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SINGULAR SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS AND A LEMMA OF ARNOLD SHAPIRO

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INTRODUCTION

WE CONSIDER differential equations of the form $A = 0$ where A is a differential polynomial in finitely many differential indeterminates $y_1, \dots, y_n$ with coefficients in a differential field $\mathcal{F}$ of characteristic zero; that is, A is an element of the differential polynomial algebra $\mathcal{A} = \mathcal{F}\{y_1 \dots, y_n\}$. We denote the derivation operators of $\mathcal{F}$ by $\delta_1, \dots, \delta_m$. We suppose fixed, once for all, a universal extension $\mathcal{U}$ of $\mathcal{F}$ (see [3], pp. 768–771).

Ritt showed (see [6], p. 13 and pp. 165–166) that if $\mathfrak{a}$ is any perfect differential ideal of $\mathcal{A}$ then $\mathfrak{a}$ is the intersection of finitely many prime differential ideals of $\mathcal{A}$ none of which contains any other; these primes, which are unique, are the *prime components* of $\mathfrak{a}$. The prime components are especially interesting when $\mathfrak{a}$ is the perfect differential ideal $\{A\}$ generated by an irreducible $A \in \mathcal{A}$.

In order to describe the situation in that case we consider a total ordering of the set of all derivatives $\delta_1^{i_1} \dots \delta_m^{i_m} y_j$ $(0 \leq i_1 < \infty, \dots, 0 \leq i_m < \infty, 1 \leq j \leq n)$ such that for all such derivatives u, v and all δ_i

$$u < \delta_i u,$$
$$u < v \Rightarrow \delta_i u < \delta_i v.$$

We call such an ordering a *ranking* of $y_1, \dots, y_n$; rankings exist (e.g. we may order the derivatives $\delta_1^{i_1} \dots \delta_m^{i_m} y_j$ lexicographically with respect to $(\Sigma i_\mu, j, i_1, \dots, i_m)$) but are in general not unique. Given a ranking, the highest derivative u present in A is called the *leader* of A, and the partial derivative $\partial A/\partial u$ is called the *separant* of A; of course, a different choice of ranking may give to A a different leader and separant.

Ritt called a zero of A (i.e., a solution of the differential equation $A = 0$) *singular* if it is a zero of every separant of A. He showed (see [6], p. 31 and p. 167) that among the prime components of A there is one, which we shall denote by $\mathfrak{P}(A)$, with the following property: $\mathfrak{P}(A)$ contains *no* separant of A whereas each other prime component of A contains *every* separant of A. $\mathfrak{P}(A)$ is called the *general* component of A, the others are called the *singular*

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components of A. Thus, every zero of a singular component of A is a singular zero of A, and every nonsingular zero of A is a zero of $\mathfrak{P}(A)$, but a singular zero of A may be a zero of $\mathfrak{P}(A)$.

It is a remarkable result of Ritt (see [6], pp. 57–62 and pp. 167–170, and also Hillman [1], p. 163) that every singular component of A is the general component of another irreducible differential polynomial in $\mathscr{A}$. Furthermore, he gave ([6], p. 109 and pp. 175–176) an algorithm (modulo the possibility of factorization of polynomials over $\mathscr{F}$) for finding a finite set of irreducible polynomials the general components of which include among them the singular components of A, and then established a criterion (the famous low-power theorem (see [6], pp. 64–70 and pp. 170–172)) for determining, given an irreducible $B \in \mathscr{A}$, whether $\mathfrak{P}(B)$ is a singular component of A.

There remains the problem, posed by Ritt, of determining, for a given zero of A, the components of A which admit that zero. In the light of the above this reduces to a number of problems of the following type: given a zero of A, to determine whether or not it is a zero of $\mathfrak{P}(A)$. It is not difficult to see, moreover, that it suffices to be able to solve this problem when the zero is $(0, \ldots, 0)$. Thus, we are led to the following problem:

Given an irreducible differential polynomial $A \in \mathscr{F}\{y_1, \ldots, y_n\}$ *which vanishes at* $(0, \ldots, 0)$, *to determine whether* $(0, \ldots, 0)$ *is a zero of* $\mathfrak{P}(A)$.

This problem is wide open. As yet, only very special cases have been solved. Two principal tools have been used in these special cases, as follows:

To prove that $(0, \ldots, 0)$ *is a zero of* $\mathfrak{P}(A)$. Let $P_j, Q_{jj'}$ $(1 \leqslant j \leqslant n,\ 1 \leqslant j' \leqslant n)$ be power series over $\mathscr{U}$ in an indeterminate constant c, which vanish at 0, such that $\det(Q_{jj'} \neq 0$; for each nonzero $F \in \mathscr{F}\{y_1, \ldots, y_n\}$ let $F*$ denote the leading coefficient of

$$F(P_1 + \sum_{j_1} Q_{1j_1}y_{j_1}, \ldots, P_n + \sum_{j_n} Q_{nj_n}y_{j_n})$$

(i.e., the lowest nonzero coefficient when considered as a power series in c over $\mathscr{U}\{y_1, \ldots, y_n\}$). *If the leader of* $A(P_1 + \sum_{j_1} Q_{1j_1}y_{j_1}, \ldots, P_n + \sum_{j_n} Q_{nj_n}y_{j_n})$ *is present in* $F*$, *or if* $S* \notin \{A*\}$ *for some separant* S *of* A, *then* $(0, \ldots, 0)$ *is a zero of* $\mathfrak{P}(A)$. This result, generalizing results of Hillman and of Ritt, is an almost immediate consequence of Hillman's leading coefficient theorem. (For an efficient proof of the leading coefficient theorem see Hillman and Mead [2]; for an indication of how this theorem leads to the above result see Hillman [1], §§ 7–8.).

To prove that $(0, \ldots, 0)$ *is not a zero of* $\mathfrak{P}(A)$. Suppose that $m = 1$ (ordinary differential equations) and that A has more than one term. It is a consequence of two results of Levi ([4], §§ 38–41 and §§ 44–52) that if A has a term $y_1^{e_1} \ldots y_n^{e_n}$ of order 0 such that for every other term T and each y_k either the degree of T in $y_k, y_k', y_k'', \ldots$ is $> e_k$ or T is divisible by $y_k^{e_k}$, or if $n = 1$ and A has a term $y_1^{f_0}y_1'^{f_1} \ldots y_1^{(r)f_r}$ such that for every other term T and $y_1^{(k)}(0 \leqslant k \leqslant r)$ the degree of T in $y_1^{(k)}, y_1^{(k+1)}, y_1^{(k+2)}, \ldots$ is $> f_k + f_{k+1} + \ldots + f_r$, then $(0, \ldots, 0)$ is not a zero of $\mathfrak{P}(A)$.

We come at last to the point of the present paper. This is to present a result which broadens considerably the class of differential polynomials A for which it is known that

$(0, \ldots, 0)$ is not a zero of $\mathfrak{P}(A)$. To do this we introduce the notion of 'domination' of one differential monomial over another (§ 3 below), and establish a key lemma (§ 5) about this notion which generalizes both of Levi's results mentioned above. The proof depends on a lemma of Levi (§ 2 below). The domination lemma yields our proposition (§ 6), a special case of which can be stated as follows (m and n now being arbitrary).

Let A have more than one term. If A has a term which is dominated by every other term of A then $(0, \ldots, 0)$ *is not a zero of* $\mathfrak{P}(A)$.

A crucial role in the proof of the domination lemma is played (§ 4) by a combinatorial lemma proved by Arnold Shapiro (unpublished). His lemma is presented in § 1.

NOTATION

We consistently use the following notation:

N, **Q**, **R** denote respectively the set of natural numbers, of rational numbers, of real numbers.

For any set K, $\mathfrak{P}(K)$ denotes the set of all subsets of K, and card K denotes the cardinal number of K. The empty set is denoted by ϕ.

Θ denotes the set of all derivative operators $\delta_1^{i_1} \ldots \delta_m^{i_m}$ $(i_1 \in \mathbf{N}, \ldots, i_m \in \mathbf{N})$ of the differential field $\mathscr{F}$ (and of all the various differential rings and fields considered).

$[A_1, \ldots, A_r]$ denotes the differential ideal generated by the elements $A_1, \ldots, A_r$ of a given differential ring. $\{A_1, \ldots, A_r\}$ denotes the perfect differential ideal generated by these elements; since in the cases considered the differential ring contains **Q**, $\{A_1, \ldots A_r\}$ is the set of all elements B such that $B^s \in [A_1, \ldots, A_r]$ for some $s \in \mathbf{N}$.

§ 1. SHAPIRO'S LEMMA

SHAPIRO'S LEMMA. *Let K be a finite set, let $(a_k)_{k\in K}$ be a family with $a_k \in \mathbf{R}$* and $a_k \geqslant 0$ $(k \in K)$, *let $(x_J)_{J\in\mathfrak{P}(K)-\mathfrak{P}(\phi)}$ be a family with $x_J \in \mathbf{R}$ and $x_J \geqslant 0$ $(J \in \mathfrak{P}(K) - \mathfrak{P}(\phi))$, and suppose that*

$$\sum_{J\in\mathfrak{P}(K)-\mathfrak{P}(K-I)} x_J > \sum_{i\in I} a_i \qquad (I \in \mathfrak{P}(K) - \mathfrak{P}(\phi)). \tag{1}$$

Then there exist numbers $x_{J,j} \in \mathbf{R}$ with $x_{J,j} \geqslant 0$ $(J \in \mathfrak{P}(K) - \mathfrak{P}(\phi), j \in J)$ such that

$$\sum_{j\in J} x_{J,j} = x_J \qquad (J \in \mathfrak{P}(K) - \mathfrak{P}(\phi))$$

and

$$\sum_{J\in j} x_{J,j} > a_j \qquad (j \in K).$$

Remarks. (1) We may think of the elements of K as representing the vertices of a simplex, the nonempty subsets of K as representing the faces of that simplex, the numbers a_j as forming a system of masses located at the vertices, and the numbers x_J as forming a system of masses located on the faces. The lemma then asserts that if, for each face I, the sum of the masses of the second system located on the faces touching I exceeds the sum of the masses of the first system located at the vertices of I, then the mass on each face can be redistributed among the vertices of that face in such a way that, for each vertex, the redistributed mass of the second system at the vertex exceeds the mass of the first system there.

(2) The proof shows that $x_{J,j}$ may be taken in the field $\mathbf{Q}((a_k)_{k\in K}, (x_J)_{J\in\mathfrak{P}(K)-\mathfrak{P}(\phi)})$.

Proof. Let $s = \operatorname{card} K$; we may suppose that $s > 0$ as otherwise the result is trivial. Then there exists a $J \in \mathfrak{P}(K) - \mathfrak{P}(\phi)$ with $x_J > 0$, and therefore there exists a unique $r \in \mathbf{N}$ such that $x_J \neq 0$ for some J with card $J = r$ but $x_J = 0$ for all J with card $J > r$; of course $1 \leqslant r \leqslant s$. Let t denote the number of elements $J \in \mathfrak{P}(K)$ with card $J = r$ and $x_J \neq 0$; then $t > 0$. If $r = 1$ then the nonzero masses of the second system are already all at the vertices, so that the result is trivial. Therefore we may assume that $r > 1$ We assume, too, that the result has been proved for lower values of (s, r, t) in the lexicographically well-ordered set $\mathbf{N}^3$.

Fix some $I_0 \in \mathfrak{P}(K)$ with card $I_0 = r$ and $x_{I_0} \neq 0$, and fix some $k \in I_0$; let I_1 denote the set of elements of I_0 other than k. Then $K - I_0 \subset K - I_1 \subset K$, so that the system of inequalities (1) can be written as three subsystems:

(1a) corresponding to $I \in \mathfrak{P}(K - I_0) - \mathfrak{P}(\phi)$;
(1b) corresponding to $I \in \mathfrak{P}(K - I_1) - \mathfrak{P}(K - I_0)$;
(1c) corresponding to $I \in \mathfrak{P}(K) - \mathfrak{P}(K - I_1)$.

The left members in (1a) contain neither of the terms x_{I_0}, x_{I_1} and the left members in (1c) contain both these terms; the left members in (1b) contain x_{I_0} but not x_{I_1}. It follows that if $\xi \in \mathbf{R}$, $\xi > 0$, and if we replace x_{I_0} by $x_{I_0} - \xi$ and x_{I_1} by $x_{I_1} + \xi$, then (1a) and (1c) remain valid. The system (1b) remains valid provided ξ is sufficiently small. If (1b) remains valid for $\xi = x_{I_0}$ then the replacement transforms the original system (1) into a similar system with a lower value for (s, r, t). We may therefore suppose that at least one of the inequalities (1b) fails after the replacement using $\xi = x_{I_0}$. Then there is a smallest value for ξ, and we denote it simply by ξ, such that after the replacement (1b) fails to hold; using this ξ we see that (1b) becomes a system (1b') of *weak* inequalities in the same direction. Obviously $0 < \xi \leqslant x_{I_0}$, and at least one of the weak inequalities (1b') is an equality.

From among all the $I \in \mathfrak{P}(K) - I_1) - \mathfrak{P}(K - I_0)$ for which the corresponding inequality (1b') is an equality, choose a maximal one, say K', and set $K'' = K - K'$. Then

$$\sum_{J\in\mathfrak{P}(K)-\mathfrak{P}(K'')} x_J = \sum_{i\in K'} a_i. \tag{2}$$

Consider any $I'' \in \mathfrak{P}(K'') - \mathfrak{P}(\phi)$; writing $I = K' \cup I''$, we see that either $I \in \mathfrak{P}(K - I_1)$ and I corresponds to an equality (1c) or else $I \in \mathfrak{P}(K - I_1) - \mathfrak{P}(K - I_0)$ and I corresponds to an inequality (1b'). In either case the inequality is strict. Subtracting from it the equation (2) we obtain

$$\sum_{J\in\mathfrak{P}(K'')-\mathfrak{P}(K''-I'')} x_J > \sum_{i\in I''} a_i \qquad (I'' \in \mathfrak{P}(K'') - \mathfrak{P}(\phi)). \tag{1''}$$

On the other hand, if we start with some $I \in \mathfrak{P}(K') - \mathfrak{P}(\phi)$ then either $I \in \mathfrak{P}(K - I_0) - \mathfrak{P}(\phi)$ and we have a strict inequality (1a) or else $I \in \mathfrak{P}(K - I_1) - \mathfrak{P}(K - I_0)$ and we have a weak inequality (1b′). If we now reduce ξ slightly, still keeping it positive, then the inequalities (1a) remain valid, and the inequalities (1b′) all become strict; that is, we regain (1b); furthermore, if the amount by which we reduce ξ is sufficiently small than (1″) remains valid, too. Then, in addition to (1″) we obtain (denoting I by I')

$$(3) \qquad \sum_{J \in \mathfrak{P}(K) - \mathfrak{P}(K - I')} x_J > \sum_{i \in I'} a_i \qquad (I' \in \mathfrak{P}(K') - \mathfrak{P}(\phi)).$$

If $J \in \mathfrak{P}(K) - \mathfrak{P}(K - I')$ then $J \in \mathfrak{P}(K) - \mathfrak{P}(K - K')$ and therefore this J does not occur in the left side of (1″). For each $J \in \mathfrak{P}(K) - \mathfrak{P}(K - K')$ we now decrease x_J and increase $x_{J \cap K'}$ by the same amount x_J (that is, we shift the entire mass x_J from the face J to the face $J \cap K'$). This does not affect the inequalities (1″), and replaces the inequalities (3) by

$$(1') \qquad \sum_{J \in \mathfrak{P}(K') - \mathfrak{P}(K' - I')} x_J > \sum_{i \in I'} a_i \qquad (I' \in \mathfrak{P}(K') - \mathfrak{P}(\phi)).$$

Since card $K' < s$ and card $K'' < s$, the lemma holds for each of the two systems (1′) and (1″). It is now a simple matter to see that the lemma holds for the original system (1).

COROLLARY. *Let K be a finite set, let $a_k \in \mathbf{N}(k \in K)$, let $x_J \in \mathbf{N}(J \in \mathfrak{P}(K) - \mathfrak{P}(\phi))$, and suppose that*

$$\sum_{J \in \mathfrak{P}(K) - \mathfrak{P}(K - I)} x_J > \sum_{i \in I} a_i \qquad (I \in \mathfrak{P}(K) - \mathfrak{P}(\phi)).$$

Then, for each sufficiently big $h \in \mathbf{N}$, there exist $y_{J,j} \in \mathbf{N}(J \in \mathfrak{P}(K) - \mathfrak{P}(\phi), j \in J)$ such that

$$\sum_{j \in J} y_{J,j} = hx_J \qquad (J \in \mathfrak{P}(K) - \mathfrak{P}(\phi))$$

and

$$\sum_{J \ni j} y_{J,j} > ha_j \qquad (j \in K).$$

Proof. There exist (see second remark after Shapiro's lemma) *rational* numbers $x_{J,j}$ satisfying the conclusion of that lemma. There obviously exists a $\xi > 0$, smaller than every nonzero $x_{J,j}$, such that if we set

$$x'_{J,j} = \begin{cases} x_{J,j} - \xi & (x_{J,j} \neq 0) \\ 0 & (x_{J,j} = 0) \end{cases}$$

then $\sum_{J \ni j} x'_{J,j} > a_j$ $(j \in K)$; of course $\sum_{j\ J} x'_{J,j} \leqslant x_J$. For any $h \in \mathbf{N}$ with $h > \xi^{-1}$ there exist $x''_{J,j} \in h^{-1}\mathbf{N}$ such that $x'_{J,j} \leqslant x''_{J,j} \leqslant x_{J,j}$ and for such $x''_{J,j}$ obviously $\sum_{J \in j} x''_{J,j} \leqslant x_J(J \in \mathfrak{P}(K) - \mathfrak{P}(\phi))$ and $\sum_{J \ni j} x''_{J,j} > a_j(j \in K)$. For each $J \in \mathfrak{P}(K) - \mathfrak{P}(\phi)$ fix an element $i(J) \in J$ and set $y_{J,j} = hx''_{J,j}(j \in J, j \neq i(J))$, $y_{J,i(J)} = hx''_{J,i(J)} + hx_J - \sum_{j \in J} hx''_{J,j}$. It is easy to see that the numbers $y_{J,j}$ have the required properties.

§ 2. LEVI'S LEMMA

Our point of departure is the following result concerning differential polynomials in $y_1, \ldots, y_n$ and a number of other differential indeterminates $u_{ij}(1 \leqslant i \leqslant n, 0 \leqslant j \leqslant r_i)$.

LEVI'S LEMMA. *Let $G_1, \ldots, G_n$ be differential polynomials in $\mathbf{Q}\{y_1, \ldots, y_n, (u_{ij})_{1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant r_i}\}$ given by $G_i = u_{i0}y_i^{q_i} + \sum_{1 \leqslant j \leqslant r_i} u_{ij}M_{ij}$ $(1 \leqslant i \leqslant n)$ where, for each i, $q_i \in \mathbf{N}$ and $M_{i1}, \ldots, M_{ir_i}$ are* *differential monomials in* $y_1, \ldots, y_n$ *of degree* $> q_i$. Then there exist a monomial

$$U = u_{10}^{d_1} \ldots u_{n0}^{d_n} \text{ in } u_{10}, \ldots, u_{n0}$$

and a differential polynomial $Y \in \mathbf{Q}\{y_1, \ldots, y_n, (u_{ij})_{1 \leqslant i \leqslant n, 0 \leqslant j \leqslant r_i}\}$ *with the following properties*:

Y is homogeneous in $(\theta u_{ij})_{\theta \in \Theta, 0 \leqslant j \leqslant r_i}$ *of degree* d_i $(1 \leqslant i \leqslant n)$;
the degree of Y in $(\theta u_{i0})_{\theta \in \Theta, 1 \leqslant i \leqslant n}$ *is* $< d_1 + \ldots + d_n$;
$Y \in [y_1, \ldots, y_n]$;
$y_i(U + Y) \in \{G_1, \ldots, G_r\}$ $(1 \leqslant i \leqslant n)$.

Levi proved this result for ordinary differential polynomials ([4] §§ 32–36, § 47), and for partial differential polynomials in the case $r = 1$ ([5], p. 118). The general lemma can be established in the same way with little extra difficulty.

§ 3. DOMINATION

We deal with differential monomials in $y_1, \ldots, y_n$, that is, with products of powers of derivatives θy_i $(\theta \in \Theta, 1 \leqslant i \leqslant n)$.

By a *prime factor* of such a differential monomial M we mean a derivative θy_i which divides M. If V is *any* set of derivatives θy_i we let ΘV denote the set of all derivatives of the y_i which can be written in the form θv $(\theta \in \Theta, v \in V)$. The product of all the prime factors w of M with $w \in \Theta V$, each w taken the same number of times as it occurs in M, is a differential monomial which we denote by M_V.

Let M and N be differential monomials. We shall say that N *dominates* M if, for every set V, the following condition is satisfied:

$$\text{either } \deg M_V < \deg N_V \text{ or } M_V = N_V.$$

Since $M_V = M_{\Theta V} = M_{(\Theta V) \cap V(M)}$, where $V(M)$ denotes the set of all prime factors of M, it suffices to verify this condition for every nonempty set V with $V \subset V(M)$. If, for every nonempty V with $V \subset V(M)$, N satisfies the stronger condition

$$\deg M_V < \deg N_V$$

then we shall say that N *strongly* dominates M.

It is easy to see that there exists a biggest set W of prime factors of M such that $M_W = N_W$. If N dominates M then a necessary and sufficient condition that N strongly dominate M is that W be empty. We shall call W the *weakness* of N over M.

If N_k dominates M and W_k denotes the weakness of N_k over M $(1 \leqslant k \leqslant r)$ then the weakness of $\prod_{1 \leqslant k \leqslant r} N_k$ over M^r is $\bigcap_{1 \leqslant k \leqslant r} W_k$.

§ 4. FACTORIAL DOMINATION

If N_k dominates (resp. strongly dominates) M_k $(1 \leqslant k \leqslant r)$ then $\prod_{1 \leqslant k \leqslant r} N_k$ dominates (resp. strongly dominates) $\prod_{1 \leqslant k \leqslant r} M_k$. If N_k dominates M $(1 \leqslant k \leqslant r)$ and, for at least one k,

N_k strongly dominates M then $\prod_{1 \leqslant k \leqslant r} N_k$ strongly dominates M^r. It follows from the former statement that if $M = \prod_{1 \leqslant k \leqslant r} v_k^{a_k}$, where $v_1, \ldots, v_r$ are the distinct prime factors of M, and if $N = \prod_{1 \leqslant k \leqslant r} N_k$ where N_k dominates (resp. strongly dominates) $v_k^{a_k}$ $(1 \leqslant k \leqslant r)$, then N dominates (resp. strongly dominates) M. We shall say in such a case that N dominates (resp. strongly dominates) M *factorially*.

If N_l dominates (resp. strongly dominates) M_l factorially $(1 \leqslant l \leqslant s)$ then $\prod_{1 \leqslant l \leqslant s} N_l$ dominates (resp. strongly dominates) $\prod_{1 \leqslant l \leqslant s} M_l$ factorially. If N_l dominates M factorially $(1 \leqslant l \leqslant s)$ and, for at least one l, N_l strongly dominates M factorially then $\prod_{1 \leqslant l \leqslant r} N_l$ strongly dominates M^s factorially.

Shapiro's lemma enters at this point.

First Preliminary Lemma. *If N_l strongly dominates M $(1 \leqslant l \leqslant s)$ then, for all $(i_1, \ldots, i_s) \in \mathbf{N}^s$ for which the sum $h = i_1 + \ldots + i_s$ is sufficiently big, $\prod_{1 \leqslant l \leqslant s} N_l^{i_l}$ strongly dominates M^h factorially.*

Proof. Write $M = \prod_{k \in K} v_k^{a_k}$ with K a finite set and the elements v_k $(k \in K)$ the distinct prime factors of M. For each nonempty set $J \subset K$ let x_{lJ} denote the number of prime factors v of N_l such that v is a derivative of v_k for every $h \in J$ and is not a derivative of v_k for any $k \in K - J$ (each v being counted as many times as it occurs in N_l). Because N_l strongly dominates M we have for each l

$$\sum_{J \in \mathfrak{P}(K) - \mathfrak{P}(K-I)} x_{lJ} > \sum_{i \in I} a_i \qquad (I \in \mathfrak{P}(K) - \mathfrak{P}(\phi)).$$

By the corollary to Shapiro's lemma there exist an $h_0 \in \mathbf{N}$ and, for each (J, j) with $J \in \mathfrak{P}(K) - \mathfrak{P}(\phi)$ and $j \in J$, a $y_{lJj} \in \mathbf{N}$ such that

$$\sum_{j \in J} y_{lJj} = h_0 x_J \quad \text{and} \quad \sum_{J \ni j} y_{lJj} > h_0 a_j.$$

It follows that we may write $N_l^{h_0} = \prod_{j \in K} N_{lj}$ where, for each $j \in K$, N_{lj} is a differential monomial of which the degree in $(\theta v_j)_{\theta \in \Theta}$ is $\geqslant h_0 a_j + 1$. Let $(i_1, \ldots, i_s) \in \mathbf{N}^s$ and write $i_l = q_l h_0 + r_l$ with $q_l, r_l \in \mathbf{N}$ and $r_l < h_0$. Then $\prod_{1 \leqslant l \leqslant s} N_l^{i_l} = \prod_{j \in K} \prod_{1 \leqslant l \leqslant s} N_{lj}^{q_l} \cdot \prod_{1 \leqslant l \leqslant s} N_l^{r_l}$. For each $j \in K$ the degree of $\prod_{1 \leqslant l \leqslant s} N_{lj}^{q_l}$ in $(\theta v_j)_{\theta \in \Theta}$ is

$$\geqslant \sum_{1 \leqslant l \leqslant s} q_l(h_0 a_j + 1) = \sum_{1 \leqslant l \leqslant s} (i_l - r_l) h_0^{-1}(h_0 a_j + 1) \geqslant (h - s(h_0 - 1)) h_0^{-1}(h_0 a_j + 1),$$

where $h = \sum_{1 \leqslant l \leqslant s} i_l$, so that this degree is $> a_j h$ provided $h > s(h_0 - 1)(h_0 a_j + 1)$. Whenever this is the case then $\prod_{1 \leqslant l \leqslant s} N_l^{i_l}$ strongly dominates M^h factorially.

Second Preliminary Lemma. *Let $F = \sum_{0 \leqslant l \leqslant s} u_l M_l \in \mathbf{Q}\{y_1, \ldots, y_n, u_0, \ldots, u_s\}$, where $M_0, M_1, \ldots M_s$ are differential monomials in $y_1, \ldots, y_n$ such that $M_0 \neq 1$ and $M_l \neq M_0$ $(1 \leqslant l \leqslant s)$. If each M_l with $l \neq 0$ dominates (resp. strongly dominates) M_0 then the ideal (F)*

contains a differential polynomial $G = u_0^a M_0^a + \sum_{1 \leq l \leq t} U_l N_l$, *where* $a \in \mathbf{N}$, $a \neq 0$, *each* U_l *is a monomial in* $u_0, u_1, \ldots, u_s$ *different from* u_0^a *of degree* a, *and each* N_l *is a differential monomial in* $y_1, \ldots, y_n$ *different from* M_0^a *which dominates (resp. strongly dominates)* M_0^a *factorially.*

Proof. Suppose that each M_l with $l \neq 0$ strongly dominates M_0. Raising both sides of the congruence $u_0 M_0 \equiv -\sum_{1 \leq l \leq s} u_l M_l \pmod F$ to an odd power a, we obtain a congruence $u_0^a M_0^a \equiv -\sum u_{l_1} \ldots u_{l_a} M_{l_1} \ldots M_{l_a} \pmod F$; by the first preliminary lemma we may choose a so big that each $M_{l_1} \ldots M_{l_a}$ strongly dominates M_0^a factorially, and therefore the differential polynomial $G = u_0^a M_0^a + \sum u_{l_1} \ldots u_{l_a} M_{l_1} \ldots M_{l_a}$ satisfies the 'resp.' part of the conclusion.

Now suppose merely that M_l dominates M_0 $(1 \leq l \leq s)$. Let Λ_0 denote the set of indices l with $l \neq 0$ such that M_l dominates M_0 factorially. For each l with $l \neq 0$ and $l \notin \Lambda_0$ the weakness of M_l over M_0 is a subset of the set of prime factors of M_0. Denote the distinct weaknesses of the various M_l with $l \neq 0$ and $l \notin \Lambda_0$ by $W_1, \ldots, W_k$ and for each W_j let Λ_j denote the set of indices l with $l \neq 0$ and $l \notin \Lambda_0$ such that the weakness of M_l over M_0 is W_j; we choose the notation so that $W_k \not\subset W_j$ $(1 \leq j \leq k-1)$. Then $F = u_0 M_0 + \sum_{0 \leq j \leq k} \sum_{l \in \Lambda_j} u_l M_l$. Set $\pi = \operatorname{card} \bigcup_{1 \leq j \leq k} \mathfrak{P}(W_j)$.

If $\pi = 0$ we may take $G = F$. Let $\pi > 0$ and suppose the result proved for lower values of π. Then $k > 0$. Raising both sides of the congruence $u_0 M_0 + \sum_{l \in \Lambda_0} u_l M_l \equiv -\sum_{l \leq j \leq k} \sum_{l \in \Lambda_j} u_l M_l \pmod F$ to an odd power h, we obtain on the left $u_0^h M_0^h$ plus a number of terms UN with U a monomial in u_0, u_l $(l \in \Lambda_0)$ different from u_0^h of degree h and N a differential monomial in $y_1, \ldots, y_n$ different from M_0^h which dominates M_0^h factorially. On the right we obtain a sum of terms $-UN = -u_{l_1} \ldots u_{l_h} M_{l_1} \ldots M_{l_h}$; for any such term either some index l_i is in a Λ_j with $1 \leq j \leq k-1$ or $l_i \in \Lambda_k$ $(1 \leq i \leq h)$. In the former case the weakness of N over M_0^h is a subset of W_j for some j with $1 \leq j \leq k-1$. In the latter case we may write $M_{l_i} = M'_{l_i} M_{l_i W_k} = M'_{l_i} M_{0 W_k}$ $(1 \leq i \leq h)$ and $M_0 = M'_0 M_{0 W_k}$, and each M'_{l_i} strongly dominates M'_0; by the first preliminary lemma we may choose h so that $M'_{l_1} \ldots M'_{l_h}$ strongly dominates $M_0'^h$ factorially, and then $N = M_{l_1} \ldots M_{l_h} = M'_{l_1} \ldots M'_{l_h} M_{0W_k}^h$ dominates $M_0^h = M_0'^h M_{0W_k}^h$ factorially. Transposing to the left side all the terms on the right we obtain on the left a differential polynomial.

$$F^* = U_0 M_0^* + \sum_{0 \leq j \leq k^*} \sum_{l \in \Lambda_j^*} U_j M_j^* \in (F)$$

where $\Lambda_0^*, \ldots, \Lambda_{k^*}^*$ are disjoint finite sets not containing 0, $U_0 = u_0^h$, each U_l with $l \neq 0$ is a monomial in $u_0, u_1, \ldots, u_s$ different from u_0^h of degree h, $M_0^* = M_0^h$, every M_l^* with $l \in \Lambda_0^*$ is a differential monomial in $y_1, \ldots, y_n$ different from M_0^h which dominates M_0^h factorially, for each index j with $1 \leq j \leq k^*$ all the M_l^* with $l \in \Lambda_j^*$ are differential monomials in $y_1, \ldots, y_n$ different from M_0^h which dominate M_0^h and have over M_0^h one and the'same weakness W_j^*, and each of these weaknesses $W_1^*, \ldots, W_{k^*}^*$ is a subset of some W_j with $1 \leq j \leq k-1$. It follows from the last remark that the number $\pi^* = \operatorname{card} \bigcup_{1 \leq j \leq k^*} \mathfrak{P}(W_j^*)$ has the property that $\pi^* < \pi$. Therefore we may apply the present lemma to F^*, and the existence of a differential polynomial $G \in (F)$ with the required property quickly follows.

§5. THE DOMINATION LEMMA

We now come to the main lemma from which our results on singular solution will quickly follow.

DOMINATION LEMMA. *Let* $F = \sum_{0 \leqslant l \leqslant s} u_l M_l \in \mathbf{Q}\{y_1, \dots, y_n, u_0, \dots, u_s\}$ *where* $M_0, \dots, M_s$ *are differential monomials in* $y_1, \dots, y_n$ *such that* $M_0 \neq 1$ *and* $M_l \neq M_0$ $(1 \leqslant l \leqslant s)$. *If each* M_l *with* $l \neq 0$ *dominates (resp. strongly dominates)* M_0 *then there exist a nonzero* $e \in \mathbf{N}$ *and a* $Y \in \mathbf{Q}\{y_1, \dots, y_n, u_0, \dots, u_s\}$ *with the following properties*:

Y is homogeneous in $(\theta u_l)_{\theta \in \Theta,\, 0 \leqslant l \leqslant s}$ *of degree* e;

the degree of Y in $(\theta u_0)_{\theta \in \Theta}$ *is* $< e$;

$Y \in [y_1, \dots, y_n]$ *(resp.* $Y \in \{M_0\}$*)*;

$M_0 (u_0^e + Y) \in \{F\}$.

Proof. Write $M_0 = v_1^{q_1} \dots v_t^{q_t}$, where $v_1, \dots v_t$ are the distinct prime factors of M_0. Suppose first that $t = 1$. For each l either M_l is divisible by $v_1^{q_1}$ or the degree of M_l in Θv_1 is $> q_1$. Therefore we may write

$$F = (u_0 + \sum_{l \in \Lambda'} u_l L_l) v_1^{q_1} + \sum_{l \in \Lambda''} u_l L_l N_l(v_1)$$

where Λ', Λ'' are disjoint sets the union of which is the set of indices $1, 2, \dots, s$ (with $\Lambda' = \phi$ if each M_l with $l \neq 0$ strongly dominates M_0), each L_l is a differential monomial in $y_1, \dots, y_n$, $\deg L_l > 0$ $(l \in \Lambda')$, and each N_l with $l \in \Lambda''$ is a differential monomial (in some new differential indeterminate z) of degree $> q_1$. We may apply Levi's lemma (case $n = 1$) to the differential polynomial $G = \bar{u}_0 z^{q_1} + \sum_{l \in \Lambda''} \bar{u}_l N_l \in \mathbf{Q}\{z, \bar{u}_0, (\bar{u}_l)_{l \in \Lambda''}\}$ to show the existence of a differential polynomial $z(\bar{u}_0^e + Z) \in \{G\}$ with Z homogeneous in $((\theta \bar{u}_0)_{\theta \in \Theta}, (\theta \bar{u}_l)_{\theta \in \Theta,\, l \in \Lambda''})$ of degree e, the degree of Z in $(\theta \bar{u}_0)_{\theta \in \Theta}$ strictly smaller than e, and $Z \in [z]$. Since the substitution of $(v_1, u_0 + \sum_{l \in \Lambda'} u_l L_l, (u_l L_l)_{l \in \Lambda''})$ for $(z, \bar{u}_0, (u_l)_{l \in \Lambda''})$ maps G onto F, the desired result follows.

Now suppose that $t > 1$ and that the lemma has been proved for lower values of t. By the second preliminary lemma, (F) contains a differential polynomial $G = u_0^a v_1^{q_1 a} \dots v_t^{q_t a} + \sum_{1 \leqslant j \leqslant r} U_j N_{1j} M'_j$, where each U_j is a monomial in $u_0, \dots, u_s$ other than u_0^a of degree a, each N_{1j} is a differential monomial in $y_1, \dots, y_n$ which dominates (resp. strongly dominates) $v_1^{q_1 a}$, each M'_j is a differential monomial in $y_1, \dots, y_n$ which dominates (resp. strongly dominates) $v_2^{q_2 a} \dots v_t^{q_t a}$, and $N_{1j} M'_j \neq M_0^a$. Replacing G by $(u_0^a M_0^a)^h + (\sum_{1 \leqslant j \leqslant r} U_j N_{1j} M'_j)^h$ if necessary, h denoting an odd natural number > 1, we may even suppose that $M'_j \neq v_2^{q_2 a} \dots v_t^{r_t a}$ $(1 \leqslant j \leqslant r)$. Then we may apply the present lemma (case $t - 1$) to the differential polynomial $F' = u'_0 v_2^{q_2 a} \dots v_t^{q_t a} + \sum_{1 \leqslant j \leqslant r} u'_j M'_j \in \mathbf{Q}\{y_1, \dots, y_n, u'_0, \dots, u'_r\}$ to prove the existence of a nonzero $e' \in \mathbf{N}$ and a $Y' \in \mathbf{Q}\{y_1, \dots, y_n, u'_0, \dots, u'_r\}$ with Y' homogeneous in $(\theta u'_j)_{\theta \in \Theta,\, 1 \leqslant j \leqslant r}$ of degree e', the degree of Y' in $(\theta u'_0)_{\theta \in \Theta}$ strictly smaller than e', $Y' \in [y_1, \dots, y_n]$ (resp. $Y' \in \{v_2^{q_2 a} \dots v_t^{q_t a}\} = \{v_2 \dots v_t\}$), and $v_2 \dots v_t (u_0'^{e} + Y') \in \{F'\}$. Substituting $(u_0^a v_1^{q_1 a}, U_1 N_{11}, \dots, U_r N_{1r}$ for $(u'_0, u'_1, \dots, u'_r)$ we see that $\{F\}$ contains a differential polynomial $v_2 \dots v_t (u_0^{a_1} v_1^{c_1} + \sum_{l \in \Lambda_1} U_{1l} M_{1l})$. where a_1 and c_1 are nonzero natural numbers, each U_{1l} is the product if a rational number with a differential monomial in

$u_0, \dots, u_s$ of degree a_1 and of degree in $(\theta u_0)_{\theta\in\Theta}$ strictly smaller than a_1, and each M_{1l} is a differential monomial in $y_1, \dots, y_n$ different from $v_1^{c_1}$ which dominates (resp. strongly dominates) $v_1^{c_1}$. Let Λ_1'' denote the set of indices $l \in \Lambda_1$ such that M_{1l} strongly dominates $v_1^{c_1}$, and set $\Lambda_1' = \Lambda_1 - \Lambda_1''$ (so that under the 'resp.' hypothesis $\Lambda_1' = \phi$). For each $l \in \Lambda_1'$ we may write $M_{1l} = L_{1l}v_1^{c_1}$ with L_{1l} a differential monomial in $y_1, \dots, y_n$ of degree > 0. Thus, the perfect differential ideal $\{F\}: M_0$ contains $(u_0^{a_1} + \sum_{l\in\Lambda_1'} U_{1l}L_{1l})v_1^{c_1} + \sum_{l\in\Lambda_1''} U_{1l}M_{1l}$. Similarly, for each $k \in \mathbf{N}$ with $1 \leqslant k \leqslant t$, $\{F\}: M_0$ contains a differential polynomial $(u_0^{a_k} + \sum_{l\in\Lambda_k'} U_{kl}L_{kl})v_k^{c_k} + \sum_{l\in\Lambda_k''} U_{kl}M_{kl}$ with entirely analogous properties. An easy application of Levi's lemma (case $n = t$) now completes the proof.

§ 6. SINGULAR SOLUTIONS

Let A be an irreducible differential polynomial in $\mathscr{F}\{y_1, \dots, y_n\}$, and suppose that $(0, \dots, 0)$ is a singular solution of the differential equation $A = 0$. A sufficient condition that $(0, \dots, 0)$ not be a zero of the general component $\mathfrak{P}(A)$ is provided by the domination lemma. We formulate the result in the following more general setting.

PROPOSITION. *Let $\mathfrak{P}$ be a prime differential ideal of $\mathscr{F}\{y_1, \dots, y_n\}$, and suppose that $\mathfrak{P}$ contains a differential polynomial $\sum_{0\leqslant l\leqslant s} C_l M_l(B_1, \dots, B_r)$, where: $C_l \in \mathscr{F}\{y_1, \dots, y_n\}$ $(0 \leqslant l \leqslant s)$ and $C_0(0, \dots 0) \neq 0$; $B_k \in \mathscr{F}\{y_1, \dots, y_n\}$ and $B_k(0, \dots, 0) = 0$ $(1 \leqslant k \leqslant r)$; $M_0, M_1, \dots, M_s$ are differential monomials in differential indeterminates $z_1, \dots, z_r$ with $M_k \neq M_0$ $(1 \leqslant k \leqslant r)$ such that M_k dominates $M_0 (1 \leqslant k \leqslant r)$;* and $M_0(B_1, \dots, B_r) \notin \mathfrak{P}$. *Then $(0, \dots, 0)$ is not a zero of $\mathfrak{P}$.*

Proof. By the domination lemma $\mathfrak{P}$ contains a differential polynomial $M_0(B_r, \dots, B_r)$ $(C_0^e + Y(B_1, \dots, B_r))$ where $e \in \mathbf{N}$ and $Y \in [z_1, \dots, z_r]$ in $\mathbf{Q}\{z_1, \dots, z_r, C_0, \dots, C_s\}$, so that $\mathfrak{P}$ contains the differential polynomial $C_0^e + Y(B_1, \dots, B_r)$ which does not vanish at $(0, \dots, 0)$.

REFERENCES

1. ABRAHAM HILLMAN: On the differential algebra of a single differential polynomial, *Ann. Math, Princeton* **56** (1952), 157–168.
2. ABRAHAM HILLMAN and DAVID MEAD: *Amer. J. Math.*, to be published.
3. E. R. KOLCHIN: Galois theory of differential fields, *Amer. J. Math.* **75** (1953), 753–824.
4. HOWARD LEVI: On the structure of differential polynomials and on their theory of ideals, *Trans. Amer. Math. Soc.* **51** (1942), 532–568.
5. HOWARD LEVI: The low power theorem for partial differential polynomials, *Ann. Math., Princeton* **46** (1945), 113–119.
6. J. F. RITT: Differential algebra, *Colloq. Lect. Amer. Math. Soc.* **33** (1950).

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