_n(-q,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (-1)^k \frac{q^{k^2}+q^{(k+1)^2}}{1+q}.$$

Proof. By Theorem 1.1,

$$E_n(-q,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (-q)^k q^{k(k+1)} T_k\left(-q^{-1},q^{-1}\right).$$

By (29) we get

$$(-q)^k q^{k(k+1)} T_k\left(-q^{-1},q^{-1}\right) = (-q)^{k^2+2k} \frac{1+q^{-2k-1}}{1+q^{-1}} = (-1)^k \frac{q^{k^2}+q^{(k+1)^2}}{1+q},$$

which finishes the proof. □

5.2. Formula for $T_k(\pm q^r, q)$ when $r \leqslant 0$

Suppose m and n are nonnegative integers with $m=0$ or $n=0$. We define $L'[(b,k)\to(m,n)]$ to be the set of lattice paths from (b,k) to (m,n) consisting of west steps $(-1,0)$ and south steps $(0,-1)$ without west steps on the x-axis nor south steps on the y-axis. For $p \in L'[(b,k)\to(m,n)]$ we define the weight $w(p)$ by

$$w(p) = q^{-A(R)} \prod_{i=m+1}^{b} \left(1-\epsilon q^{1-i}\right) \prod_{i=n+1}^{k} \left(-q^{2i+1}\right), \tag{30}$$

where $A(R)$ is the area of the upper region R of the rectangle with four vertices $(0,0)$, $(b,0)$, $(0,k)$, and (b,k) divided by the path p.

Lemma 5.10. *For $b,k \geqslant 0$, we have*

$$\beta_\epsilon(b,k) = \sum_{\substack{m,n\geqslant 0\\ mn=0}} \beta_\epsilon(m,n) \sum_{p\in L'[(b,k)\to(m,n)]} w(p).$$

Proof. Since this can be done similarly as in the proof of Lemma 5.3, we omit the proof. □

Notice that $L'[(b,k)\to(0,0)] = \emptyset$ unless $(b,k)=(0,0)$. Since $\beta_\epsilon(m,0)=1$, the formula in the previous lemma can be written as follows: if $(b,k)\neq(0,0)$, we have

$$\beta_\epsilon(b,k) = \sum_{n\geqslant 1} \beta_\epsilon(0,n) \sum_{p\in L'[(b,k)\to(0,n)]} w(p) + \sum_{m\geqslant 1} \sum_{p\in L'[(b,k)\to(m,0)]} w(p). \tag{31}$$

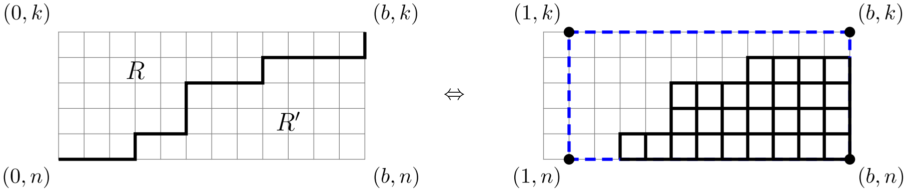


Fig. 12. An example of $p \in L'[(b,k) \to (0,n)]$. The lower region R' can be identified with a rotated partition contained in $B(k-n, b-1)$.

Lemma 5.11. *For $b, k \geqslant 0$ and $n \geqslant 1$ with $(b,k) \neq (0,0)$, we have*

$$\sum_{p \in L'[(b,k)\to(0,n)]} w(p) = (\epsilon q^{1-b}; q)_b (-q)^{(k-n)(k+n-2b+2)} \begin{bmatrix} b+k-n-1 \\ k-n \end{bmatrix}_{q^2}.$$

Proof. Let $p \in L'[(b,k) \to (0,n)]$. From the definition of $w(p)$ in (30), we have

$$w(p) = q^{-A(R)} (\epsilon q^{1-b}; q)_b (-1)^{k-n} q^{(k+1)^2-(n+1)^2}.$$

Note that R is contained in the rectangle with four vertices $(0,n)$, $(0,k)$, (b,n), and (b,k), see Fig. 12. Let R' be the region of this rectangle minus R. Then $-A(R) = -2b(k-n) + A(R')$. Since the last step of p is a west step, R' can be identified with a partition $\lambda \subset B(k-n, b-1)$, which is rotated by an angle of 180°, and $A(R') = 2|\lambda|$. Therefore,

$$\begin{aligned}\sum_{p \in L'[(b,k)\to(0,n)]} w(p) &= (\epsilon q^{1-b}; q)_b (-1)^{k-n} q^{(k-n)(k+n+2)-2b(k-n)} \sum_{\lambda \subset B(k-n,b-1)} q^{2|\lambda|} \\ &= (\epsilon q^{1-b}; q)_b (-q)^{(k-n)(k+n-2b+2)} \begin{bmatrix} b+k-n-1 \\ k-n \end{bmatrix}_{q^2}. \qquad \square\end{aligned}$$

Lemma 5.12. *For $b, k \geqslant 0$ and $m \geqslant 1$ with $(b,k) \neq (0,0)$, we have*

$$\sum_{p \in L'[(b,k)\to(m,0)]} w(p) = (\epsilon q^{1-b}; q)_{b-m} (-q)^{k(k-2b+2)+2(b-m)} \begin{bmatrix} k+b-m-1 \\ b-m \end{bmatrix}_{q^2}.$$

Proof. This can be done by the same argument as in the proof of the previous lemma. □

Now we can find a formula for $\beta_\epsilon(b,k)$.

Theorem 5.13. *For $b, k \geqslant 0$ with $(b,k) \neq (0,0)$, we have*

$$\begin{aligned}\beta_\epsilon(b,k) = &\sum_{i=0}^{k-1} (\epsilon q^{1-b}; q)_b (-q)^{i(2k-2b-i+2)} \begin{bmatrix} b+i-1 \\ i \end{bmatrix}_{q^2} \beta_\epsilon(0, k-i) \\ &+ \sum_{i=0}^{b-1} (\epsilon q^{1-b}; q)_i (-q)^{k(k-2b+2)+2i} \begin{bmatrix} k+i-1 \\ i \end{bmatrix}_{q^2}.\end{aligned}$$

Proof. By (31) and Lemmas 5.11 and 5.12, we have

$$\beta_\epsilon(b,k)=\sum_{n\geqslant 1}\left(\epsilon q^{1-b};q\right)_b(-q)^{(k-n)(k+n-2b+2)}\begin{bmatrix} b+k-n-1 \\ k-n\end{bmatrix}_{q^2}\beta_\epsilon(0,n)$$
$$+\sum_{m\geqslant 1}\left(\epsilon q^{1-b};q\right)_{b-m}(-q)^{k(k-2b+2)+2(b-m)}\begin{bmatrix} k+b-m-1 \\ b-m\end{bmatrix}_{q^2}.$$

In the first sum the summand is zero unless $k-n\geqslant 0$, and in the second sum the summand is zero unless $b-m\geqslant 0$. By replacing $k-n$ with i in the first sum and $b-m$ with i in the second sum, we get the desired formula. □

By Theorem 1.1 with $\epsilon=+$ and (26), we get a formula for $T_k(q^{-b},q)$.

Corollary 5.14. *For $b\geqslant 1$ and $k\geqslant 0$, we have*

$$T_k\left(q^{-b},q\right)=\sum_{i=0}^{b-1}\left(q^{1-b};q\right)_i(-q)^{k(k-2b+2)+2i}\begin{bmatrix} k+i-1 \\ i\end{bmatrix}_{q^2}.$$

If $b=1$ in Corollary 5.14, we get

$$T_k(1/q,k)=(-q)^{k^2},$$

which together with Theorem 1.1 implies

$$E_n(1/q,q)=\frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)(-1)^k,$$

which is equal to 1 if $n=0$, and 0 otherwise. Notice that this corresponds to the trivial identity:

$$\sum_{n\geqslant 0}E_n(1/q,q)x^n=1.$$

By Theorem 1.1 with $\epsilon=-$ and (25), we get a formula for $T_k(-q^{-b},q)$.

Corollary 5.15. *For $b\geqslant 1$ and $k\geqslant 0$, we have*

$$T_k\left(-q^{-b},q\right)=\sum_{i=0}^{k-1}\left(-q^{1-b};q\right)_b(-q)^{i(2k-2b-i+2)}\begin{bmatrix} b+i-1 \\ i\end{bmatrix}_{q^2}$$
$$+(-q)^{k^2+2k-2kb}\sum_{i=0}^{b-1}\left(-q^{1-b};q\right)_i q^{2i}\begin{bmatrix} k+i-1 \\ i\end{bmatrix}_{q^2}.$$

If $b=1$ in Corollary 5.15, we get

$$T_k(-1/q,q) = 2\sum_{i=0}^{k-1}(-q)^{i(2k-i)} + (-q)^{k^2}$$
$$= (-q)^{k^2} + 2\sum_{i=1}^{k}(-q)^{(k-i)(k+i)}$$
$$= (-q)^{k^2}\sum_{i=-k}^{k}(-q)^{-i^2},$$

which together with Theorem 1.1 implies the following formula for $E_n(-1/q,q)$.

Proposition 5.16. *We have*

$$E_n(-1/q,q) = \frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)\sum_{i=-k}^{k}(-q)^{i^2}.$$

We note that Proposition 5.16 was first discovered by Josuat-Vergès (personal communication).

6. The original formula of Josuat-Vergès for $E_n(q)$

The original formula for $E_n(q)$ in [9] is the following:

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)\sum_{i=0}^{2k}(-1)^{i+k}q^{i(2k-i)+k}, \tag{32}$$

$$E_{2n+1}(q) = \frac{1}{(1-q)^{2n+1}}\sum_{k=0}^{n}\left(\binom{2n+1}{n-k} - \binom{2n+1}{n-k-1}\right)\sum_{i=0}^{2k+1}(-1)^{i+k}q^{i(2k+2-i)}. \tag{33}$$

In this section we prove that (2) and (3) are equivalent to (32) and (33) respectively. By changing the index i with $i+k$ in (32) we obtain (2). For the second identity, let

$$f(k) = \frac{1}{1-q}\sum_{i=0}^{2k+1}(-1)^{i+k}q^{i(2k+2-i)}.$$

Using Pascal's identity, we obtain that $(1-q)^{2n}E_{2n+1}(q)$ is equal to

$$\sum_{k=0}^{n}\left(\binom{2n+1}{n-k} - \binom{2n+1}{n-k-1}\right)f(k)$$
$$= \sum_{k=0}^{n}\left(\binom{2n}{n-k-1} - \binom{2n}{n-k-2}\right)f(k) + \sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)f(k)$$
$$= \sum_{k=1}^{n+1}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)f(k-1) + \sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)f(k).$$

Since $\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)f(k-1) = 0$ when $k=0$ and $k=n+1$, we have

$$E_{2n+1}(q)=\frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\big(f(k)+f(k-1)\big). \tag{34}$$

Thus in order to get (3) it suffices to show $f(k)+f(k-1)=q^{k(k+2)}A_k(q^{-1})$. Since

$$(1-q)f(k)=\sum_{i=0}^{2k+1}(-1)^{i+k}q^{i(2k+2-i)}=\sum_{i=-(k+1)}^{k}(-1)^{i+2k+1}q^{(i+k+1)(k+1-i)}$$
$$=-q^{(k+1)^2}\sum_{i=-(k+1)}^{k}(-q)^{-i^2}=(-1)^k-q^{(k+1)^2}\sum_{i=-k}^{k}(-q)^{-i^2},$$

we have $f(k)+f(k-1)=1$ if $k=0$, and for $k\geqslant 1$,

$$f(k)+f(k-1)=-\frac{q^{(k+1)^2}}{1-q}\sum_{i=-k}^{k}(-q)^{-i^2}-\frac{q^{k^2}}{1-q}\sum_{i=-(k-1)}^{k-1}(-q)^{-i^2},$$

which is easily seen to be equal to $q^{k(k+2)}A_k(q^{-1})$. Thus we get (3).

7. Concluding remarks

In this paper we have found a formula for the coefficient $E_n(t,q)$ of x^n in the continued fraction

$$\cfrac{1}{1-\cfrac{[1]_q[1]_{t,q}x}{1-\cfrac{[2]_q[2]_{t,q}x}{\cdots}}}.$$

Since $E_n(t,q)$ is a generalization of the q-Euler number, it is natural to consider a similar generalization of (5). Thus we propose the following problem.

Problem 1. Find a formula for the coefficient of x^n in the following continued fraction:

$$\cfrac{1}{1-\cfrac{[1]_{t,q}x}{1-\cfrac{[2]_{t,q}x}{\cdots}}}.$$

Also, we can consider a generalization of $E_n(t,q)$ as follows.

Problem 2. Find a formula for the coefficient of x^n in the following continued fraction:

$$\cfrac{1}{1-\cfrac{[1]_{y,q}[1]_{t,q}x}{1-\cfrac{[2]_{y,q}[2]_{t,q}x}{\cdots}}}.$$

Recently Prodinger [13] expressed the continued fractions (in fact the corresponding T-fractions, see [11, Lemma 6.1] for the relation between S-fractions and T-fractions) in the above two problems as fractions of formal power series when both y and t are equal to q^d for a positive integer d. From another result of Prodinger [13, Section 11], one can obtain the following formula for $T_k(q^b, q)$ for a positive integer b:

$$T_k\left(q^b, q\right) = \sum_{i=0}^{b} q^{\binom{i+1}{2}} \begin{bmatrix} b \\ i \end{bmatrix}_q \sum_{j=-k}^{k-i} (-1)^j q^{j^2+i(k+j)} \begin{bmatrix} k+j+b \\ b \end{bmatrix}_q. \tag{35}$$

Problem 3. Find a direct proof of the equivalence of (9) and (35).

In the introduction we have two formulas (7) and Corollary 1.2 for $E_n(t, q)$. Using hypergeometric series Kim and Stanton [12] showed that these are equivalent and simplified to the following formula:

$$E_n(t, q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (-q)^k \sum_{i=0}^{k} t^i q^{\binom{k-i}{2}} \left(q; q^2\right)_i \begin{bmatrix} k+i \\ k-i \end{bmatrix}_q. \tag{36}$$

Han et al. [7] introduced the polynomials $P_n^\alpha(x, q)$ defined by $P_0^\alpha(x, q) = 1$ and

$$P_n^\alpha(x, q) = [x, a]_q \frac{[x, b]_q P_{n-1}^\alpha([x, c]_q, q) - [x, d]_q P_{n-1}^\alpha(x, q)}{1 + (q-1)x},$$

where $\alpha = (a, b, c, d)$ is a tuple of nonnegative integers and $[x, n]_q = xq^n + [n]_q$. They proved that

$$\sum_{n \geqslant 0} P_n^\alpha(x, q) z^n = \cfrac{1}{1 - \cfrac{q^d [b-d]_q [x, a]_q z}{1 - \cfrac{q^a [c]_q [x, b]_q z}{1 - \cfrac{q^d [b-d+c]_q [x, a+c]_q z}{1 - \cfrac{q^a [2c]_q [x, b+c]_q z}{\cdots}}}}}.$$

One can easily check that $E_n(t, q) = P_n^{(0,1,2,0)}([1]_{t,q}, q)$. Thus as a special case of [7, Proposition 1] we have

$$\sum_{n \geqslant 0} E_n(t, q) z^n = \sum_{m \geqslant 0} \frac{tq^{2m+1} [2m]_{t,q}!}{\prod_{i=0}^{m} (tq^{2i+1} + [2i+1]_{t,q}^2 z)} z^m.$$

Using the idea in the last section of [15] Zeng proved the following formula (personal communication):

$$E_n(t, q) = t^{-n} \sum_{m=0}^{n} \sum_{i=0}^{m} (-1)^{n-i} \frac{q^{2m-2in+i^2-n-i} [2m]_{t,q}! [2i+1]_{t,q}^{2n}}{[2i]_q!! [2m-2i]_q!! \prod_{k=0, k \neq i}^{m} [2k+2i+2]_{t^2,q}}, \tag{37}$$

where $[2m]_{t,q}! = \prod_{i=1}^{2m} [i]_{t,q}$ and $[2i]_q!! = \prod_{k=1}^{i} [2k]_q$.

Problem 4. Find a direct proof of the equivalence of (36) and (37).

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