Topology and its Applications 12 (1981) 267–282
North-Holland Publishing Company

CELLULARITY IN POLYHEDRA

James P. HENDERSON
Department of Mathematics, University of Tennessee, Knoxville, TN 37916, USA

Received 17 March 1980
Revised 4 June 1980

A compact subset X of a polyhedron P is cellular in P if there is a pseudoisotopy of P shrinking precisely X to a point. A proper surjection between polyhedra $f: P \to Q$ is cellular if each point inverse of f is cellular in P. It is shown that if $f: P \to Q$ is a cellular map and either P or Q is a generalized n-manifold, $n \neq 4$, then f is approximable by homeomorphisms. Also, if P or Q is an n-manifold with boundary, $n \neq 4, 5$, then a cellular map $f: P \to Q$ is approximable by homeomorphisms. A cellularity criterion for a special class of cell-like sets in polyhedra is established.

AMS (MOS) Subj. Class. (1979): Primary: 57N80, 57Q55; Secondary: 57N60, 57P99, 57Q30

polyhedron	cellular map	intrinsic stratification
cellular set	pseudoisotopy	generalized n-manifold
engulfing	approximable by homeomorphisms	

Introduction

One of the more active areas of study has been that of trying to identify those maps $f: M^n \to Y$ which are approximable by homeomorphisms, where M^n is a topological n-manifold. In particular the class of cellular maps from a manifold onto a topological space has been extensively studied. Siebenmann proved that a cellular map $f: M^n \to N^n$ is approximable by a homeomorphism, where N^n is also an n-manifold, $n \neq 4$ [14]. More recently, Edwards has shown that a cellular map $f: M^n \to Y$, $n \geq 5$, is approximable by homeomorphisms if Y is a finite dimensional ANR such that maps $f, g: B^2 \to Y$ may be approximated by $\tilde{f}, \tilde{g}: B^2 \to Y$ such that $\tilde{f}(B^2) \cap \tilde{g}(B^2) = \emptyset$ [8].

Relatively little has been done on the approximation of maps between polyhedra by homeomorphisms. Handel used an intrinsic stratification of polyhedra to approximate a certain type of cellular map $f: P \to Q$ between polyhedra by homeomorphisms. He required that for a point y in an n-dimensional stratum of Q, the set $f^{-1}(y)$ must be a cellular subset of the manifold which is the n-dimensional stratum of P [9].

Cannon proposed a much broader class of maps for study by defining a more general concept of cellularity [4]. He defined a cellular set X in a polyhedron P to be a compact subset of P for which there is a pseudoisotopy of P shrinking precisely X. Hence a cellular set in a polyhedron may intersect more than one stratum. Cellular maps between polyhedra are those proper maps $f: P \to Q$ such that for each $y \in Q$, the set $f^{-1}(y)$ is a non-empty cellular set in P.

Cannon asked if a cellular map $f: P \to Q$ is approximable by homeomorphisms if P or Q is a manifold. That question is answered by the following theorem.

Theorem. *Let $f: P \to Q$ be a cellular map with either P or Q a generalized n-manifold, $n \neq 4$. Then f is approximable by homeomorphisms.*

In light of McMillan's criterion for cellularity in a P.L. manifold [13], it seems natural to ask if there is a cellularity criterion for polyhedra. In the last section of this paper, a restricted concept of a cell-like set in a polyhedron is introduced, and a polyhedral cellularity criterion given for that class of cell-like sets.

This paper is an expansion of some of the results of my thesis completed under the supervision of Professor J.W. Cannon.

0. Definitions and background

A *polyhedron* is a subset of some Euclidean space $\mathbb{R}^n$ such that each point $b \in P$ has a neighborhood $N = bL$, the join of b and a compact set L. Throughout, P and Q will denote polyhedra.

A *pseudoisotopy* is a homotopy $H_t: P \to P$ such that H_t is a homeomorphism for $0 \leq t < 1$ and $H_1: P \to P$ is a surjection.

A compact subset X of P is said to be cellular in P if there is a pseudoisotopy $H_t: P \to P$ such that X is the only nondegenerate point pre-image of H_1. As an example, let P be a 2-simplex and X an arc in P. If X lies in the interior of P or if X meets the boundary of P in one point, then X is cellular in P. However, if X meets the boundary of P in a set which is not connected, then X is not cellular in P. This example makes it clear that the property of a set being cellular depends not only on the set itself, but also on its embedding in P. In fact, if P is a manifold, then the above definition is equivalent to the usual definition of cellularity.

A *proper surjection $f: P \to Q$ is cellular* if for each $y \in Q$, $f^{-1}(y)$ is cellular in P. Again, if P is a manifold, then this corresponds to the standard concept of a cellular map.

At this point, Siebenmann's approximation theorem will be stated for later reference.

Theorem 0.1 [14]. *Let $f: M^n \to N^n$ be a cellular map where M^n and N^n are n-manifolds, possibly with boundary, such that $f|\partial M$ gives a cellular map $f|\partial M: \partial M \to \partial N$. Suppose that one of the following holds.*

(a) $n \neq 4, 5$
(b) $n = 5$ *and* $f|\partial M$ *is a homeomorphism.*
Then f is approximable by homeomorphisms.

Two engulfing theorems will also be given here for later use. The first is Stalling's engulfing theorem [15], and the second is a slightly modified form of an engulfing theorem discussed by Ancel [2].

Theorem 0.2. *Let M^n be a P.L. n-manifold, $n \geq 5$, U an open set in M, K a complex in M of dimension $\leq n-3$ such that K is closed in M, and L a subcomplex of K in U such that* $\text{cl}(K-L)$ *is a polyhedron of a finite r-subcomplex R of K. Let (M, U) be r-connected. Then there is a compact set $E \subset M$ and an isotopy $h_t : M \to M$ such that $K \subset h_1(U)$ and $h_t|(M-E) \cup L = \text{id}$.*

Theorem 0.3. *Let M^n be a P.L. n-manifold, $n \leq 5$. Suppose that for an integer r, $0 \leq r \leq n-3$, there exist open sets U_i and V_{i+1}, $-1 \leq i \leq r$, such that $U_{i+1} \subset U_i$, $V_{i+1} \subset U_i$, and each i-complex in U_i may be homotoped into V_i* rel *V_{i+1} by a homotopy in U_{i-1}.*

Then given a closed complex K in U_r of dimension $\leq n-3$, and a subcomplex L of K in V_{r+1} such that $\text{cl}(K-L)$ *is the polyhedron of a finite r-subcomplex R of K, there is an isotopy $h_t : M \to M$ and a compact subset E of U_{-1} such that $h_1(V_0) \supset K$ and $h_t|(M-E) \cup L = \text{id}$.*

1. A stratification of polyhedra

Given a polyhedron P and $x \in P$, define the intrinsic dimension of x in P, denoted $I(x, P)$, by $I(x, P) = \max\{n \in \mathbb{Z} \mid \text{there is an open embedding } h : \mathbb{R}^n \times cL \to P \text{ with } L \text{ a compact polyhedron and } h(\mathbb{R}^n \times cL) \text{ a neighborhood of } h(0 \times c) = x\}$, where cL is the open cone on L. The intrinsic n-skeleton of P is $P^{(n)} = \{x \in P \mid I(x, P) \leq n\}$ and the intrinsic n-stratum of P is $P[n] = P^{(n)} - P^{(n-1)}$. The depth of the stratification is

$$d(P) = \max\{i - j \mid P[i] \neq \emptyset \text{ and } P[j] \neq \emptyset\}.$$

The next three propositions are standard stratification results. Proofs of similar propositions may be found in [1].

Proposition 1.1. *$P[n]$ is a topological manifold.*

Two points x and y are said to lie in the same isotopy class of P if there is an isotopy $g_t : P \to P$ with $g_0 = \text{id}$ and $g_1(x) = y$. One of the more useful properties of the given stratification of a polyhedron is that the strata and the isotopy classes coincide.

Proposition 1.2. *Two points x and y lie in the same isotopy class of P iff x and y lie in the same component of some stratum $P[n]$ of P.*

The next proposition provides a justification for labeling this stratification intrinsic.

Proposition 1.3. *Given a triangulation T of P and an integer n, there is a subcomplex K_n of T such that $K_n = P^{(n)}$.*

Proposition 1.4. *Let $x \in P[n]$ and $h: \mathbb{R}^n \times cL \to P$ be an embedding providing a neighborhood $h(\mathbb{R}^n \times cL)$ of $h(0 \times D) = x$. There is a compact polyhedron $L_i \subset L$ such that $h(\mathbb{R}^n \times cL) \cap P^{(i)} = h(\mathbb{R}^n \times cL_i)$ for $i \geq n$.*

Proof. Let S be a triangulation of L, and define

$$L_i = \bigcup\{\sigma \in S \mid h(\mathbb{R}^n \times c\sigma) \cap P^{(i)} \neq \emptyset\}.$$

For $z \in L_i$, $h(\mathbb{R}^n \times cz) \cap P^{(i)} \neq \emptyset$ implies $h(\mathbb{R}^n \times cz) \subset P^{(i)}$. Therefore

$$h(\mathbb{R}^n \times cL_i) \subset h(\mathbb{R}^n \times cL) \cap P^{(i)}.$$

If $w \in h(\mathbb{R}^n \times cL) \cap P^{(i)}$ and $w \notin P[n]$, then

$$w \in h(\mathbb{R}^n \times (0, \infty) \times L) \cap P^{(i)},$$

where $(\mathbb{R}^n \times cL) - (\mathbb{R}^n \times c)$ is regarded as $\mathbb{R}^n \times [(0, \infty) \times L]$. It then follows that there is some $\sigma \in S$ such that

$$w \in h(\mathbb{R}^n \times (0, \infty) \times \sigma) \subset h(\mathbb{R}^n \times cL_i).$$

The next proposition is not so much concerned with the stratification as with the ability to isotopically move sets in neighborhoods of the form $\mathbb{R}^n \times cL$ close to the n-stratum. This result is used repeatedly.

Proposition 1.5. *Let $x \in P[n]$ and U be a neighborhood of x homeomorphic to $\mathbb{R}^n \times cL$. Then for each compact set E in U and $\varepsilon > 0$, there is an isotopy $h_t: U \to U$ with compact support in $U - P[n]$ such that $h_0 = \mathrm{id}$ and $h_1(E) \subset U \cap N_\varepsilon(P[n])$.*

Proof. If we consider cL as the identification space

$$\frac{[0, \infty) \times L}{\{0\} \times L},$$

define $cL(m)$ to be the open subset of cL given by

$$\frac{[0, m) \times L}{\{0\} \times L}.$$

Let g be a homeomorphism from U onto $\mathbb{R}^n \times cL$. Since $g(E)$ is a compact subset of $\mathbb{R}^n \times cL$, there is a P.L. n-cell B^n in $\mathbb{R}^n$ and an integer m such that $g(E) \subset B^n \times cL(m) \subset \mathbb{R}^n \times cL$. Choose another P.L. n-cell D^n such that $B^n \subset \operatorname{int} D^n$, and positive numbers k, s, and t such that $m < k$, $s < t$, and $B^n \times cL(t) \subset g(U \cap N_\varepsilon(P[N]))$. We may now use the various product structures to define an isotopy $\tilde{h}_t : \mathbb{R}^n \times cL \to \mathbb{R}^n \times cL$ taking $B^n \times cL(m)$ onto $B^n \times cL(t)$ which is the identity outside of $(D^n \times cL(k)) - (D^n \times cL(s))$. The desired isotopy is then $g^{-1}\tilde{h}_t g : U \to U$.

The last theorem of this section is an isotopy extension theorem due to Aiken.

Theorem 1.6 [1]. *Let P be a polyhedron such that each stratum of P inherits a P.L. manifold structure from P. If* $\tilde{g}_t : P[n] \to P[n]$ *is a P.L. isotopy with compact support E, then for each neighborhood U of E in P, there is an isotopy* $g_t : P \to P$ *which extends* $\tilde{g}_t$ *and has support in U.*

2. Cellular sets in polyhedra

In this section the basic properties of cellular sets are developed. One should note the strong similarities between the properties of cellular sets in polyhedra and the properties of the traditional cellular sets in a manifold.

Proposition 2.1. *Let* $g_t : P \to P$ *be a pseudoisotopy shrinking precisely the cellular set X. Then if* $P[l]$ *is the lowest dimensional stratum that X intersects, X intersects only one component A of* $P[l]$ *and* $g_1(X) \in A$.

Proof. Let $x \in X \cap P[l]$. Since g_t is isotopic to the identity for $t < 1$, $g_t(x)$ lies in $P[l]$ for $t < 1$. Therefore $g_1(X) = g_1(x)$ must lie in $P^{(l)}$. Suppose that $g_1(X) \in P[k]$, $k < l$. If U is a neighborhood of X such that $U \cap P^{(l-1)} = \emptyset$, then $g_1(U)$ is a neighborhood of $g_1(X)$ such that $g_1(U - X) \cap P^{(l-1)} = \emptyset$. It then follows that $k = 0$. But if $g_1(X) = y \in P[0]$, then $y \in X$, for $g_t(y) = y$ for $t > 1$. This contradicts the assumption that $P[l]$ was the lowest dimensional stratum that X intersects.

Suppose now that there is a component B of $P[l]$ such that $X \cap B \neq \emptyset$ and $g_1(X) \notin B$. Then for $w \in X \cap B$, the path $g_t(w)$, $0 \leq t \leq 1$, will be a path in $P[l]$ having end points in different components of $P[l]$. Hence X intersects only one component of $P[l]$ and $g_1(X)$ lies in that component.

The definition of cellularity for polyhedra was given in terms of pseudoisotopies, not cells or decomposition spaces. At this point, we would like to show that with the proper interpretation the concepts of a defining sequence and the shrinking of decomposition spaces, which may be used to define cellularity in manifolds, carry over to polyhedral cellularity. We will prove that a cellular set in a polyhedron has a defining sequence of neighborhoods all homeomorphic to an open cone neighborhood of a specific point in the polyhedron. Such neighborhoods will be homeomorphic to $\mathbb{R}^n \times cL$ for some integer n and compact polyhedron L, and are called *cellular neighborhoods*.

Theorem 2.2. *The following are equivalent*:

(1) *X is cellular in P.*

(2) *The projection $\pi : P \to P/X$ is approximable by homeomorphisms.*

(3) *$X = \bigcup_{i=1}^{\infty} N_i$, where the N_i's are homeomorphic cellular neighborhoods with $\bar{N}_{i+1} \subset N_i$.*

Proof. (1)→(3). Let g_t be the pseudoisotopy shrinking X to the point $y \in P[l]$. It is sufficient to prove that for each neighborhood U of X, there is a neighborhood N of X homeomorphic to the open cone neighborhood $V = h(\mathbb{R}^l \times cL)$ of y such that $X \subset N \subset \bar{N} \subset U$. (Here h is the embedding given by the fact that $y \in P[l]$.) We may assume that $\bar{V} \subset g_1(U)$. Choose $s < 1$ such that for $t > s$, $g_t^{-1}(V) \subset U$. There is an r between s and 1 so that $g_r(X) \subset V$. We now have $X \subset g_r^{-1}(V) \subset g_r^{-1}(\bar{V}) \subset U$, and $g_r^{-1}(V)$ is the desired cellular neighborhood N.

(3)→(1). Using the product structure of N_1 in a manner similar to that in Proposition 1.5, we can define an isotopy $H_1 : P \times I \to P$ with the initial stage of H_1 being the identity, H_1 having compact support in N_1, and $\operatorname{diam} H_1(X \times 1) < 1$. Inductively then, define $H_i : P \times I \to P$ to be an isotopy such that $H_i(P \times 0) = H_{i-1}(P \times 1)$, the support of H_i lies in a compact subset of $H_{i-1}(N_i \times 1)$, and $\operatorname{diam} H_i(X \times 1) < 1/i$. The desired pseudoisotopy $g : P \times I \to P$ is defined by

$$g(x \times t) = \begin{cases} H_i(x \times t) & \text{if } t \in \left[\dfrac{i-1}{i}, \dfrac{i}{i+1}\right], \\ \lim\limits_{i \to \infty} H_i(x \times 1) & t = 1. \end{cases}$$

(3)→(2). Let $\pi : P \to P/X$ be the projection map, and $\varepsilon > 0$ be given. For each neighborhood U of X, there is a neighborhood N_i of X such that $N_i \subset U$ and $\pi(N_i) \subset N_\varepsilon(\pi(X))$. As before, we can use the product structure of N_i to construct a homeomorphism $h : P \to P$ with compact support in N_i such that $\operatorname{diam} h(x) < \varepsilon$. It now follows from the Bing Shrinking Criterion that $\pi : P \to P/X$ is approximable by homeomorphisms.

(2)→(3). Since P is homeomorphic to P/X, $\pi(X)$ has a neighborhood homeomorphic to $\mathbb{R}^k \times cL$. Given a neighborhood U of X, choose a neighborhood N of $\pi(X)$ homeomorphic to $\mathbb{R}^k \times cL$ such that $\pi(X) \subset N \subset \bar{N} \subset \pi(U)$. Since π is approximable by homeomorphisms, we may choose a homeomorphism $h : P \to P/X$ such that $X \subset h^{-1}(N) \subset h^{-1}(\bar{N}) \subset U$. As before, this shows that X is cellular in P.

The third characterization of cellular sets in Theorem 2.2 allows us to choose cellular neighborhoods of a cellular set X which have a predetermined product structure in a neighborhood of $P[l]$, the lowest dimensional stratum X intersects.

Corollary 2.3. *Let X be a cellular subset of P, W a neighborhood of X, and $P[l]$ as above. If $h : \mathbb{R}^l \times cL \to P$ is an open embedding such that $X \cap P[l] \subset h(\mathbb{R}^l \times cL) \subset W$, then there is an open embedding $g : \mathbb{R}^l \times cL \to P$ such that $X \subset g(\mathbb{R}^l \times cL) \subset W$ and $h = g$ in an ε-neighborhood of $\mathbb{R}^l \times c$ in $\mathbb{R}^l \times cL$ for some $\varepsilon > 0$.*

Proof. Let D^l be an l-cell in $\mathbb{R}^l$ such that $X \cap P[l] \subset h((\operatorname{int} D^l) \times cL)$. Choose $k: \mathbb{R} \times cL \to P$ to be an open embedding such that $k(\mathbb{R}^l \times cL) \cap P[l] \subset h(D^l \times c)$ and $X \subset k(\mathbb{R}^l \times cL) = W$. We may apply Proposition 1.5 to obtain an isotopy h_t with compact support in $k(\mathbb{R}^l \times cL)$ such that $h_1(X) \subset h(\mathbb{R}^l \times cL)$. The homeomorphism $g = h_1^{-1}h: \mathbb{R}^l \times cL \to W$ is the desired embedding. Note that g and h will agree in a neighborhood of $\mathbb{R}^l \times c$ since h_1 has compact support missing $h(\mathbb{R}^l \times c)$.

3. Cellular maps and generalized manifolds

We first consider cellular maps between polyhedra where either the domain or range is a generalized n-manifold, $n \neq 4$. A polyhedron P is a generalized n-manifold if for each $x \in P$, $H_*(P, P-x) \approx H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$. Two results form the basis for studying such maps. The first is a form of the Vietoris–Begle mapping theorem. The second is a theorem of J.W. Cannon and follows from his solution to the double suspension problem.

Theorem 3.1. [12]. *If G is an upper semi-continuous decomposition of a polyhedron P into cellular sets, then the natural projection $\pi: P \to P/G$ induces an isomorphism $\pi_*: H_q(P) \to H_q(P/G)$ for every q.*

Theorem 3.2 [3]. *A polyhedral generalized n-manifold P is locally an n-manifold except possibly at the vertices of P. If $n \leq 3$, or $n \geq 5$ and $\pi_1(lk(v, P)) = 0$, then P is locally an n-manifold at the vertex of P.*

The first approximation theorem for cellular mappings between polyhedra is a relatively straightforward application of the above results.

Theorem 3.3. *Let $f: P \to Q$ be a cellular map with P a generalized n-manifold, $n \neq 4$. Then f is approximable by homeomorphisms.*

Proof. We first show that Q is also a generalized n-manifold. Let $y \in Q$ and $g: P \to P$ be the final stage of a pseudoisotopy shrinking $f^{-1}(y)$ to the point $z \in P$. We consider the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & H_k(P-z) & \longrightarrow & H_k(P) & \longrightarrow & H_k(P, P-z) & \longrightarrow \\
& \uparrow g_* & & \uparrow g_* & & \uparrow \alpha & \\
\longrightarrow & H_k(P-f^{-1}(y)) & \longrightarrow & H_k(P) & \longrightarrow & H_k(P, P-f^{-1}(y)) & \longrightarrow \\
& \downarrow f_* & & \downarrow f_* & & \downarrow \beta & \\
\longrightarrow & H_k(Q-y) & \longrightarrow & H_k(Q) & \longrightarrow & H_k(Q, Q-y) & \longrightarrow
\end{array}$$

The restriction $\bar{f}$ of f to $P-f^{-1}(y)$ is also a cellular map onto $Q-y$, so both $\bar{f}_*$ and f_* are isomorphisms. The restriction $\bar{g}$ of g to $P-f^{-1}(y)$ is a homeomorphism while g is homotopic to the identity map. Therefore $\bar{g}_*$ and g_* are isomorphisms. It now follows from the five lemmas that α and β are isomorphisms, and Q is a generalized n-manifold.

As a result of Theorem 3.2, the non-manifold points of P and Q must lie in $P[0]$ and $Q[0]$. Let $D=Q[0]\cup f(P[0])$. Since $f^{-1}(D)$ is a discrete collection of cellular sets in P, we may assume that f is 1–1 over D. We now approximate the cellular map $f|P-f^{-1}(D): P-f^{-1}(D)\to Q-D$ be a homeomorphism h which may be extended to agree with f on $f^{-1}(D)$. The existence of the approximating homeomorphism h follows from Theorem 0.1.

The next step is to consider cellular maps where Q is a generalized n-manifold. The cellular maps whose image is a generalized n-manifold, $n\neq 4$, are approximable by homeomorphisms, but the proof of this fact involves an investigation of how the point preimage of f intersect the strata of P. The following proposition is the key to the proof.

Proposition 3.4. *Let $f:P\to Q$ be a cellular map with Q a generalized n-manifold, $n\neq 4$. Then for each component A of $P[k]$, there exists $y_A\in Q$ with $f^{-1}(y_A)\cap A\neq\emptyset$ and $f^{-1}(y_A)\cap P^{(k-1)}=\emptyset$.*

Proof. The proof will be by induction on the depth of the stratification of P, $d(P)$.

If $d(P)=0$, then P is an n-manifold for some n. For each component A of P, choose $x\in A$ and let $y_A=f(x)$.

Assume now that the proposition is true for polyhedra with depth of stratification less than d, and that $d(P)=d$. The proof will be completed by showing that if $P[l]$ is the lowest dimensional stratum of P, then $f^{-1}(f(P[l]))$ contains no component B of $P[j]$, $j>l$. Once this is done, we apply the inductive hypothesis to the cellular map

$$\bar{f}:P-f^{-1}(f(P[l]))\to Q-f(P[l]),$$

where $\bar{f}$ is the obvious restriction.

Suppose that $l=0$ and there is some component B of $P[j]$, $j>0$, such that $B\subset f^{-1}(f(P[0]))$. Since B is connected, $B\subset f^{-1}(f(x))$ for some $x\in P[0]$. If $g_t:P\to P$ is a pseudoisotopy shrinking $f^{-1}(f(x))$, then $g_t(B)\approx B$ for $t<1$. Therefore diam $g_t(B)>\varepsilon$ for some $\varepsilon>0$ and $t<1$. Thus the assumption that $g_1(B)=x$ is not possible.

Assume now that $l>0$ and B is a component of $P[j]$ such that $B\subset f^{-1}(f(P[l]))$. We take B to be minimal in the sense that no component of a lower dimensional stratum $P[m]$ has the same property for $m>l$. Since $P[l]$ is closed, we have that $f(\bar{B})\subset f(P[l])$. The set $\bar{B}-B$ will consist of the union of components of lower dimensional strata, so $\bar{B}-B\subset P[l]$. Consider the polyhedron $K=P[l]\cup B$, and $G=\{f^{-1}(y)\cap K\,|\,y\in Q\}$. Then G will be a cellular, upper semi-continuous decomposition of K. Let $g \in G$ be such that $g \cap B \neq \emptyset$, and $x = h_1(g)$, where h_1 is the final state of a pseudoisotopy shrinking g. We have the following diagram:

$$\begin{array}{ccccccc}
\longrightarrow & H_*(K-x) & \longrightarrow & H_*(K) & \longrightarrow & H_*(K, K-x) & \longrightarrow \\
& \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & H_*(K-g) & \longrightarrow & H_*(K) & \longrightarrow & H_*(K, K-g) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_*(K/G-\pi(g)) & \longrightarrow & H_*(K/G) & \longrightarrow & H_*(K/G, K/G-\pi(g)) & \longrightarrow \\
& \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & H_*(P[l]-g) & \longrightarrow & H_*(P[l]) & \longrightarrow & H_*(P[l], P[l]-g) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_*(P[l]-x) & \longrightarrow & H_*(P[l]) & \longrightarrow & H_*(P[l], P[l]-x) & \longrightarrow
\end{array}$$

Using arguments similar to those in the proof of Theorem 3.3, we conclude that all verticle arrows are isomorphisms. But $H_l(P[l], P[l]-x) \approx \mathbb{Z}$ and

$$\begin{aligned} H_l(K, K-x) &\approx H_l(\mathbb{R}^l \times c\tilde{L}, \mathbb{R}^l \times c\tilde{L} - (0 \times c)) \\ &\approx H_{l-1}(\Sigma^l \tilde{L}) \approx H_{-1}(\tilde{L}) = 0. \end{aligned}$$

Therefore it is false that $K/G = P[l]/G^*$, where $G^* = \{g \cap P[l] \| g \in G\}$, and $B \not\subset f^{-1}(f(P[l]))$.

Theorem 3.5. *Let $f: P \to Q$ be a cellular map with Q a generalized n-manifold, $n \neq 4$. Then f is approximable by homeomorphisms.*

Proof. The idea of the proof is somewhat similar to that of the previous theorem. We first want to show that P is also a generalized n-manifold. Since each stratum of P is an isotopy class of P, we need only prove that each component of every stratum of P contains a point with the desired local homology.

Let A be a component of $P[k]$. Then Proposition 3.4 provides $y_A \in Q$ such that $f^{-1}(y_A) \cap A \neq \emptyset$ and $f^{-1}(y_A) \cap P^{(k-1)} = \emptyset$. Let $g_t: P \to P$ be a pseudoisotopy shrinking $f^{-1}(y_A)$ to the point x, which Proposition 2.1 tells us is in A. As before,

$$H_*(P, P-x) \approx H_*(P, P-f^{-1}(y_A)) \approx H_*(Q, Q-y_A) \approx H_*(\mathbb{R}^n, \mathbb{R}^n - 0).$$

Again, let $D = f(P[0]) \cup [0]$. We assume that f is 1–1 over D and approximate $\tilde{f}: P - f^{-1}(D) \to Q - D$ by a homeomorphism which extends the map f on $f^{-1}(D)$.

4. Cellular maps and manifolds with boundary

The first theorem of this section is probably the key to studying cellular maps between polyhedra. Although a cellular map $f: P \to Q$ need not be stratum preserving in the sense that for $y \in Q[n]$, $f^{-1}(y) \subset P[n]$, we prove that $P[n] \neq \emptyset$ if and only if $Q[n] \neq \emptyset$, with one restriction on P and Q.

Theorem 4.1. *Let $f: P \to Q$ be cellular with $P[4] = Q[4] = \emptyset$. Then $P[i] \neq \emptyset$ if and only if $Q[i] \neq \emptyset$, and $f|P^{(i)} = f_i : P^{(i)} \to Q^{(i)}$ is a cellular map with $Q[i] = f_i(P[i] - f_i^{-1}(f_i(P^{(i-1)})))$.*

Proof. The proof will be by induction on $d(Q)$.

If $d(Q) = 0$, then the map is approximable by homeomorphisms, so if $Q = Q[n]$, $f = f_n$, and $P = P[n]$.

Assume now that $d(Q) = k$ and the theorem is true if the depth of the stratification of the range of a cellular map is less than k. Let $Q[m]$ be the highest dimensional stratum of Q. Then $f^{-1}(Q[m])$ is an open subset of P which is homeomorphic to $Q[m]$ by Theorem 3.5. Therefore $P[m] \neq \emptyset$. Let $P_{m-1} = P - A_m$, where $A_m = \bigcup\{A \mid A$ is a component of $P[m]$ intersecting $f^{-1}(Q[m])\}$. Then P_{m-1} is a subpolyhedron of P, and $f|P_{m-1} = \bar{f}_{m-1} : P_{m-1} \to Q^{(m-1)}$ is a cellular map. We apply the induction assumption to this map and conclude that $P_{m-1} = P_{m-1}^{(m-1)}$, and $\dim P_{m-1} \leq m-1$. Therefore $P[m] = A_m$, and $P = P^{(m)}$. We now have $\bar{f}_{m-1} = f_{m-1}$. We would like to conclude that $Q[i] \neq \emptyset$ if $P[i] \neq \emptyset$ directly from the inductive assumption applied to $f_{m-1} : P^{(m-1)} \to Q^{(m-1)}$. However, the stratification of $P^{(m-1)}$ might not agree with that of P. Similarly there might be differences between the stratification of Q and $Q^{(m-1)}$. We can conclude that if $Q[j]$ is the highest dimensional stratum of Q contained in $Q^{(m-1)}$, then $Q[j]$ is open in $Q^{(m-1)} = Q^{(j)}$, and $Q[j]$ is a subset of $Q^{(j)}[j]$. As before, $f_{m-1}^{-1}(Q[j])$ is open in $P^{(m-1)}$, and homeomorphic to $Q[j]$. Also, $Q^{(j)}[j] - Q[j] \subset Q^{(j-1)}$ is a polyhedron of dimension less than j. If we construct A_j and P_{j-1} as we constructed A_m and P_{m-1}, we have that $f_m | P_{j-1} = \bar{f}_{j-1} : P_{j-1} \to Q^{(j-1)}$ is cellular. Again by induction P_{j-1} is a polyhedron of dimension less than j. Therefore A_j will be a subpolyhedron of dimension j. We now have that $P_{j-1} = P^{(j-1)}$. We continue in this manner, working down the intrinsic skeleton of Q to conclude the proof.

Theorem 4.2. *Let $f: P \to Q$ be a cellular map. If P or Q is an n-manifold, possibly with boundary, and with $n \neq 4, 5$, then f is approximable by homeomorphisms.*

Proof. Let P be a manifold with boundary. Since $P = P[n] \cup P[n-1]$, we know that $Q = Q[n] \cup Q[n-1]$. Each $y \in Q[n-1]$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times cL$. L must be a compact, 0-dimensional polyhedra, and hence be finite. We need only show that L consists of a single point.

Suppose $L=\emptyset$. Then $y\in B$, where B is a component of $Q[n-1]$ such that $B\cap \overline{Q[n]}=\emptyset$. Let $x\in P[n-1]$ be such that $f(x)=y$. We may choose a path from x to a point $z\in f^{-1}(Q[n])\subset P[n]$. But $f(x)\in B$, $f(z)\in Q[n]$, and $B\cap\overline{Q[n]}=\emptyset$. Therefore the image of the path is not connected, so $L\neq\emptyset$.

Assume $L=\{p_1,p_2,\ldots,p_j\}$, $j\geq 2$. We have

$$\tilde{H}_n(Q,Q-y)\approx\tilde{H}_{n-1}(\Sigma^{n-1}L)\approx\tilde{H}_0(L)\neq 0.$$

But

$$\tilde{H}_n(Q,Q-y)\approx\tilde{H}_n(P,P-f^{-1}(y))\approx\tilde{H}_{n-1}(B^n)=0.$$

Therefore L must be a single point, and Q is a manifold with boundary. We may now apply Theorem 0.1 to get the approximating homeomorphisms.

Assume now that Q is an n-manifold with boundary. Then $Q=Q[n]\cup Q[n-1]$ implies that $P=P[n]\cup P[n-1]$. Let $x\in P[n-1]$, and U be a cellular neighborhood of $f^{-1}(f(x))$, with U homeomorphic to $\mathbb{R}^{n-1}\times cL$. Then $f(U)$ is a connected neighborhood of $f(x)\in Q[n-1]$. There is neighborhood V of $f(x)$ such that V is homeomorphic to $\mathbb{R}^{n-1}\times[0,\infty)$ and $f(x)\subset V\subset U$. Now $f^{-1}(V)$ will be a neighborhood of x in U, with $f^{-1}(V)\cap P[n]$ connected. Therefore $U\cap P[n]$ must be connected, and L is a singleton. Therefore P is a manifold with boundary. We may again apply Theorem 0.1.

5. Polyhedral cellularity criterion

In [13], McMillan proved the following important result:

Theorem 5.1. *Let X be a cell-like set in the interior of a P.L. n-manifold M^n, $n\geq 5$. Then a necessary and sufficient condition that X be the intersection of a sequence of P.L. n-cells $\{F_i\}$ where $F_{i+1}\subset \operatorname{Int} F_i\subset M$ is that the following property holds:*

For each open set U containing X, there exists an open set V such that $X\subset V\subset U$ and each loop in $V-X$ is null homotopic in $U-X$.

We develop a generalization of McMillan's result. However, we will not be able to consider arbitrary cell-like sets in polyhedron. We need a restricted concept of a cell-like set.

A homotopy $h_t: Y\to P$ is *stratum respecting* if $I(h_t(y),P)\geq I(h_s(y),P)$ for $t\leq s$ and $y\in Y$. If for each $y\in Y$ $I(h_t(y),P)=I(h_s(y),P)$ for $t\leq s$, then the homotopy h_t is *stratum preserving*. A compact subset X of P is a *rooted cell-like* set in P if for each neighborhood U of X there is a neighborhood V of X with $X\subset V\subset U$ and a stratum respecting contraction of V in U.

If one is familiar with the proof of McMillan's theorem, then the reason for the restricted definition of a cell-like set should be clear. We will want to use the contraction of the neighborhood V of X in U to construct homotopies in each stratum.

Lemma 5.2. *Let X be a rooted cell-like set in P. Suppose that $P[i]$ is the highest dimensional stratum that X intersects, and that $X^* = X \cap P^{(i-1)}$ is a cellular set in P. Then for each neighborhood U of X and cellular neighborhood D of X^* in U, there is a neighborhood W of X, a cellular neighborhood N of X^*, and a stratum preserving homotopy $h_t : W \to U$ such that $h_0 = \mathrm{id}$, $h_1(W) \subset D$, and h_t is supported off of N.*

Proof. Choose $\bar{W}$ to be a neighborhood of X such that $\bar{W}$ is a compact subpolyhedron of U, $\bar{W} \cap P^{(i-1)} \subset D$, and there is a stratum respecting homotopy $g_t : \bar{W} \to U$ contracting $\bar{W}$ in U with $g_t(\bar{W} \cap P^{(i-1)}) \subset D$. There is some $\varepsilon > 0$ such that $g_t(N_\varepsilon(\bar{W} \cap P^{(i-1)}) \subset D$. Letting A_0 be a simplicial neighborhood of $\bar{W} \cap P^{(i-1)}$ in $\bar{W}$ such that $A_0 \subset N_{\varepsilon/2}(\bar{W} \cup P^{(i-1)})$, triangulate $\bar{W}$ in such a manner that A_0 is a full subcomplex and that $st(A_0, T) \subset N_\varepsilon(\bar{W} \cap P^{(i-1)})$. Choose $k : \bar{W} \to [0, 1]$ to be a simplicial map such that $A_0 = k^{-1}(0)$. We redefine the homotopy g_t on $\bar{W}$ as follows:

For $x \in \bar{W}$,

$$H_1(x, t) = \begin{cases} g_t(x) & \text{if } k(x) = s \geq t, \\ g_s(x) & \text{if } k(x) = s \leq t. \end{cases}$$

The homotopy H_1 is a stratum respecting homotopy which is the identity on A_0. Let $t(1)$ be the first time such that $H_1(x, t(1)) \in p^{(i-1)}$ for some $x \in \overline{W - A_0}$. Define

$$B_1 = \{x \in \overline{W - A_0} \mid H_1(x, t(1)) \in P^{(i-1)}\}.$$

Choose $s(1) < t(1)$ such that for $s(1) \leq s \leq t(1)$, $H_1(B_1, s) \subset N_{\varepsilon/2}(P^{(i-1)}) \cap D$. There is a simplicial neighborhood A_1 of B_1 in $\bar{W}$ such that for $s \geq s(1)$, $H_1(A_1, s) \subset N_\varepsilon(P^{(i-1)}) \cap D$. We now define the homotopy $H_2 : \bar{W} \times I \to U$ on $A_0 \cup A_1$. For $x \in A_0 \cup A_1$,

$$H_2(x, t) = \begin{cases} H_1(x, t) & t \leq s(1), \\ H_1(x, s(1)) & t \geq s(1). \end{cases}$$

This can be extended to the rest of $\bar{W}$ by using a simplicial map from $\bar{W}$ to $[0, 1]$ and the homotopy H_1 in the same manner that we extended H_1 from the identity on A_0. If $H_2(\bar{W}, s(1)) \not\subset D$, we define $s(2)$ in the same way that $s(1)$ was defined for H_1. We inductively define $s(n)$ if $H_{n-1}(\bar{W}, s(n-1)) \not\subset D$. The compactness of $\bar{W}$ and the fact that $H_{n-1}(\bar{W}, 1) \subset D$ guarantee that after a finite number of steps, there is an integer m such that $H_m(\bar{W}, s(m)) \subset D$. Note that the homotopy $H_m : \bar{W} \times [0, s(m)]$ is stratum preserving. We now choose N to be a cellular neighborhood of $X \cap P^{(k-1)}$ in D such that $N \cap H_m(\overline{W - A_0}, t) = \phi$ for $0 \leq t \leq s(m)$. The homotopy $h_t : W \to U$ is now given by $h_t(x) = H_m(x, t)$ for $0 \leq t \leq s(m)$.

We now state the polyhedra cellularity criterion.

Polyhedral Cellularity Criterion: A compact subset X of a polyhedron P satisfies the polyhedral cellularitiy criterion (PCC) if for each open set $U \supset X$, there is an open set $V \supset X$ such that $V \subset U$ and for every stratum $P[i]$, $i \geq 3$, each singular k-cell

$D^k \subset U \cap P[i]$ with bd $D^k \subset (V-X) \cap P[i]$ is homotopic rel bd D^k in $U \cap P[i]$ to a singular k-cell $B^k \subset (U-X) \cap P[i]$ for $k = 1, 2$.

Theorem 5.3. *Let P be a polyhedron such that each $P[n]$ inherits a P.L. n-manifold structure from P, and let X be a rooted cell-like set in P which intersects no stratum of dimension less than 5. Then X is cellular in P if and only if X satisfies the polyhedral cellularity criterion.*

Proof. Assume that X is cellular in P and that U is a neighborhood of X. Since X is cellular, there is a cellular neighborhood V of X such that $X \subset V \subset \bar{V} \subset U$. Let D^k be a singular k-disk in $U \cap P[i]$ with bd $D^k \subset (V-X) \cap P[i]$. There is a cellular neighborhood of X such that $X \subset N \subset \bar{N} \subset V$ and $N \subset V - \text{bd}\, D^k$. Let $h_t : P \to P$ be a product structure isotopy as in Proposition 1.5 such that h_t has compact support in N and $h_1(X) \cap D^k = \emptyset$. The desired k-cell is $h_1^{-1}(D^k)$.

We now prove the converse. Assume that X is a rooted cell-like set intersecting no stratum of dimension less than 5 and that X satisfies the PCC. The proof will be completed by induction on

$$e(X) = \max\{i - j \mid P[i] \cap X \neq \emptyset \text{ and } P[j] \cap X \neq \emptyset\}.$$

If $e(X) = 0$, then X lies in a P.L. n-manifold $P[n]$. We first show that X is a cellular subset of $P[n]$. Let W be a neighborhood of X. There is a neighborhood U of X such that $U \subset W$ and $U \cap P^{(n-1)} = \emptyset$. There is a neighborhood V of X for which there is a stratum respecting contraction of V in U. It follows then that $V \cap P[n]$ contracts in $U \cap P[n]$, and X is a cell-like subset of $P[n]$. Since X satisfies the PCC, we may also assume that V is in the neighborhood of X provided by the PCC with respect to U. Each loop in $(V-X) \cap P[n]$ bounds a singular disk D in $U \cap P[n]$. Since bd $D \subset (U-X) \cap P[n]$, bd D also bounds a singular disk in $(U-X) \cap P[n]$. It now follows from Theorem 5.1 that X is cellular in $P[n]$. There are P.L. n-cells F_0, F_1 and F_2 in $P[n]$ and a cellular neighborhood N of a point in $P[n]$ such that $X \subset \text{int}\, F_1 \subset F_1 \subset \text{int}\, F_2 \subset F_2 \subset U$ and $F_0 \subset N \cap P[n] \subset \bar{N} \cap P[n] \subset \text{int}\, F_1$. Let $\tilde{g}_t : P[n] \to P[n]$ be a P.L. isotopy with compact support in int F_2 such that $\tilde{g}_0 = \text{id}$ and $\tilde{g}_1(F_0) = F_1$. Applying Theorem 1.6, we obtain an isotopy $g_t : P \to P$ with compact support in U such that $X \subset g_1(N) \subset g_1(\bar{N}) \subset U$. Thus X is cellular in P.

We now assume that X is a rooted cell-like set in P with $e(X) = m$, and that the theorem is true for rooted cell-like sets Y with $e(Y) < m$.

Let $P[i]$ be the highest dimensional stratum that X intersects, and let $X^* = X \cap P^{(i-1)}$. Now X^* is a rooted cell-like set with $e(X^*) < m$, and hence is cellular.

Given an open neighbourhood U of X, we need only show that there is a cellular neighborhood N of X with $\bar{N} \subset U$. Let N^* be a cellular neighborhood of X^* with $\bar{N}^* \subset U$. The remainder of the proof consists of engulfing arguments designed to pull N^* out to cover all of X with an isotopy having compact support in U. We may assume that $U \cap P[i]$ has one component.

The needed engulfing lemmas will be stated now, with their proofs delayed until the completion of the proof of this theorem.

Lemma 5.4. *There exist neighborhoods M_1 and M_2 of X^* and a cellular neighborhood D of X^* for which $U \supset M_1 \supset M_2 \supset D$ with $X \subset M_1$ and $\bar{M}_2 \subset N^*$ such that for each closed subpolyhedra $K \subset (M_1 - \bar{D}) \cap P[i]$ and $L \subset (M_2 - \bar{D}) \cap P[i]$ with $\dim K \leq i-3$, $L \subset K$, and $\overline{K-L}$ compact, there is a homeomorphism $h_1 : U \cap P[i] \to U \cap P[i]$ isotopic to the identity with compact support $E_1 \subset (U - \bar{D}) \cap P[i]$ for which $h_1(N^* \cap P[i]) \supset K$ and $h_1|L = \mathrm{id}$.*

Lemma 5.5. *Given M_1 in the above lemma, for each closed subpolyhedra $K \subset M_1 \cap P[i]$ and $L \subset (M_1 - X) \cap P[i]$ such that $\dim K \leq 2$, $L \subset K$, and $\overline{K-L}$ is compact, there is a homeomorphism $h_2 : U \cap P[i] \to U \cap P[i]$ isotopic to the identity with compact support $E_2 \subset M_1 \cap P[i]$ such that $h_2[(M_1 - X) \cap P[i]] \supset K$ and $h_2|L = \mathrm{id}$.*

We now complete the proof of the theorem. We assume that $\bar{D} \cap P[i]$ and $M_1 \cap P[i]$ are the underlying spaces for subcomplexes of a triangulation T of $U \cap P[i]$, that $st(X \cap P[i], T) \subset M_1 \cap P[i]$, and $st(\bar{D} \cap P[i], T) \subset M_2 \cap P[i]$. Let K be the 2-skeleton of $\{\sigma \in T \mid \sigma \subset M_1 \cap P[i],\ \sigma \not\subset D \cap P[i]\}$ and define

$$L = \{\sigma \mid \sigma \in K, \sigma \subset (M_1 - X) \cap P[i]\}.$$

Applying Lemma 5.5 we obtain $h_2 : U \cap P[i] \to U \cap P[i]$ such that $h_2((U - X) \cap P[i] \supset K$ with h_2 the identity off of $E_2 \subset M_1 \cap P[i]$.

Define $K_1 = K \cup \{\sigma \in T \mid \sigma \subset [(U - (X \cup E_2)) \cup D]\}$, and let J be the dual skeleton to K_1. Then J is a compact, codimension 3 polyhedron. It follows from Lemma 5.4 that there is a homeomorphism $h_1 : U \cap P[i] \to U \cap P[i]$ such that $J \subset h_1(N^* \cap P[i])$ with h_1 being the identity over D.

We now use the technique of Stallings [15] to get a homeomorphism $h_3 : U \cap P[i] \to U \cap P[i]$ wheih is isotopic to the identity with compact support such that

$$h_3 h_2[(U - X) \cap P[i]] \cup h_1(N^* \cap P[i]) = U \cap P[i].$$

Therefore $X \cap P[i] \subset h_2^{-1} h_3^{-1} h_1(N^* \cap P[i])$. Since $h_2^{-1} h_3^{-1} h_1$ is isotopic to the identity with compact support, we may apply Theorem 1.6 to extend this homeomorphism to $h : U \to U$ with compact support in U such that $U \supset h(\bar{N}^*) \supset h(N^*) \supset X$.

Proof of Lemma 5.4. We want to apply Theorem 0.3, with $n = i$, $M^n = U_{-1} = P[i] \cap U$, and $r = i - 3$. We first define the U_i's. Let D_0 be a cellular neighborhood of X^* in U, and apply Lemma 5.2 to the sets X, X^*, D_0, and U to get a neighborhood W_0 of X and a stratum preserving homotopy $h_t^0 : \bar{W}_0 \to U$ such that $h_1^0(\bar{W}_0) \subset D_0$ and h_t^0 is supported off of a cellular neighborhood D_1 of X^* with $\overline{D_1} \subset D_0$. The set U_0 is then $W_0 \cap P[i]$. Similarly, we inductively choose the sets W_j for which there is a stratum preserving homotopy $h_t^j : \bar{W}_j \to W_{j-1}$ such that $\overline{h_1^j(\bar{W}_j)} \subset D_j$ and h_t^j is supported off of a cellular neighborhood D_{j+1} of X^* with $\overline{D_{j+1}} \subset D_j$. The set U_j is then $W_i \cap P[i]$. Let $\tilde{V}_0$ be a cellular neighborhood of X^* which lies in D_0. Using the cone structure on D_0, let $g_t^0 : h_1^0(W_0) \to D_0$ be a stratum preserving homotopy such that $g_1^0(h_1^0(W_0)) \subset \tilde{V}_0$ and g_t^0 is fixed on a neighborhood of X^*. Now let $\tilde{V}_1$ be a cellular neighborhood of X^* such that the support of g_t^0 lies outside of $\tilde{V}_1$ and $\tilde{V}_1 \subset D_1$. We then define g_t^j and $\tilde{V}_j$ similarly for $1 \le j \le n-2$. Finally, D may be chosen to be a cellular neighborhood of X^* whose closure lies in $\tilde{V}_{n-2}$. Letting $V_i = \tilde{V}_i \cap P[i]$, we see that the hypothesis of Theorem 0.3 are satisfied, and that the desired M_1 is $\tilde{U}_{i-3}$ and $M_2 = \tilde{V}_{i-2}$, where $\tilde{U}_{i-3}$ is an open set in P such that $\tilde{U}_{i-3} \cap P[i] = U_{i-3}$.

Proof of Lemma 3.5. We wish to apply Theorem 0.2. It suffices to show that $(M_1 \cap P[i], (M_1 - X) \cap P[i])$ is 2-connected.

Since each component A of $M_1 \cap P[i]$ is path connected and $A - X$ is non-empty, the pair is 0-connected.

Assume now that $g : (I^1, \text{bd}\, I^1) \to (M_1 \cap P[i], (M_1 - X) \cap P[i])$ is given. We may choose an open set V such that $X \subset V \subset M_1$, and each 1-disk $D^1 \subset M_1 \cap P[i]$ with $\text{bd}\, D^1 \subset (V - X) \cap P[i]$ is homotopic rel $\text{bd}\, D^1$ to a 1-disk B^1 in $(M_1 - X) \cap P[i]$. Let $D_1, D_2, \ldots, D_n$ be 1-cells in I^1 such that $g(\text{bd}\, D_i) \subset (V - X) \cap P[i]$ and $g^{-1}(X) \subset \bigcup_{i=1}^n D_i$. We may now homotopically move each $g(D_i)$ off of X keeping $g(\text{bd}\, D_i)$ fixed. Piecing these homotopies together, we achieve the desired result.

Let $g : (I^2, \text{bd}\, I^2) \to (M_1 \cap P[i], (M_1 - X) \cap P[i])$ be given. There is a neighborhood V_2 of X such that singular 2-disks in $V_2 \cap P[i]$ whose boundary misses X may be homotoped off of X in $M_1 \cap P[i]$ keeping the boundary of the singular 2-disk fixed. Let V_1 be a neighborhood of X such that singular 1-disks in $V_1 \cap P[i]$ whose boundary misses X can be homotoped off of X in $V_2 \cap P[i]$ keeping the boundary fixed.

Cover $g^{-1}(X)$ in I^2 with the interiors of a finite number of punctured 2-cells $\tau_1, \ldots, \tau_m$ such that $g(\tau_j) \subset V_1$. We will show how to homotop $g(\tau_j)$ off of X in U. Let T_j be a triangulation of τ_j. Using arguments like those in the 0-connected and 1-connected cases, we know that there is a homotopy which moves $g(T_j^1)$ off of X in V_2 and keeps $g(\text{bd}\, \tau_j)$ fixed, where T_j^1 is the 1-skeleton of T_j. We may extend this homotopy to all of $g(\tau_j)$. Let $\emptyset_t^j : g(\tau_j) \to V_2$ be that homotopy. For each 2-simplex $\sigma^2 \in T_j$, we now use the choice of V_2 in $M_1 \cap P[i]$ to homotopically move $\emptyset_1^j(g(\sigma^2))$ off of X in $M_1 \cap P[i]$ keeping $\emptyset_1^j(g(\text{bd}\, \sigma^2))$ fixed. We have thus moved $g(\tau_j)$ off of X in $M_1 \cap P[i]$ keeping $g(\text{bd}\, \tau_j)$ fixed. It now follows that $(M_1 \cap P[i], (M_1 - X) \cap P[i])$ is 2-connected.

References

[1] E. Aiken, Manifold phenomena in the theory of polyhedra, Trans. Amer. Math. Soc. 143 (1969) 413-473.

[2] F.D. Ancel, Engulfing the tracks of a proper homotopy, notes.

[3] J.W. Cannon, The recognition problem: what is a topological manifold? Bull. Amer. Math. Soc. 84 (1978) 832–866.
[4] J.W. Cannon, Shrinking cell-like decompositions of manifolds: codimension three, Ann. of Math. 110 (1979) 83–112.
[5] J.W. Cannon, J.L. Bryant, and R.C. Lacher, The structure of generalized manifolds having non-manifold sets of trivial dimension, in: J.C. Cantrell, ed., Geometric Topology (Academic Press, New York, 1979).
[6] F.D. Crary, Some new engulfing theorems, Ph.D. Thesis, Univ. of Wisconsin at Madison, 1973.
[7] Albrecht Dold, Lectures on Algebraic Topology (Springer-Verlag, Berlin, New York, 1972).
[8] R.D. Edwards, Approximating certain cellular maps by homeomorphisms, manuscript.
[9] M. Handel, Approximating stratum preserving CE maps between CS sets by stratum preserving homeomorphisms, in: L.C. Glaser and T.B. Rushing, eds., Geometric Topology, Lecture Notes in Mathematics, Vol. 438 (Springer-Verlag, New York, 1975) 245–250.
[10] M. Handel, A resolution of stratification conjectures concerning CS sets, Topology 17 (1978) 167–175.
[11] R.C. Lacher, Locally flat strings and half strings, Proc. Amer. Math. Soc. 18 (1967) 299–304.
[12] R.C. Lacher, Cell-like mappings and their generalizations, Bull. Amer. Math. Soc. 83 (1977) 495–552.
[13] D.R. McMillan, Jr., A criterion for cellularity in a manifold, Annals of Math. 79 (1964) 327–337.
[14] L.C. Siebenmann, Approximating cellular maps by homeomorphisms, Topology 11 (1972) 271–294.
[15] J.R. Stallings, The piecewise-linear structure of Euclidean space, Proc. Camb. Phil. Soc. 58 (1962) 481–488.
[16] F.C. Tinsley, Cell-like decompositions of manifolds and the 1-LC property, Ph.D. Thesis, Univ. of Wisconsin at Madison, 1977.