Stable marriage problem mapped to Lotka-Volterra model: Stable equilibria mapped from stable matchings

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Abstract

Stable Marriage Problem (SMP) is a matching problem which seeks a stable matching between in N women and N men. We propose a Lotka-Volterra Model (LVM) mapped from SMP. Further, we show the equilibria of LVM mapped from the stable matchings of SMP are stable and those not corresponding to the stable matchings are unstable by the local stability analysis.

Keywords: Dynamical System; Lotka-Volterra Model; Stable Marriage Problem;

1. Introduction

The Stable Marriage Problem (SMP) is a two-sided matching problem to find a stable matching\(^1\). With N men and N women given, a matching consists of N pairs of a man and a woman where each man and woman appear only once in the matching. Each man has an ordered preferences without tie over all the women, and vice versa. The stable matching is a matching without any blocking pair. The blocking pair is a pair in which a man prefers a woman to his current partner and the woman also prefers the man to her current partner in the matching. At least one stable matching is known to exist in SMP\(^1,2\). A stable matching can be obtained by Gale-Shapley Algorithm (GSA)\(^2\). In GSA, men propose women and women accepts, rejects, or keeps as a pending proposal. This process is repeated until each man can find an woman who men accepts the proposal. SMP can be applied to several practical problems such as assignments of medical intern to hospitals or an allocation of college students to college classes\(^2\).

Vande Vate formulated SMP as the mathematical programming problem\(^3\). He formulated SMP as a 0-1 integer programming problem, and converted it to a linear programming problem by a linear relaxation. All extreme points of this linear programming problem are expressed as integer values. Balinski and Ratier defined an SMP graph and suggested a graph theoretic approach\(^4\). Roth introduced a game theoretic framework to SMP\(^5\). He assumed a market of marriages between men and women can be modeled as cooperative and strategic games.

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We studied an approach for SMP by mapping SMP to a dynamical system. The dynamical system assumes interactions between pairs, and that system is capable of choosing a matching (a set of pairs) among all possible pairs. In order to express SMP as a dynamical model, we use the Lotka-Volterra model (LVM), in which blocking pairs feed upon the pairs being blocked, and pairs sharing the same person must be excluded. We show that stable equilibria correspond to stable matchings in all the SMP instances with size of three by solving initial-value problems of the proposed dynamical model.

In this study, we show the equilibria and these stability of SMP mapped to LVM(SMP-LVM) with size three or less, and suggest the concrete values and the stability of the equilibrium in SMP-LVM.

Section 2 concisely introduces SMP. Section 3 defines LVM mapped from SMP. Section 4 investigates the local stability of the dynamical model of SMP with size two and three.

### Nomenclature

- **SMP**: Stable Marriage Problem
- **LVM**: Lotka-Volterra Model
- **GSA**: Gale-Shapley Algorithm
- **ODE**: Ordinary Differential Equations
- **SMP-LVM**: Stable Marriage Problem mapped to Lotka-Volterra Model
- \( x_{m,w} \): a variable called a pair variable indicating whether a pair \((m, w)\) is in the matching \(M(x_{m,w} = 1)\) or not \((x_{m,w} = 0)\).

### 2. Stable marriage problem

The original SMP consists of a set \(M\) containing \(N\) men and a set \(W\) containing \(N\) women. The number \(N\) of all men or women is called as the size of SMP. Let all members \(p \in M \cup W\) have a preference list containing all the members in the opposite sex, denoted by symbol \(\succ_p\). In particular, \(a \succ_p b\) indicates that a person \(p\) prefers \(a\) to \(b\). A matching is a one-one correspondence between the men and women and is a set of pairs \(M \subseteq M \times W\).

If a man \(m\) and a woman \(w\) are paired in \(M\), then \(m\) and \(w\) are called partners, and these partners in \(M\) are described as one-to-one mappings between \(M\) and \(W\) by \(m = p_M(w)\) and \(w = p_W(m)\). A pair \((m', w') \in M \times W\) is a blocking pair for a matching \(M\) if

\[
\begin{align}
(i) & \quad (m', w') \notin M \text{ and } m' \in M \text{ and } w' \in W \\
(ii) & \quad w' \succ_{m'} p_M(m') \\
(iii) & \quad m' \succ_w p_W(w').
\end{align}
\]

Table 1 shows an SMP instance of size three specified by the preference list (a) and (b). Each row shows the preference order of a man to women in Table 1 (a). The leftmost woman in rows is the most preferable for each man, and the rightmost woman in rows is the most undesirable for each man. Table 1 (b) shows preference order of a woman to a man.

The stable matching in Table 1 is \(\{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}\). The matching \(\{(m_1, w_3), (m_2, w_2), (m_3, w_1)\}\) is not the stable matching because the blocking pair \((m_3, w_3)\) exists. Another two stable matchings are \(\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}\) and \(\{(m_1, w_2), (m_2, w_3), (m_3, w_1)\}\). All the stable matchings are only those three stable matchings.

The rank of woman \(w\) in the preference of man \(m\) is defined as \(\text{Rank}_w(m)(m \in M, w \in W)\). For example, in Table 1, \(\text{Rank}_w(w_3) = 1\) and \(\text{Rank}_w(w_2) = 3\). In the similar way, the rank of man \(m\) in the preference of woman \(w\) is defined as \(\text{Rank}_m(w)(m \in M, w \in W)\).

As a special case of SMP, the preference matrix in SMP is a Latin square. The Latin preference is defined by \(\text{Rank}_w(w) = N + 1 - \text{Rank}_w(m)\). For example, the preference in Table 1 is the Latin preference.
3. Lotka-Volterra model mapped from stable marriage problem

3.1. Lotka-Volterra model

The Lotka-Volterra model shows a population dynamics of predator-prey interactions expressed as ordinary differential equations (ODE)\(^8,9\). The ODE describes dynamics of interacting populations between predators and prey.

The original two populations model can be generalized to \(n\) populations model. The generalized Lotka-Volterra model for \(n\) populations is of the form

\[
\frac{dN_i}{dt} = N_i \left( r_i + \sum_{j=1}^{n} a_{ij} N_j \right) \quad i = 1, \ldots, n.
\]

(2)

The variable \(N_i\) denotes the density of the \(i\)-th species. The parameter \(r_i\) is an intrinsic growth (or decay) rate of the \(i\)-th increasing prey population (or predator mortality rate). The interaction matrix \(a_{ij}\) describes the effect of the \(j\)-th upon the \(i\)-th population, which is positive if it enhances the growth of \(i\)-th species and negative if inhibits it.

3.2. Formulation of the Lotka-Volterra model mapped from SMP

We regard a pair of a man and a woman as a species in Lotka-Volterra Model. An essential pair inhibits nonessential pairs. As a result, the density of essential pairs increases while the density of nonessential pairs decreases. In SMP, the nonessential pairs are:

- the polygyny or the polyandry pair,
- the blocking pair.

We map a specific LVM from an SMP of (2). We use a vector notation of the pair variables \(x = (x_{m,w})^T \in \mathbb{R}^{|M| \times |W|} \ (m \in M, w \in W)\). If \((m, w) \in M\), then \(x_{m,w} = 1\) and if \((m, w) \notin M\), then \(x_{m,w} = 0\). We expect that an attractor of the LVM corresponds to the stable matching in SMP. To normalize the pair variables \(x_{m,w}\), the following logistic equation can be used.

\[
\frac{dx_{m,w}}{dt} = (1 - x_{m,w}) x_{m,w} \quad (m \in M, w \in W).
\]

(3)

The vector \(x\) satisfies the constraint that one man pairs with only one woman and one woman pairs with only one man in the matching. The pair of a man (woman) must inhibit other pairs including the man (woman). The equation involves terms inhibiting the polygyny and the polyandry as follows:

\[
\frac{dx_{m,w}}{dt} = 1 - x_{m,w} - \sum_{j \in W, j \neq w} x_{m,j} - \sum_{i \in M, i \neq m} x_{i,w} x_{m,w} \quad (m \in M, w \in W).
\]

(4)

The constraint for men (women) is described by 3rd term (4th term) in (4). The pair satisfying the constraints (1) must be inhibited by predator pairs. The predator may be specified by

\[
\{(m’, j)|(m’, j) \in M, (m’, w’) \notin M, m’ \in M, w’ \in W, j \in W, j >_{m’} w’\},
\]

(5)
In (5) and (6), \((m', w')\) is a prey pair (blocking pair). Thus, involving there predator pairs and prey pairs leads to the dynamical model as follows:

\[
\frac{dx_{m,w}}{dt} = \begin{cases} 
1 - x_{m,w} - \sum_{j \in W, j \neq w} x_{m,j} - \sum_{i \in M, i \neq m} x_{i,w} \\
\quad + \sum_{w \neq w'} x_{m,j} + \sum_{j \neq w} x_{i,w} - \sum_{j \neq w'} x_{i,w} - \sum_{i \neq m} \end{cases} x_{m,w} \quad (m \in M, w \in W).
\]

The equations (7) can be simplified to:

\[
\frac{dx_{m,w}}{dt} = \begin{cases} 
1 - x_{m,w} - 2 \sum_{j \neq w} x_{m,j} - 2 \sum_{i \neq m} x_{i,w} \end{cases} x_{m,w} \quad (m \in M, w \in W).
\]

We call the ODE (8) as SMP-LVM. The matrix form of the SMP-LVM is as follows:

\[
\frac{dx}{dt} = (e - Ax)x.
\]

Matrix \(A = \{a_{i,j}\}\) is an interaction matrix. The column vector \(e\) consists of all one element \(e = (1, \ldots, 1)^T\). The matrix element \(a_{i,j}\) with \(i = (m, w), j = (m', w')\) are:

\[
a_{(m, w), (m', w')} = \begin{cases} 
1 & \text{if } m = m' \text{ and } w = w', \\
2 & \text{else if } (w' >_m w \text{ and } m = m') \text{ or } (m' >_w m \text{ and } w = w'), \\
0 & \text{otherwise.}
\end{cases}
\]

4. Equilibria of proposed model

4.1. Equilibria of SMP-LVM with size two

The proposed model (8) has the following two types of equilibria:

1) An equilibrium corresponding to a matching in the original SMP:

   (a) the matching is stable,

   (b) the matching is unstable,

2) An equilibrium which does not correspond to any matching.

In the following, we discuss above two types in details.

1) Let the right-hand side of (8) be \(F(x(t))\). If \(\sum_{j \in W} x_{m,j} \leq 1\) and \(\sum_{i \in M} x_{i,w} \leq 1\) \((\forall m \in M, \forall w \in W)\) and \(x(t) \to v\) \((t \to \infty)\) where \(v \in [0, 1]^N\), then \(F(v) = 0\). Because \(v\) satisfies \(\sum_{j \in W} v_{m,j} \leq 1\) and \(\sum_{i \in M} v_{i,w} \leq 1\) \((m \in M, w \in W)\),

\[
\sum_{j \in W} v_{m,j} = \sum_{j \neq w} v_{m,j} + v_{m,w} + \sum_{w > w'} v_{m,w'} \leq 1,
\]

\[
\sum_{i \in M} v_{i,w} = \sum_{i > w} v_{i,w} + v_{i,w} + \sum_{m > w'} v_{m,w'} \leq 1.
\]
The Jacobian of this SMP-LVM is:

\[
\begin{align*}
\frac{\partial F}{\partial x} &= J(x) = \\
&= \begin{pmatrix}
1 - 2x_{m_1,w_1} - 2x_{m_1,w_2} & -2x_{m_1,w_1} & 0 & 0 \\
0 & 1 - 2x_{m_1,w_2} - 2x_{m_2,w_2} & 0 & 0 \\
-2x_{m_2,w_1} & 0 & 1 - 2x_{m_1,w_1} - 2x_{m_2,w_1} & 0 \\
0 & 0 & -2x_{m_2,w_2} & 1 - 2x_{m_2,w_1} - 2x_{m_2,w_2}
\end{pmatrix}.
\end{align*}
\]

The equilibria \( v_1^* \) and \( v_2^* \) are known to be locally stable, for the eigenvalue of Jacobian (14) is \(-1\). Similarly, the equilibria \( v_3^* \), \( v_4^* \), \( v_5^* \), \( v_6^* \) are not stable, for the Jacobian (14) has the eigenvalues \( 1, -1 \). The equilibria \( v_7^* \) is also locally unstable because the eigenvalue of Jacobian (14) is \( 1 \).

2) The model (8) mapped from SMP with Latin preference may have an equilibrium \( v = \alpha e \) where

\[
\alpha = \frac{1}{2N - 1}.
\]

For example, the SMP in Table 2 satisfies Latin preference: \( \text{Rank}_m(w) = N + 1 - \text{Rank}_w(m) \). The interaction matrix \( A \) of the SMP-LVM (9) are shown below (16).

\[
A = \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 0 & 2 \\
2 & 0 & 1 & 0 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( w_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( w_1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( w_2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. (a) instance of the preference table of men for SMP with size two; (b) instance of the preference table of women for SMP with size two.
The equilibrium $v^*_8$ can be also obtained as $v^*_8 = A^{-1}e = \frac{1}{2}e$. By solving the characteristic equation of the Jacobian (14), we can get the eigenvalues $\frac{1}{2}, -1, -\frac{1}{2} - \frac{1}{2}i, -\frac{1}{2} + \frac{1}{2}i$. The equilibrium is locally unstable.

### 4.2. Equilibria of SMP-LVM with size three

Table 4 shows all the equilibria and their stability of SMP-LVM (8) mapped from size three SMP. All instances of SMP with size three are classified into six classes in the Table 3 by a homomorphism of the reduced stable marriage graph\(^\text{10}\). All the equilibria corresponding to the stable matching in original SMP with size three or less is locally stable from the result of Table 4. Other equilibria which is not related to stable matchings of the original SMP are unstable. All the equilibria corresponding to the stable matching are also stable because the signs of the eigenvalues of Jacobian matrix of the SMP-LVM are all negative.

Table 3. An instance from six classes classified according to a graph homomorphism of the reduced stable marriage graph, in SMP with size three\(^\text{10}\)

<table>
<thead>
<tr>
<th>Class</th>
<th>Men’s preferences</th>
<th>Women’s preferences</th>
<th>Stable matchings</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$m_1$: $w_1, w_2, w_3$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_2), (m_3, w_3)}$</td>
</tr>
<tr>
<td>2</td>
<td>$m_1$: $w_2, w_1, w_3$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_2), (m_3, w_3)}, {(m_1, w_2), (m_2, w_1), (m_3, w_1)}$</td>
</tr>
<tr>
<td>3</td>
<td>$m_1$: $w_3, w_1, w_2$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_3), (m_3, w_2)}, {(m_1, w_3), (m_2, w_2), (m_3, w_1)}$</td>
</tr>
<tr>
<td>4</td>
<td>$m_1$: $w_1, w_3, w_2$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_3), (m_3, w_2)}, {(m_1, w_3), (m_2, w_2), (m_3, w_1)}, {(m_1, w_3), (m_2, w_1), (m_3, w_2)}$</td>
</tr>
<tr>
<td>5</td>
<td>$m_1$: $w_2, w_3, w_1$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_2), (m_3, w_3)}, {(m_1, w_3), (m_2, w_2), (m_3, w_1)}, {(m_1, w_3), (m_2, w_1), (m_3, w_2)}$</td>
</tr>
<tr>
<td>6</td>
<td>$m_1$: $w_3, w_1, w_2$</td>
<td>$w_1, m_1, m_2, m_3$</td>
<td>${(m_1, w_1), (m_2, w_3), (m_3, w_2)}, {(m_1, w_3), (m_2, w_2), (m_3, w_1)}, {(m_1, w_3), (m_2, w_1), (m_3, w_2)}$</td>
</tr>
</tbody>
</table>
### Table 4. All the equilibria in SMP-LVM with size three and their local stability. A element $v_{i,j}$ of 0-1 vector $v$ indicates a pair variable between $m_i \in M$ and $w_j \in W$. All the classes in Table 3 are shown. The column of the local stability shows the local stability of equilibria, and the sign of these eigenvalues by Jacobian matrix. The rightmost column shows the number of equilibria which has the equivalent equilibrium pattern.

<table>
<thead>
<tr>
<th>Class</th>
<th>Equilibrium pattern</th>
<th>Local stability</th>
<th>Number of equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>common to all the classes</td>
<td>$0$ $0$ $0$</td>
<td>unstable (+)</td>
<td>$1$</td>
</tr>
<tr>
<td>only one pair $v_{i,j} = 1$</td>
<td>$0$</td>
<td></td>
<td>$9$</td>
</tr>
<tr>
<td>two pairs subjected to monogamous relationship</td>
<td>$1/3$ $0$</td>
<td>unstable (−, +)</td>
<td>$18$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$ $1/3$ $1/3$</td>
<td>stable (−)</td>
<td>$5$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$ $1/3$ $1/3$</td>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$ $1/3$ $1/3$</td>
<td>stable (−)</td>
<td>$4$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$ $1/3$ $1/3$</td>
<td></td>
<td>$2$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$</td>
<td>unstable (−, +)</td>
<td>$4$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$</td>
<td></td>
<td>$2$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$</td>
<td>unstable (−, +)</td>
<td>$3$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$</td>
<td></td>
<td>$3$</td>
</tr>
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<td>matchings with blocking pair</td>
<td>$1/3$</td>
<td>stable (−)</td>
<td>$1$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$</td>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$</td>
<td>unstable (−, +)</td>
<td>$2$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$</td>
<td></td>
<td>$2$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/3$</td>
<td>stable (−)</td>
<td>$4$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/3$</td>
<td></td>
<td>$4$</td>
</tr>
<tr>
<td>number of unstable</td>
<td>$1/5$</td>
<td>unstable (−, +)</td>
<td>$3$</td>
</tr>
<tr>
<td>matchings with blocking pair</td>
<td>$1/5$</td>
<td></td>
<td>$3$</td>
</tr>
<tr>
<td>all of the stable matchings</td>
<td>$1/5$</td>
<td>unstable (−, +)</td>
<td>$6$</td>
</tr>
<tr>
<td>all of stable matchings</td>
<td>$1/5$</td>
<td></td>
<td>$6$</td>
</tr>
</tbody>
</table>

4.3. **Diagram in SMP-LVM mapped from SMP with size two**

Fig. 1 (a) and (b) depict the relation between initial values and equilibria in SMP-LVM of Table 2 by using the decision making difficulty. The initial values $x_{m,w}(0)$ of each pair variable in the SMP-LVM are specified by two parameters $r_M$ and $r_W$ as follows:

$$x_{m,w}(0) = \frac{r_M(n + 1 - \text{Rank}_m(w)) + r_W(n + 1 - \text{Rank}_w(m))}{2n} \quad (m \in M, w \in W).$$

$r_M$ ($r_W$) is a priority for weighting the initial values for males (females) ranging $[0, 1]$.

Fig. 1 (a) shows the number of time steps required to reach an equilibrium using initial values given by (17). The vertical axis is used to indicate $r_W$ and the horizontal axis is used to indicate $r_M$ in Fig. 1 (a).

Fig. 1 (b) shows the equilibria when the dynamics of SMP-LVM with initial values given by (17) converges. In this figure, the vertical axis and horizontal axis are the same as the axes of Fig. 1 (a) and the colours denotes...
the equilibria exist in SMP-LVM of Table 2. Fig. 1 (b) shows convergences on four equilibria which are the most preferred equilibrium \( v^*_1 \) by women where \( r_W > r_M \), the most preferred equilibrium \( v^*_2 \) by men where \( r_M > r_W \), the equilibrium \( v^*_7 \) of no pair where \( r_M = r_W = 0 \), the equilibrium \( v^*_8 \) where \( r_M = r_W (\neq 0) \). These figures suggest that the dynamics with initial values ranging between 0.0 and 1.0 converges on equilibria corresponding to stable matchings except the case of all initial values be equal.

5. Conclusion

This paper proposes a specific Lotka-Volterra Model mapped from the Stable Marriage Problem. The equilibria corresponding to stable matchings in the original SMP of size two and three are locally stable. Similarly, the equilibria corresponding to matchings with blocking pair are locally unstable. Furthermore, the SMP-LVM mapped from a size \( N \) SMP with Latin preference has an unstable equilibrium \( (1/2N, \ldots, 1/2N) \).

Acknowledgements

This work was partly supported by a Grant-in-Aid for Scientific Research (B): “Foundation on Immunity-Based Complex Systems and Sensor Agents” (23300057) 2011.
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