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SPACE GRAPHS AND SPHERICITY

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For a finite point set in Euclidean n-space, if we connect each pair of points by a line segment whenever the distance between them is less than a certain positive constant, we obtain a *space graph* in n-space. The *sphericity* of a graph G is defined to be the minimum number n such that G is isomorphic to a space graph in n-space. In this paper we study the sphericities of graphs and present upper bounds on the sphericity for several types of graphs.

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and the order of G, i.e. the size of $V(G)$, is denoted by $|G|$.

Let $x_1, \dots, x_m$ be m different points in Euclidean n-space R^n and let t be a positive number. If we connect each pair $x_i, x_j, i \neq j$, by a line segment whenever $\|x_i - x_j\| < t$, we obtain a geometric graph ($\|\cdot\|$ denotes the Euclidean norm). The graph so formed is called a *space graph* in n-space, and is denoted by $\mathbb{G}(x_1, \dots, x_m; t)$. The number t is called its *adjacency limit*. A space graph in n-space is, as an abstract graph, nothing but the intersection graph of a family of equiradial n-discs in n-space.

It is easily seen that for any graph G, there exists a space graph in some dimension which is isomorphic to G. The smallest n such that G is isomorphic to a space graph in n-space is called the *sphericity*[1] of G, and is denoted by $\mathrm{sph}(G)$. The sphericity of a graph is a type of dimension of a graph, similar to that of the boxicity and the cubicity of a graph [11].

The graphs of sphericity 1 are intersection graphs of families of unit intervals, and are studied in [7] and [10] (see also [4, Ch. 8]). The coloring problem for graphs of sphericity two has an important application to the frequency assignment problem [5]. To characterize those graphs of sphericity ≤ 3 is very important in the calculation of molecular conformation, as described in Havel [6], but it is a difficult problem.

For a few types of graphs, the sphericities can be easily found. For example, $\mathrm{sph}(K_1) = 0$, $\mathrm{sph}(K_n) = \mathrm{sph}(P_n) = 1$ for $n > 1$, and $\mathrm{sph}(C_n) = 2$ for $n > 3$, where K_n,

[1] More precisely, the unit sphericity.

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P_n, and C_n denote the complete graph, the path graph, and the circuit graph, of order n, respectively. As a less trivial example, we have $\mathrm{sph}(K_{m(2)})=2$ for $m>1$, where $K_{m(2)}$ is the complete m-partite graph on m sets of size 2. However, to estimate $\mathrm{sph}(G)$ for a general graph G is difficult.

In this paper, we present some upper bounds on sphericity. The most general result is that if G is not a complete graph, then $\mathrm{sph}(G)\le|G|-\omega(G)$, where $\omega(G)$ denotes the clique number of G, i.e. the size of the largest complete subgraph of G. If G is a split graph [2] and $|G|-\omega(G)>2$, then it is shown that $\mathrm{sph}(G)\le|G|-\omega(G)-1$. For a tree T whose every vertex has degree $\le k$, $k\ge 3$, we have $\mathrm{sph}(T)\le\frac{1}{2}(k+1)\log_k|T|$. For the complete bipartite graph $K_{m,n}$, we have $\mathrm{sph}(K_{m,n})\le m-1+\{\frac{1}{2}n\}$, where $\{r\}$ denotes the upper integral approximation of r.

2. The sphericity

Let $x_1,\ldots,x_m$ be (not always different) points in R^n, $n>0$, and let $t>0$. Then we can take y_i close to x_i, $i=1,\ldots,m$ so that (i) $y_1,\ldots,y_m$ are all different, and (ii) $\|x_i-x_j\|<t$ if and only if $\|y_i-y_j\|<t$, for all $i\neq j$. If all x_i's are not different, we will use the notation $\mathbb{G}(x_1,\ldots,x_m;t)$ to represent one of the space graphs $\mathbb{G}(y_1,\ldots,y_m;t)$.

If G is a graph with vertex set $\{v_1,\ldots,v_n\}$, then its adjacency matrix $\mathbb{A}(G)$ is defined to be the $n\times n$ matrix (a_{ij}), in which $a_{ij}=1$ or 0 according as v_i and v_j are adjacent or not. We denote by $\mathbb{I}_n$ the identity $n\times n$ matrix and by $\mathbb{J}_n$ the $n\times n$ matrix each entry of which is 1.

Theorem 1. *Every graph of order n is isomorphic to some space graph in n-space.*

Proof. Let $\{v_1,\ldots,v_n\}$ be the vertex set of G and let x_i be the point in R^n corresponding to the ith row of the matrix $\mathbb{A}(G)+n\mathbb{I}_n$, $i=1,\ldots,n$. Then

$$\begin{aligned}\|x_i-x_j\|^2 &\le 2n^2-3n && \text{if } a_{ij}=1,\\ &\ge 2n^2 && \text{if } a_{ij}=0.\end{aligned}$$

Hence $\mathbb{G}(x_1,\ldots,x_n;\sqrt{2n^2-n})$ is isomorphic to G.

This theorem permits us to define the sphericity of a graph G, $\mathrm{sph}(G)$, as the smallest n such that G is isomorphic to a space graph in n-space.

Example. Let $K_{m(2)}$ be the complete m-partite graph on m sets of size 2. Then $\mathrm{sph}(K_{m(2)})=2$ for $m>1$.

To see this, let x_i, y_i, $i=1,\ldots,m$ be the endpoints of m distinct diameters of a unit disc in the plane. Then $K_{m(2)}$ is isomorphic to $\mathbb{G}(x_1,y_1,\ldots,x_m,y_m;2)$. Hence $\mathrm{sph}(K_{m(2)})\le 2$ but clearly $\mathrm{sph}(K_{m(2)})\ge 2$ for $m>1$.

A subset U of the vertex set $V(G)$ of a graph G is called *independent* if no edge of G has both end vertices in U. A set $U \subset V(G)$ is called *complete* if any two distinct vertices of U are adjacent.

Theorem 2. *If G is not a clique,* $\mathrm{sph}(G) \le |G| - \omega(G)$.

Proof. Let $k = \omega(G)$, $l = |G| - \omega(G)$, and let $V(G) = \{v_1, \ldots, v_{k+l}\}$. Assume that $\{v_1, \ldots, v_k\}$ is the complete set determining $\omega(G)$. Then the adjacency matrix $\mathbb{A}(G) = (a_{ij})$ of G takes the form of

$$\begin{pmatrix} \mathbb{K}_k & \mathbb{A} \\ \mathbb{A}' & \mathbb{B} \end{pmatrix},$$

where $\mathbb{K}_k = \mathbb{J}_k - \mathbb{I}_k$ and $\mathbb{A}'$ denotes the transpose of $\mathbb{A}$. Let x_i be the point in R^{l+1} corresponding to the ith row of the $(k+l) \times (l+1)$ matrix

$$\left(\begin{array}{c|c} l & \\ \vdots & \mathbb{A} \\ l & \\ \hline 0 & \\ \vdots & \mathbb{B} + l\mathbb{I}_l \\ 0 & \end{array}\right).$$

Then for $i < j \le k$, $\|x_i - x_j\|^2 \le l$. For $k < i < j$, we have $\|x_i - x_j\|^2 \le 2l^2 - 3l$ or $\ge 2l^2$ according as $a_{ij} = 1$ or 0. And for $i \le k < j$, we have $\|x_i - x_j\|^2 \le 2l^2 - l$ or $\ge 2l^2$ according as $a_{ij} = 1$ or 0. Hence

$$\mathbb{G}(x_1, \ldots, x_{k+l}; \sqrt{2l^2 - l^2/(l+1)})$$

is isomorphic to G.

To see $\mathrm{sph}(G) \le l$, we project $x_1, \ldots, x_{k+l}$ into the hyperplane in R^{l+1} passing through the origin and perpendicular to $(1, \ldots, 1)$, by the projection with matrix

$$\mathbb{P} = \mathbb{I}_{l+1} - \frac{1}{l+1}\mathbb{J}_{l+1}.$$

By a simple calculation we have

$$\|(x_i - x_j)\mathbb{J}_{l+1}\|/(l+1) \le l/\sqrt{l+1}.$$

Hence, if we let $x_i^* = x_i\mathbb{P}$, $i = 1, \ldots, k+l$, we have

$$\mathbb{G}(x_1^*, \ldots, x_{k+l}^*; \sqrt{2l^2 - l^2/(l+1)}) \cong G.$$

Therefore, $\mathrm{sph}(G) \le l = |G| - \omega(G)$.

3. Split graphs

A graph is called a *split graph* [2] if the vertex set can be partitioned into a complete set and an independent set. If G is a split graph, then it is easily seen that $V(G)$

can be partitioned into a complete set of size $\omega(G)$ and an independent set (see e.g. [4, p 150]).

Theorem 3. *If G is a split graph and $|G|-\omega(G)>2$, then*

$$\mathrm{sph}(G) \le |G|-\omega(G)-1.$$

We cannot drop the assumption $|G|-\omega(G)>2$, for $|K_{1,3}|-\omega(K_{1,3})-1=1$ but $\mathrm{sph}(K_{1,3})=2$.

Before we proceed to the proof, we state a lemma without proof. The *circimradius* of a bounded subset in R^n is the radius of the smallest closed n-disc that includes all points of the subset.

Lemma 1. *Let $\Delta^m \subset R^m$ be a regular m-simplex of circumradius 1 with center at the origin of R^m and let $p_1, \ldots, p_{m+1}$ be the vertices of Δ^m. Then the inner product (p_i, p_j), $i \ne j$, equals $-1/m$ and the side-length of Δ^m equals $\sqrt{2(m+1)/m}$.*

Proof of Theorem 3. Let $k=\omega(G)$, $m=|G|-\omega(G)-1$ and let $V(G)=\{u_1, \ldots, u_k, v_1, \ldots, v_{m+1}\}$, where $\{u_1, \ldots, u_k\}$ is a complete set of size k and $\{v_1, \ldots, v_{m+1}\}$ is an independent set. Then the adjacency matrix $\mathbb{A}(G)=(a_{ij})$ of G takes the form of

$$\begin{pmatrix} \mathbb{K}_k & \mathbb{A} \\ \mathbb{A}' & \mathbb{O} \end{pmatrix},$$

where $\mathbb{K}_k=\mathbb{J}_k-\mathbb{I}_k$, and $\mathbb{O}$ is 0-matrix of size $(m+1)\times(m+1)$. Let $\mathbb{A}=(b_{ij})$, i.e. $b_{ij}=a_{i(k+j)}$.

We begin with the case $m>2$. Let $\Delta^m \subset R^m$ be a regular m-simplex of circumradius 1 centered at the origin, and let $p_1, \ldots, p_{m+1}$ be its vertices, as in Lemma 1. Denote by s_i the sum of the entries in the ith row of $\mathbb{A}$. We define the point x_i by

$$(*) \qquad x_i = \begin{cases} -3p_{m+1} & \text{if } s_i=0, \\ \sum_{j=1}^{m} (b_{ij}/s_i)p_j & \text{if } s_i \ne 0 \end{cases}$$

for $i=1, \ldots, k$. (If $s_i \ne 0$, then x_i is the barycenter of those p_j's such that $b_{ij}=1$.) Let $y_j=4p_j$, $j=1, \ldots, m+1$. Then x_i, y_j are points in R^m. Clearly

$$\text{(i)} \qquad \|x_i-x_j\| \le 4,$$

and by Lemma 1, we have

$$\text{(ii)} \qquad \|y_i-y_j\| = 4\sqrt{2(m+1)/m} > 4 \quad \text{for } i \ne j.$$

If $b_{ij}=1$ (i.e. u_i and v_j are adjacent), then using Lemma 1 we have

$$\begin{aligned} \|x_i-y_j\|^2 &= \left\| \sum_{h \ne j} (b_{ih}/s_i)p_h + (1/s_i-4)p_j \right\|^2 \\ &= 4^2-7(m+1-s_i)/(ms_i) \le 4^2. \end{aligned}$$

If $s_i \neq 0$ and $b_{ij}=0$, then we have

$$\|x_i - y_j\|^2 = \left\| \sum_h (b_{ih}/s_i) p_h - 4p_j \right\|^2$$
$$= 4^2 + (m+1+7s_i)/(ms_i) > 4^2.$$

And if $s_i = 0$, then we have

$$\|x_i - y_j\|^2 = \|-3p_{m+1} - 4p_j\|^2 = 3^2 + 4^2 + 24(p_{m+1}, p_j)$$
$$\geq 3^2 + 4^2 - 24/m > 4^2.$$

Thus we have

(iii) $\|x_i - y_j\| \leq 4$ or >4 according as u_i and v_j are adjacent or not.

Now by (i), (ii), and (iii), it follows

$$\mathbb{G}(x_1, \ldots, x_k, y_1, \ldots, y_{m+1}; 4+\varepsilon) \cong G$$

for sufficiently small $\varepsilon > 0$. Hence $\mathrm{sph}(G) \leq m = |G| - \omega(G) - 1$.

To prove the theorem for $m=2$, we now let $p_1 = (-1, 0)$, $p_2 = (1, 0)$, $p_3 = (0, 1)$, $y_1 = (-4, 0)$, $y_2 = (4, 0)$, $y_3 = (0, 13/3)$ in R^2, and define the x_i's by $(*)$. Then, by a simple calculation, it can be verified that $\mathbb{G}(x_1, \ldots, x_k, y_1, y_2, y_3; \sqrt{33/2})$ is isomorphic to G. Hence $\mathrm{sph}(G) \leq 2$ in this case. This completes the proof.

The next theorem is proved by a type of argument similar to that used in [1, p. 110].

Theorem 4. *For every positive integer m, there exists a split graph G such that* $\mathrm{sph}(G) = |G| - \omega(G) - 1 = m$.

Proof. For $m=1$, we may take $G = P_4$, the path graph of order 4. Suppose now $m > 1$. Let $k = 2^{m+1}$ and let $A_1, \ldots, A_k$ be k distinct subsets of $\{1, 2, \ldots, m+1\}$. (There are just k distinct subsets.) Define G to be the graph obtained from the complete graph with vertex set $\{u_1, \ldots, u_k\}$ by adding $m+1$ vertices $v_1, \ldots, v_{m+1}$ and adding edges $\{u_i, v_j\}$ such that $j \in A_i$, $i = 1, \ldots, k$. Then G is a split graph with $\omega(G) = k$. To see $\mathrm{sph}(G) \geq m$, suppose that $\mathbb{G}$ is a space graph in R^n of adjacency limit t which is isomorphic to G, and let $y_1, \ldots, y_{m+1}$ be the vertices of $\mathbb{G}$ corresponding to $v_1, \ldots, v_{m+1}$. Denote by D_j the open n-disc in R^n of radius t centered at y_j, $j = 1, \ldots, m+1$. Since u_i is adjacent to all of the v_j's such that $j \in A_i$, but none of the v_j's such that $j \notin A_i$, the set

$$\left\{ \bigcap_{j \in A_i} D_j \right\} \cap \left\{ \bigcap_{j \notin A_i} D_j^{\mathrm{c}} \right\}$$

(where D_j^{c} denotes the complement of D_j, i.e. $D_j^{\mathrm{c}} = R^n - D_j$), must contain the vertex corresponding to u_i, and hence, must be nonempty, for $i = 1, \ldots, k$. This means $\{D_1, \dots, D_{m+1}\}$ is an independent family [9] of discs in R^n. Since the number of n-discs in an independent family of n-discs in R^n is at most $n+1$ (see [9]), we obtain $m+1 \le n+1$. Hence

$$m \le \mathrm{sph}(G) \le |G| - \omega(G) - 1 = m.$$

4. Trees

If G is a graph having sphericity $\le n$, then it is clear that there is a space graph isomorphic to G whose very vertex is *on* a given n-sphere S^n in R^{n+1}. Furthermore, if the radius of the n-sphere S^n is large enough, then it is possible to take $|G|$ points on S^n in such a way that they generate, with adjacency limit 1, a space graph isomorphic to G.

Lemma 2. *Let G_1, G_2, G_3 be three connected graphs. Connect them by two edges in any way, and let G be the resulting connected graph. Then*

$$\mathrm{sph}(G) \le \max\{\mathrm{sph}(G_i); i = 1, 2, 3\} + 1.$$

Proof. Let $n = \max\{\mathrm{sph}(G_i); i = 1, 2, 3\}$. Let S_1^n, S_2^n be two n-spheres of sufficiently large radii, and let H^n be a hyperfine, in R^{n+1}. Take three point sets, V_1 in S_1^n, V_2 in S_2^n, V_3 in H^n such that V_i generates, with adjacency limit 1, a space graph isomorphic to G_i, $i = 1, 2, 3$. Then it is possible to move and rotate S_1^n and S_2^n so that $V_1 \cup V_2 \cup V_3$ generates, with adjacency limit 1, a space graph isomorphic to G. (See Fig. 1, in which two new edges are indicated by e_1 and e_2.) Hence $\mathrm{sph}(G) \le n+1$.

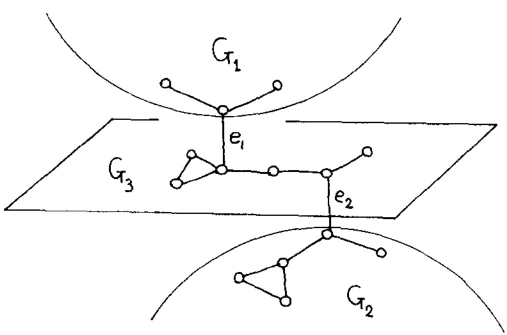

Fig. 1.

Let T be a tree. A *branch* at a vertex v of T is a maximal subtree of T containing v as an end-vertex (i.e. a vertex of degree 1). The *weight* at a vertex v of T is the maximum number of edges in any branch at v. A vertex u is a *mass center* of T if u has minimum weight. If T is of order n then the weight at a mass center is $\le \frac{1}{2}n$, see [8, p. 66].

We denote by $f_k(n)$ the maximum sphericity among the trees of order n whose every vertex has degree $\leq k$, and let $f_k(0)=0$. The integer part of a real number $r>0$ is denoted by $[r]$, and $f_k(r)$ is shorthand for $f_k([r])$.

Lemma 3. *For $n,k\geq 3$, $f_k(n)\leq \max\{f_k(n/i)+[\frac{1}{2}(i+1)];\ 2\leq i\leq k,\ i\neq 3\}$.*

Proof. Let T be a tree of order n such that the maximum degree of T is $\leq k$ and $\mathrm{sph}(T)=f_k(n)$. Let v be a mass center of T and d be its degree. Then $2\leq d\leq k$. Let $B_1, B_2, \dots, B_d$ be the branches at v and let n_i be the number of edges in B_i (i.e. $n_i=|B_i|-1$), $i=1,\dots,d$. We may assume $n_1\geq n_2\geq\cdots\geq n_d$. Since v is a mass center of T, we have $n_1\leq\frac{1}{2}n$. Denote by B_i^* the tree B_i-v (the removal of v from B_i) and denote by T_i the tree consisting of branches $B_i, B_{i+1},\dots,B_d$; $i=1,\dots,d$. Let T_{d+1} be the single-vertex-tree 'v'. Note that $|B_i^*|=n_i$, and that T_i is obtained from B_i^*, B_{i+1}^*, and T_{i+2} by adding two edges, each $i<d$. Let j be the minimum of i such that $n_i\geq|T_{i+1}|$, that is, $n_j\geq|T_{j+1}|$, and $n_i<|T_{i+1}|$ for $i<j$. Then $j\leq d$ and we must have $n_i\leq n/(i+1)$ for $i<j$. If $n_2>\frac{1}{4}n$ or $j=1$, then $|T_3|<\frac{1}{2}n$, and hence, by Lemma 2,

$$\mathrm{sph}(T)\leq\max\{\mathrm{sph}(B_1^*),\ \mathrm{sph}(B_2^*),\ \mathrm{sph}(T_3)\}+1\leq f_k(\tfrac{1}{2}n)+1.$$

Thus the lemma holds in this case. Now suppose $n_2\leq\frac{1}{4}n$ and $j>1$. Then

$$\mathrm{sph}(T_{j-1})\leq\max\{\mathrm{sph}(B_{j-1}^*),\ \mathrm{sph}(B_j^*),\ \mathrm{sph}(T_{j+1})\}+1\leq f_k(n_{j-1})+1.$$

Similarly (if $j\geq 4$),

$$\begin{aligned}\mathrm{sph}(T_{j-3})&\leq\max\{\mathrm{sph}(B_{j-3}^*),\ \mathrm{sph}(B_{j-2}^*),\ \mathrm{sph}(T_{j-1})\}+1\\&\leq\max\{f_k(n_{j-3})+1,\ f_k(n_{j-1})+2\}.\end{aligned}$$

Repeating this procedure $[\frac{1}{2}j]$ times, we have

$$\mathrm{sph}(T_1)\leq\max\{f_k(n_i)+\tfrac{1}{2}(i+1);\ i=1,3,5,\dots,j-1\},$$

if j is even, and

$$\mathrm{sph}(T_2)\leq\max\{f_k(n_i)+\tfrac{1}{2}i;\ i=2,4,6,\dots,j-1\},$$

if j is odd. In the latter case we need one more step to connect B_1^* and T_2. Thus, in either case, we have

$$\begin{aligned}\mathrm{sph}(T)&\leq\max\{f_k(n_i)+[\tfrac{1}{2}(i+2)];\ 1\leq i\leq k-1\}\\&\leq\max\{f_k(n/i)+[\tfrac{1}{2}(i+1)];\ 2\leq i\leq k, i\neq 3\}.\end{aligned}$$

This completes the proof.

Theorem 5. *Let T be a tree whose every vertex has degree $\leq k$, $k\geq 3$. Then* $\mathrm{sph}(T)\leq\frac{1}{2}(k+1)\log_k|T|$.

Proof. We shall show

$$(*) \qquad f_k(m) \le \tfrac{1}{2}(k+1)\log_k m$$

by induction on m. Since $\frac{1}{2}(k+1)\log_k m$ is monotone increasing with respect to k for $k \ge 4$, $m > 1$, it is easily seen that $(*)$ holds for $m \le 2$. Suppose now $(*)$ holds for $m < n$, $n \ge 3$. Since

$$[\tfrac{1}{2}(i+1)] < \tfrac{1}{2}(k+1)\log_k i \quad \text{for } k \ge i \ge 2,\ i \ne 3$$

as easily verified, we have

$$\tfrac{1}{2}(k+1)\log_k(n/i) + [\tfrac{1}{2}(i+1)] < \tfrac{1}{2}(k+1)\log_k n.$$

Therefore, by Lemma 3 and the inductive assumption,

$$\begin{aligned} f_k(n) &\le \max\{f_k(n/i) + [\tfrac{1}{2}(i+1)];\ 2 \le i \le k,\ i \ne 3\} \\ &< \tfrac{1}{2}(k+1)\log_k n. \end{aligned}$$

This completes the proof.

5. Complete bipartite graphs

A point set Y of a Euclidean space is called *1-dispersed* if $y_1, y_2 \in Y$, $y_1 \ne y_2$ implies $\|y_1 - y_2\| \ge 1$. For a bounded subset X of a Euclidean space, the largest number of points in any 1-dispersed subset of X, is denoted by $\alpha(X)$. Let $D^k(r) \subset R^k$ be the open k-disc of radius r centered at the origin, and let $d(n,r)$ be the minimum number k such that $\alpha(D^k(r)) \ge n$. Then it is clear that $d(n,r)$ is monotone nonincreasing on r. Note that $d(n,1) = \mathrm{sph}(K_{1,n})$.

Lemma 4. *If $a^2 + b^2 < 1$, $a, b > 0$, then*

$$\mathrm{sph}(K_{m,n}) \le d(m,a) + d(n,b).$$

Proof. Let $k = d(m,a)$, $l = d(n,b)$ and let $\{p_1, \dots, p_m\}$, $\{q_1, \dots, q_n\}$ be 1-dispersed subsets of $D^k(a)$, $D^l(b)$, respectively.

In $R^{k+l} = R^k \times R^l$, let

$$\begin{aligned} x_i &= (p_i, 0) \in R^k \times R^l, \quad i = 1, \dots, m, \\ y_j &= (0, q_j) \in R^k \times R^l, \quad j = 1, \dots, n. \end{aligned}$$

Then the space graph generated by these $m+n$ points with adjacency limit 1, is clearly isomorphic to $K_{m,n}$. Hence $\mathrm{sph}(K_{m,n}) \le k + l$.

For a real number r, we denote by $\{r\}$ the smallest integer not less than r.

Theorem 6. *For $n \ge m \ge 1$, $\mathrm{sph}(K_{m,n}) \le m - 1 + \{\tfrac{1}{2}n\}$.*

Proof. Let $k=\{\frac{1}{2}n\}$ and $b=\sqrt{\frac{1}{2}+1/(5m)}$. Since $2k$ points

$$(\pm\sqrt{\tfrac{1}{2}},0,\ldots,0),(0,\pm\sqrt{\tfrac{1}{2}},0,\ldots,0),\ldots,(0,\ldots,0,\ \pm\sqrt{\tfrac{1}{2}})$$

in R^k are 1-dispersed, we have $\alpha(D^k(b))\geq 2k\geq n$. Hence $d(n,b)\leq\{\frac{1}{2}n\}$. Thus, if $m=1$, then $\mathrm{sph}(K_{1,n})=d(n,1)\leq d(n,b)\leq\{\frac{1}{2}n\}$. Suppose now $m\geq 2$ and let

$$a=\sqrt{(m-1)/(2m)+1/(5m)}.$$

By Lemma 1, the side-length of a regular $(m-1)$-simplex of circumradius $\sqrt{(m-1)/(2m)}$ equals 1. Hence $\alpha(D^{m-1}(a))\geq m$ and hence $d(m,a)\leq m-1$. Since $a^2+b^2=1-1/(10m)$, the theorem follows from Lemma 4.

It can be proved that for $m\leq 3$

$$\mathrm{sph}(K_{m,n})=d(n,\sqrt{(m+1)/(2m)})+m-1$$

but I do not give the proof here, for it is long and involved. Using this equality and the results concerning 1-dispersed point sets over 2-spheres [12], and [3, Ch. 8], I have calculated $\mathrm{sph}(K_{m,n})$ for $m\leq 3$, $n\leq 10$. The values are given in Table 1. The upper bounds given by Theorem 6 compare favorably with the values in Table 1.

Table 1
The sphericities of $K_{m,n}$

m	$n=1$	2	3	4	5	6	7	8	9	10
1	1	1	2	2	2	3	3	3	3	3
2	1	2	3	3	3	4	4	4	5	5
3	2	3	4	4	5	5	5	6	6	6

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References

[1] B. Alspach and M. Rosenfeld, On embedding triangle-free graphs in unit spheres, Discrete Math. 19 (1977) 103–111.

[2] S. Foldes and P.L. Hammer, Split graphs, in: F. Hoffman et al., eds., Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Louisiana State Univ., Baton Rouge, LA (1977) 311–315.

[3] L. Fejes Tóth, Regular Figures (Pergamon Press, New York, 1964).

[4] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[5] W.K. Hale, Frequency assignment: theory and applications, Proc. IEEE 68 (1980) 1497-1514.
[6] T.F. Havel, The combinatorial distance geometry approach to the calculation of molecular conformation, Ph.D. Thesis, Group in Biophysics, University of California, Berkeley (1982).
[7] H. Maehara, On time graphs, Discrete Math. 32 (1980) 281-289.
[8] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ. 38 (AMS, Providence, RI, 1962).
[9] A. Rényi, C. Rényi and J. Surányi, Sur l'indépendance des domaines simple dans l'espace Euclidien à n dimensions, Collq. Math. 2 (1951) 130-135.
[10] F.S. Roberts, Indifference graphs, in: F. Harary, ed., Proof Techniques in Graph Theory (Academic Press, New York, 1969) 139-146.
[11] F.S. Roberts, On the boxicity and cubicity of a graph, in: W.T. Tutte, ed., Recent Progress in Combinatorics (Academic Press, New York, 1969) 301-310.
[12] K. Schütte und B.L. Van der Waerden, Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz?, Math. Ann. 123 (1951) 96-124.