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Towards compatible triangulations

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Abstract

We state the following conjecture: any two planar n-point sets that agree on the number of convex hull points can be triangulated in a compatible manner, i.e., such that the resulting two triangulations are topologically equivalent. We first describe a class of point sets which can be triangulated compatibly with any other set (that satisfies the obvious size and shape restrictions). The conjecture is then proved true for point sets with at most three interior points. Finally, we demonstrate that adding a small number of extraneous points (the number of interior points minus three) always allows for compatible triangulations. The linear bound extends to point sets of arbitrary size and shape.

Keywords: Computational geometry; Triangulation; Strong isomorphism; Steiner points

1. Introduction

Can any two planar point sets (that agree on the number of points and extreme points) be triangulated in a compatible manner? This intuitive question is the topic of the present paper. Apart from the theoretical interest in this basic problem, questions of this kind arise in various areas of application, including image analysis, morphing, and cartography.

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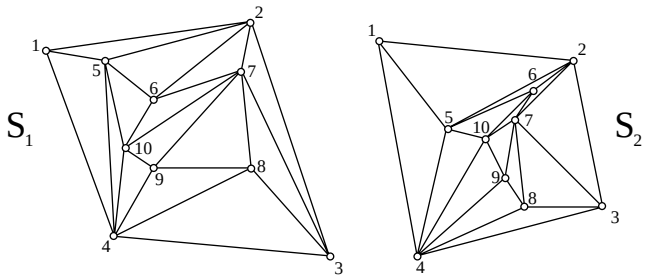

Fig. 1. These two triangulations are compatible.

Morphing, i.e., continuous transformation of one shape into another, is a particularly important area due to the wide use in animation, modeling, and computer graphics where it is also called *shape blending* or *metamorphosis* [4,13]. The problem of morphing polygons or planar triangulations has been attracting a lot of research [6–8,14,15]. In [6], Floater and Gotsman introduce an efficient approach for morphing compatible triangulations, which is further explored by Surazhsky and Gotsman [15]. For an application in cartography, see [12].

In the present work we state—and give some evidence for—the surprising conjecture that the obvious necessary conditions for compatible triangulations to exist are also sufficient. We further show that compatible triangulability can always be achieved by adding a small number of extraneous points.

1.1. Basic observations

To define the problem in question more rigorously, let S_1 and S_2 be two finite sets of points in the Euclidean plane. Two triangulations (i.e., two-dimensional simplicial complexes) T_1 of S_1 and T_2 of S_2 are called strongly isomorphic or *compatible* if the face lattices formed by their triangles, edges, and vertices (points) are isomorphic. Fig. 1 depicts an example. An equivalent and more intuitive definition is the following: the triangulations T_1 and T_2 are compatible if there exists a bijection φ between S_1 and S_2 such that ijk is a triangle in T_1, empty of points of S_1, if and only if $\varphi(i)\varphi(j)\varphi(k)$ is a triangle in T_2, empty of points of S_2. Compatible triangulations obviously exhibit isomorphic graphs of edges. The converse is not true, in general. An edge isomorphism between T_1 and T_2 need not preserve the external face; see [11] (where Fig. 2 has been taken from) for a careful discussion of this phenomenon. This led us to prefer the notion of *compatible triangulations* to *isomorphic triangulations* which sometimes has been used in the literature.

The problem of triangulating two given point sets compatibly comes in two flavors, namely where the bijection between the points of S_1 and S_2 is either fixed in advance, or variable. The case of fixed correspondence is a known problem which has been studied by Saalfeld [12]. He pointed out that compatible triangulations do not always exist and proposed several heuristic approaches for their construction. A challenging problem left open in this context is to determine the complexity of the related decision

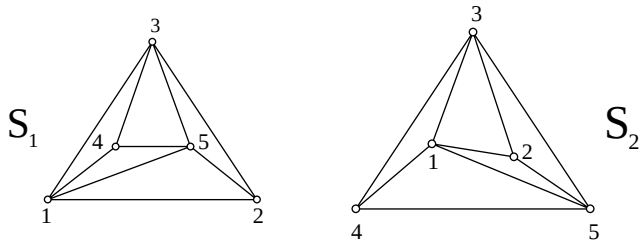

Fig. 2. Edge isomorphism need not preserve the external face.

problem, which so far is neither known to be NP-complete nor to be polynomially solvable.

The problem becomes easier if S_1 and S_2 are the (ordered) sets of vertices on the boundary of two simple n-gons. Given a bijection between the polygon vertices (there are only $2n$ cyclic shifts to choose from) the existence can be decided in time $O(n^2)$, and compatible triangulability of the polygons can always be achieved by adding $O(n^2)$ extraneous points (so-called Steiner points) in each polygon; see [3]. Improvements which take into account the number of reflex vertices or the quality of the angles of the produced triangles exist. See [10,15], respectively.

The present paper is concerned with the problem of finding compatible triangulations *without* fixed point correspondence. In other words, we ask whether there *exists* a bijection between two point sets S_1 and S_2 which allows compatible triangulations. To our knowledge this problem has not been studied before. The main question is, of course, under which propositions compatible triangulations do exist. It is clear that (1) in both sets S_1 and S_2 the number n of points must be the same, as has to be the number h of *extreme* points (points on their convex hulls $CH(S_1)$ and $CH(S_2)$), by the well-known formula $e = 3n - h - 3$ on the number e of edges of a triangulation. Moreover, (2) extreme points necessarily map to extreme points, and their cyclic order on the convex hulls is preserved. Also, (3) for each point, the sorted order of its adjacent points in the triangulation is preserved.

Properties (2) and (3) follow from a more general fact pointed out by Saalfeld [12]: (4) If two triangulations are compatible then any two corresponding (empty) triangles ijk and $\varphi(i)\varphi(j)\varphi(k)$ must have the same orientation, either clockwise or counter-clockwise. (We ignore the symmetric case where the orientation is reversed for each matching pair of triangles.) Intuitively speaking, this means that the underlying point sets S_1 and S_2 locally exhibit the same order type, as defined by Goodman and Pollack [9]. The *order type* of a finite point set is a mapping that assigns an orientation to each ordered triple of points. It is interesting to note that Properties (2)–(4) need *not* hold if only graph isomorphism rather than compatibility of the triangulations is required; compare Fig. 2.

We will assume throughout the paper that all point sets considered are in *general position*, meaning that no three points within a set lie on a common straight line. There are simple examples of point sets not in general position that do not admit compatible triangulations even if the propositions mentioned above are all met. Fig. 3 displays

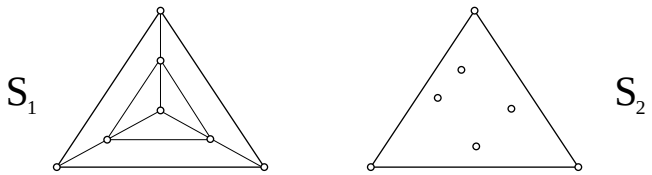

Fig. 3. No compatible triangulations exist.

such an example. All edges shown in the left-hand side triangulation are *unavoidable*, in the sense that they have to appear in every triangulation of the point set S_1. In particular, the complete graph K_4 on the four interior (i.e., non-extreme) points of S_1 is unavoidable. For the point set S_2, however, we cannot create such a K_4 without introducing edge crossings, because the four interior points of S_2 are in convex position.

1.2. The conjectures

Throughout, let $|CH(S)|$ denote the number of extreme points in a point set S. Our main conjecture can be formulated in the following way.

Conjecture 1 (Main conjecture on compatible triangulations). *Let S_1 and S_2 be two sets of points in the plane, without fixed correspondence. Then compatible triangulations for S_1 and S_2 exist if these sets fulfill the following properties*:
- $|S_1| = |S_2|$,
- $|CH(S_1)| = |CH(S_2)|$, *and*
- S_1 *and* S_2 *are in general position.*

We also consider an extended version of the compatible triangulation problem, where the correspondence between the extreme points (but not between the interior points) in the two sets is prescribed. The status of the extended problem is unresolved as well.

Conjecture 2 (Extended conjecture on compatible triangulations). *Conjecture* 1 *still holds if the bijection between the extreme points of S_1 and S_2 is prescribed by a cyclically shifted labeling.*

The motivation for stating the extended conjecture stems from the attempt to prove Conjecture 1 by induction, which naturally leads to assuming the stronger version as a hypothesis.

The intention of the present paper is to undertake first steps towards proving the conjectures above. In fact, obtaining an affirmative answer would be a strong theoretical result, showing that all planar point sets of the same size and hull size are 'topologically equivalent' in this very sense. The result, if true, gains in importance in view of the huge number of inequivalent order types for n points; see [1] for the exact numbers of order types for $n \leqslant 10$.

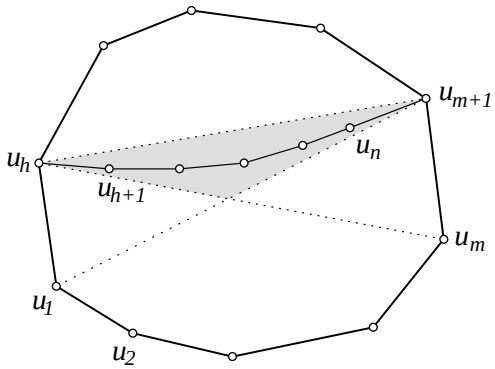

Fig. 4. Point set U.

The paper is organized as follows. Section 2 exhibits a family of point sets each of whose members allows for a compatible triangulation with any other point set (within the evident size and hull restrictions). In Section 3 we prove Conjecture 2 provided the number of interior points in either set is three or less. These restricted cases already reflect part of the intrinsic complexity of the problem: the orientation of the three interior points need to be reversed to allow for a solution in certain cases. Section 4 shows that compatible triangulability can always be achieved by adding at most $k-3$ Steiner points, where k denotes the number of interior points. In fact, a linear number of Steiner points suffices for point sets of arbitrary size and shape. This contrasts the afore-mentioned situation for simple polygons (see [3]) where a quadratic number of Steiner points is necessary in the worst case, and is a result of practical relevance. Finally, Section 5 concludes the paper with some related remarks.

2. A family of universal sets

A point set U is called *weak universal* if for every point set S with $|S|=|U|$ and $|CH(S)|=|CH(U)|$ there is a triangulation of S compatible with a triangulation of U. The set U is called *universal* if, in addition, the bijection between the extreme points of S and U may be prescribed by a cyclic shift. The existence of a (weak) universal point set does *not* resolve our conjectures, because the compatibility relation is not transitive. Conjecture 2, if true, states that *all* point sets are universal. Here we prove that certain point sets are indeed universal.

Let U be a point set with extreme points $u_1,\ldots,u_h$ (as they appear counter-clockwise on the convex hull) and internal points $u_{h+1},\ldots,u_n$ such that, for some $1\leqslant m\leqslant h-2$,

- the subset $U\backslash\{u_1,\ldots,u_m\}$ is in convex position, and
- the internal set $\{u_{h+1},\ldots,u_n\}$ lies to the left of both u_hu_m and u_1u_{m+1}; see Fig. 4.

We are going to prove that U is a universal set. The proof of the result requires a preliminary lemma.

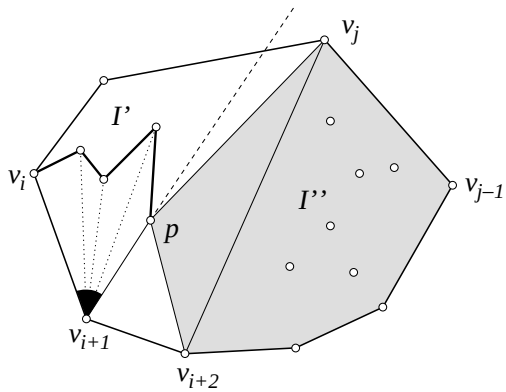

Fig. 5. Constructing the triangulation $T(\pi)$.

Lemma 1. *Let P be a convex polygon with vertices $v_1,\ldots,v_h$ in counter-clockwise order, and let I be an arbitrary finite point set interior to P. For any choice of indices $1\leqslant i,\ j\leqslant h$ with $i\leqslant j-2$ there exists a simple path π of edges such that*

(a) *path π connects v_i, I, and v_j (in this order) and these are the only vertices of π;*

(b) *the simple polygon bounded by π and $v_iv_{i+1}\ldots v_j$ can be triangulated without using an edge that connects two vertices of π.*

Proof. By induction on $d=j-i\geqslant 2$, the distance between v_i and v_j on P's boundary. For $d=2$ the result is trivial: sort $I\cup\{v_i,v_j\}$ radially around v_{i+1}, to produce a path star-shaped as seen from v_{i+1}, and triangulate by connecting all points in I to this vertex.

Let now $d>2$. If the set I is totally to the right of $v_{i+2}v_j$ then connect v_i and v_{i+2} and slice off P the triangle $v_iv_{i+1}v_{i+2}$ which cannot contain any point in I. Apply induction to the remaining convex polygon; the boundary distance between v_i and v_j is now $d-1$.

If, on the other hand, there are points of I to the left of $v_{i+2}v_j$ then let among those p be the one maximizing the angle $v_iv_{i+1}p$; see Fig. 5. Connect p to v_{i+1} and v_{i+2} and observe that the resulting triangle encloses no points of I. Now consider the convex polygon $Q=pv_{i+2}\ldots v_j$ (shaded in Fig. 5). Let I' and I'' be the subsets of I exterior and interior to Q, respectively. By construction, no point of I' lies to the right of $v_{i+1}p$. So, as done for the case $d=2$ above, construct the path π' for $I'\cup\{v_i,p\}$ that is star-shaped as seen from v_{i+1}, and build the corresponding triangulation. Finally, apply induction to the polygon Q whose boundary distance between p and v_j is $d-1$. This yields a simple path π'' for $I''\cup\{p,v_j\}$ whose concatenation with π' obviously is still simple and satisfies the statement of the lemma. □

Theorem 1. *Let U be a set as described above. Then U is a universal set.*

Proof. Let S be any point set with $|S| = |U| = n$ and with extreme points $v_1, \ldots, v_h$ in counter-clockwise order. We construct for $CH(S)$ the path π from v_h to v_{m+1}, and the associated triangulation $T(\pi)$ on π's right-hand side, whose existence is given by Lemma 1. (Note that the index m is given from U and that the counter-clockwise boundary distance between v_h and v_{m+1} is at least two.) The crucial property is that $T(\pi)$ can be reproduced on the simple polygon $u_h \ldots u_n u_{m+1} u_m \ldots u_1$: only edges between non-consecutive vertices of the path $u_h \ldots u_n u_{m+1}$ lie outside this polygon, but those correspond to edges outruled by the definition of $T(\pi)$. It remains to be observed that any triangulation of $CH(S)$ on π's left-hand side can be put on top of the convex polygon $u_{m+1} u_{m+2} \ldots u_n$, making compatible the global triangulations. □

3. Few interior points

Below we prove Conjecture 2 for point sets with at most three interior points. Even in these seemingly simple situations, the illusion of a quick proof is destroyed by the fact that the orientation of the three interior points might have to be reversed to achieve compatibility. We first focus on the following class of point sets.

Lemma 2. *Let S be a point set containing an extreme point p such that $S \backslash \{p\}$ is in convex position. Then S is universal.*

Proof. The set S fulfills the requirements for the set U in Section 2 for $m = 1$ and thus is a universal set by Theorem 1. □

Theorem 2. *Let S_1 and S_2 be two n-point sets each of which contains $k \leqslant 3$ interior points. Then Conjecture 2 is true.*

Proof. The cases $k = 0$, where S_1 and S_2 are in convex position, and $k = 1$, where star-like compatible triangulations always exist, are trivial.

The case $k = 2$ is still easy. Consider two extreme points u and v of S_1 which lie on different sides of the line through the two interior points of S_1. Then u and v can be chosen such that the corresponding (extreme) points of S_2 lie on different sides of the line through the two interior points of S_2 as well. (Otherwise, by the same cyclic order of extreme points, one of S_1 and S_2 totally would lie on a fixed side of the respective line, which is impossible.) We build two triangles for S_1 by joining its interior points x and y and connecting u and v to both of them. The remaining extreme points can be connected to either x or y accordingly. Doing the same for S_2 results in compatible triangulations.

The case $k = 3$ is more intriguing. Let us refer to the triangle formed by the three interior points of S_1 (or S_2) as the *interior triangle* of S_1 (or S_2). We distinguish between two situations I and II below. The former is similar to the case $k = 2$, in that the solution can be forced to contain the interior triangles. Let xyz be this triangle for S_1. Consider three extreme points of S_1, labeled u, v, and w say, such that the four

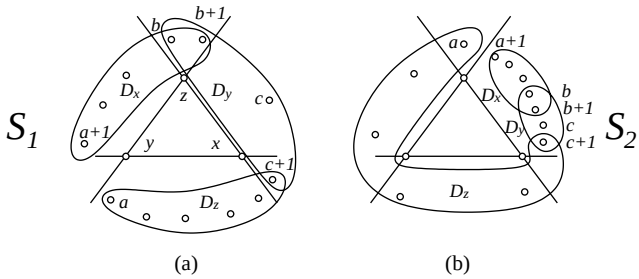

Fig. 6. Two subsets move into the same trilateral cell.

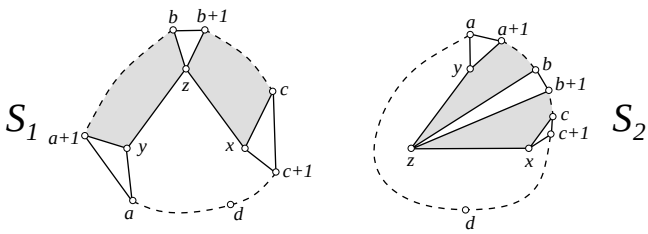

Fig. 7. Compatible edge skeletons that yield convex parts.

triangles uxy, vyz, wzx and xyz pairwise do not overlap. Observe that such points always exist.

Case I: The points labeled u, v, and w above can be chosen such that, for the four triangles spanned by the corresponding points in S_2 and the interior triangle in S_2, no overlap occurs as well. In this case, we integrate these four triangles for both sets. For extreme points with labels intermediate to u, v, and w, edges can be drawn now to a unique interior point, in a compatible manner and much like the case $k=2$.

Case II: Otherwise. The solution we are going to construct will not contain the interior triangle xyz. (In fact, no such solution exists.) Let D_x be the subset of S_1 not on x's side of the line through y and z. Define subsets D_y and D_z analogously; see Fig. 6(a). Since labels as in I do not exist, the subsets of S_2 that correspond to D_x, D_y, and D_z have to look as in Fig. 6(b). Two subsets are totally contained in the same trilateral region, whereas the third subset covers the rest of the convex hull of S_2. Without loss of generality, let the third subset correspond to D_z. We wish to reduce D_x, D_y, and D_z in size so as to become disjoint sets, by introducing 'breakpoint' labels a, b, and c such that D_x ranges from $a+1$ to b, D_y ranges from $b+1$ to c, and D_z ranges from $c+1$ to a. This is always possible unless two of the sets are singletons consisting of the same point p. In this case, however, removal of p leaves the set $S_1 \setminus \{p\}$ in convex position, and compatible triangulations exist by Lemma 2. Otherwise, we assign labels x, y, and z to the interior triangle of S_2 and draw compatible edges as shown in Fig. 7. Note that the orientation of xyz has been reversed for S_2. The two shaded polygons for S_1 are convex and thus can be triangulated compatibly with

their counterparts for S_2. It remains to partition the still untriangulated area for S_1 into convex parts. Possible reflex angles may occur at x or y. If, as in Fig. 7, this happens for x then there must exist a point labeled d in $D_z \backslash D_y$ which makes this angle convex. (If d would not exist then all extreme points of S_2 would lie on the same side of the line through x and z.) The situation for y is similar. This completes the construction of compatible triangulations. □

4. When Steiner points are allowed

Since it is still left open whether Conjectures 1 and 2 are true, the question of whether compatibility can be always achieved by adding a reasonably small number of extraneous points (called Steiner points) suggests itself. We demonstrate below that, unlike the case for polygon triangulations (where a quadratic lower bound exists, see [3]) a linear number of Steiner points suffices. An efficient algorithm for finding compatible triangulations, which is of practical relevance, is implicit in the proofs.

There is a helpful property which stems from geometric arguments.

Lemma 3. *Let A and B be two point sets with triangular convex hulls in fixed correspondence, and with k and ℓ interior points, respectively. There exist sets $A' \supset A$ and $B' \supset B$ with $k+\ell$ interior points, such that every triangulation of A' is compatible to some triangulation for B'.*

Proof. Let ϕ be the unique orientation preserving affine transformation that maps $CH(A)$ to $CH(B)$ by respecting the given bijection between the hull points. Let $B' = B \cup \phi(A)$ and $A' = A \cup \phi^{-1}(B)$. Then A' and B' are of the same order type. Therefore, any triangulation for A' maps to a valid (compatible) triangulation for B. □

Now consider two point sets S_1 and S_2 with the same number $h \geqslant 3$ of extreme points, but with possibly different numbers k_1 and k_2 of interior points. Construct compatible triangulations for their convex hulls $CH(S_1)$ and $CH(S_2)$. Then, apply Lemma 3 to each corresponding pair of triangles in these triangulations. This generates compatible triangulations for S_1 and S_2 with at most $\max\{k_1, k_2\}$ Steiner points. More generally, if the numbers of extreme points of two point sets do *not* agree, we can first add Steiner points to adapt in size the smaller convex hull to the larger one, and then proceed as before. We obtain the following general assertion.

Theorem 3. *For $i = 1, 2$ let S_i be a point set with h_i extreme points and k_i interior points. Then S_1 and S_2 can be triangulated compatibly by introducing at most $|h_1 - h_2| + \max\{k_1, k_2\}$ Steiner points per set. The total number of introduced points is at most $|h_1 - h_2| + k_1 + k_2$.*

Theorem 3 is tight for the construction it is based on, but can be slightly improved in certain cases by exploiting Theorem 2.

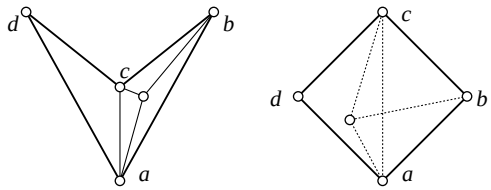

Fig. 8. Small example without compatible triangulation.

Theorem 4. *Let S_1 and S_2 be two n-point sets whose $n-k$ extreme points are in fixed correspondence. For $k \geqslant 3$, there exist compatible triangulations using at most $k-3$ Steiner points per set.*

Proof. Recall from Theorem 2 that no Steiner points are needed if $k \leqslant 3$. Let us assume $k \geqslant 4$. Select any two triples T_1 and T_2 of interior points in S_1 and S_2, respectively. Construct compatible triangulations for $S_1 \cup T_1$ and $S_2 \cup T_2$, following the proof of Theorem 2. Applying Lemma 3 now yields at most $k-3$ Steiner points, since 3 interior points have already been used in the previous construction step. □

5. Remarks

It is a challenging open problem to prove our conjectures for general point sets. Unfortunately, the constructions in the proof of Theorem 2 do not seem to generalize easily to the case of more interior points. Approaches to prove Conjecture 2 by induction suffer from the lack of appropriate methods for splitting the problem. Here is a discouraging related fact: given two quadrilaterals, one convex and the other star-shaped, plus one interior point for each, it may not be possible to triangulate them compatibly if the correspondence between their boundary vertices is fixed by a cyclic shift. See Fig. 8, where all triangulation edges for the star-shaped polygon are unavoidable, and the corresponding edges for the convex polygon cross.

We tested Conjecture 2 for small point set sizes, utilizing a database of combinatorially inequivalent point sets (i.e., order types); see [1,2]. If two point sets exhibit the same order type then every triangulation of one set also leads to a compatible triangulation for the other. By checking all different pairs of order types we could verify the conjecture for up to 9 points. Although our database provides all order types of size up to 10, exhaustive tests turned out to be time consuming and are still incomplete for size 10. In fact, determining the time complexity of computing compatible triangulations, if they exist, for two given n-point sets is an open problem.

Finally, we pose the problem of triangulating compatibly two point sets in *fixed* correspondence when Steiner points are allowed. In view of the applications mentioned in [12], a fast algorithm using a small number of Steiner points would be of practical relevance.

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