The prime number theorem is PRA-provable

Olivier Sudac

Équipe de logique mathématique, Université Paris 7 - CNRS URA 753 and LLAIC,
Université d’Auvergne-Clermont 1, France

Abstract

We introduce several theories the language of which is rich enough to talk about usual objects of real and complex analysis, and the axioms of which allow to recover a lot of classical results (sometimes with slight modifications). We prove that these theories are conservative extensions of primitive recursive arithmetic (PRA) and develop analysis within them. Moreover, as an example of their efficiency, we shall prove that the prime number theorem is PRA-provable. © 2001 Elsevier Science B.V. All rights reserved.

Résumé

On introduit plusieurs théories dont le langage est assez riche pour parler des objets usuels de l’analyse réelle et complexe, et dont les axiomes permettent de retrouver beaucoup de résultats classiques, avec parfois quelques modifications. On montre que ces théories sont des extensions conservatrices de l’Arithmétique Primitive Récursive (PRA). Enfin, à titre d‘exemple de leur intérêt, on montre que le théorème des nombres premiers est PRA-prouvâble. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Number theory has been developed since Euler making a great use of analysis, which seems to be essential. If the introduction of such transcendent methods to elucidate properties of the structure \((\mathbb{N}, +, \cdot, \leq)\) are incomparably effective, the question remains to know whether the theorems are provable in a formal theory such as Peano arithmetic (PA) or, better, in some fragments of PA, in particular within the primitive recursive arithmetic (PRA).

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E-mail address: olivier.sudac@libertysurf.fr (O. Sudac).
Mints [10] presents a system which is equivalent to PRA, and insures that the prime number theorem is provable in this system without giving details on how to deal with analysis.

Takeuti [17] gives a conservative extension of PA, called FA (finite-type arithmetic), which uses a language with higher types of variables to talk about reals, real functions, and so on, and allows to recover the elementary results of real and complex analysis, and it seems, as the author says, most of the theorems of analytical number theory. Independently, Cegielski [3, 4] had provided a complete proof in PA of the Dirichlet’s Theorem about infinity of prime numbers in an arithmetic progression, by constructing, from any model of PA, some sets of definable objects (in the language of PA), such as “real” and “complex” numbers with which he mimics real analysis. In [5], Cegielski refines this technique to write out a detailed proof of the provability within PRA of the same theorem: this is more difficult because, for example, a property as essential as the Bolzano–Weierstrass theorem disappears.

Following the work of Cegielski, we introduce an increasing sequence of (conservative extensions of) first-order arithmetical theories $R\Sigma^0_\mu$, starting with $R\Sigma^0_1 = PRA$. The language of these theories allows faithful expression of the usual analytical objects. Many classical results can be reconstructed in these theories – with some slight modifications. Of course, not all of them since computable analysis (such as presented in [Abe] [11]) is a model of $R\Sigma^0_1$ and brings its share of counter-examples.

In particular, we show how we can develop “elementary” complex analysis (Cauchy’s theorem on residues...), and, on the precise example of the prime number theorem, how these tools can mimic analytical number theory, in a way sufficient to prove this theorem.

One of the basic notions, drawn by Patrick Cegielski (also present as a watermark in Takeuti), is the notion of language of inductive fields, that is to say the language of fields with a unary predicate which represents the natural numbers and allows to express some induction rules. The other essential notion is the one of definable sequences, that is to say sequences defined by a formula of the language, so that we can talk about it.

However, as soon as we want to add to the inductive field axioms another one, which seems quite essential for the working mathematician, as “every (definable) rational Cauchy sequence converges”, we get a very strong theory: Cegielski [3, 4] proved it to be a conservative extension of second-order arithmetic. Thus we have to make restrictions on the induction rule.

1.1. The prime number theorem

The prime number theorem is one of the most famous results of analytical number theory. If we call $\pi(x)$ the number of primes lower than $x$, it asserts that (see [6]):

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$ 

This is not a purely arithmetical formula; however, thanks to an idea of Matiassevitch (see [9]), it is equivalent to an arithmetical formula (technically, a formula of $L(PA)$,
called APNT), as it will be shown, so that the question of its provability in PRA makes sense. The main result of this paper is:

**Theorem 1.1.** PRA ⊨ APNT.

### 1.2. Summary of the proof

In Section 2, given a model of RIΣp, we construct fields which play the role of the fields of rational numbers. In Section 3, we introduce a language which allows to talk about “sequences”, especially rational sequences, with which we construct fields which play the role of the field of real numbers and investigate properties of sequences of such a field. In Section 4, we extend this language so that we can also talk about functions over these fields, we examine suitable notions of continuity (Section 4.2), differentiability (Section 4.3), and construct an integral (Section 4.4) such that classical properties are still true. In Section 6, we introduce similar constructions for “complex” fields and prove an analogue of the Cauchy theorem on residues (Section 6.4). Indeed, we do not get the exact analogue of the objects corresponding to the standard model: the “real” field we construct here, when starting from \( \mathbb{N} \), is the field of recursive reals. But it turns out that this will be sufficient for what we want to prove. Section 7 is devoted to the proof, with the previous tools, of the prime number theorem.

We shall use small characters for the proofs which are trivial adaptations of the classical one.

### 1.3. Preliminaries

We will be concerned only with first-order logic (see [13] for precise definitions), and especially with first-order theories of Arithmetic (see [7]). Let \( L \) be a first-order language containing the binary predicate symbol \( \leq \). Recall the following definitions:

**Definition 1.2.** (1) The \( A_0 \) formulae of \( L \) are defined as follows:
- The open formulae (i.e. without quantifier) are \( A_0 \) formulae;
- If \( \phi \) and \( \psi \) are \( A_0 \) formulae then so is every boolean combination of \( \phi \) and \( \psi \);
- A formula formed by a \( A_0 \) formula and some bounded quantifiers (i.e. \( \forall x \leq y, \exists x \leq y \)) is still a \( A_0 \) formula.

(2) A formula is said to be \( \Sigma_p \) (resp. \( \Pi_p \)) for \( p \in \mathbb{N} \) iff it has the form

\[
\exists x_1 \forall x_2 \exists x_3 \ldots Q_p x_4 \phi \quad \text{(resp.} \forall x_1 \exists x_2 \ldots \exists x_4 \phi) \tag{1.2.1}
\]

with \( Q_p = \exists \) or \( \forall \) and \( \phi \) a \( A_0 \) formula.

**Definition 1.3.** (1) The language of Peano arithmetic is the first-order language \( L(PA) := \{0, 1, +, \cdot, \leq\} \) where 0 and 1 are constants, + and . are binary functions, \( \leq \) is a binary predicate symbol.

(2) Peano arithmetic is the first-order theory with language \( L(PA) \) and with the following axioms:
∀x ¬(x + 1 = 0);
∀x, y (x + 1 = y + 1 → x = y);
∀x x + 0 = x;
∀x, y x + (y + 1) = (x + y) + 1;
∀x x.0 = 0;
∀x, y x.(y + 1) = x.y + x;
∀x, y(x ≤ y ↔ ∃z y = x + z);

for every formula \( \Phi(x, \bar{y}) \) of \( L(PA) \), we have the following inference rule:

\[
\Phi(0, \bar{y}) \quad \forall x \{ \Phi(x, \bar{y}) \rightarrow \Phi(x + 1, \bar{y}) \}
\]

\[
\forall x \Phi(x, \bar{y})
\]

**Definition 1.4.** The theory \( R\Sigma^p_1 \) is the first-order theory of language \( L(PA) \) with the same axioms as \( PA \) except that the induction rule is restricted to \( \Sigma^p_1 \)-formulae.

We can, as soon as \( p = 1 \), recover all primary classical arithmetic results (i.e. involving neither analysis nor higher algebra, for example, the Unique Factorisation Theorem...), and, thanks to the Gödel function \( \beta \), define some sequences by induction of: the exponential function, the numbering of the prime number sequence, and so on... (for details, see [7] or [5]).

### 2. Fields

We first need to have, for any model \( \mathcal{N} \) of \( R\Sigma^p_1 \), a structure equivalent to the standard structure \( (\mathbb{Q}, +, \cdot, \mathbb{N}, \leq) \).

#### 2.1. Language of inductive fields

**Definition 2.1.** The language of inductive fields is the language \( L(IF) := \{ +, \cdot, \mathcal{N}, \leq \} \) where + and \( \cdot \) are binary functions, \( \mathcal{N} \) and \( \leq \) are, respectively, unary and binary predicates.

\( \mathcal{N}(x) \) will be read as “\( x \) is a natural number of the model”. We also set the following notations:

- \( x = 0 \leftrightarrow \forall y \; x + y = y; \quad x = 1 : \leftrightarrow \forall y \; x.y = y; \quad \)
- “to be an integer”: \( \exists \mathcal{N}(x), \mathcal{N}(-x) \)
- “to be a rational”: \( \exists q \neq 0 (\mathcal{N}(p) \land \mathcal{N}(q) \land qx = p) \).

We shall write \( \mathcal{N}(x), \exists \mathcal{N}(x), \exists q \neq 0 \) to be interpreted as the usual operations.

#### 2.2. \( \Sigma^p_1 \)-subinductive fields

**Definition 2.2.** We call theory of preinductive fields the first-order theory with language \( L(IF) \) and with the following axioms:

1. Axioms of ordered field;
2. \( - \mathcal{N}(0) \)
- \( \forall x \ A'(x) \rightarrow A'(x+1) \).

**Definition 2.3.** A preinductive field \( \mathcal{K} \) is said to be *archimedean* iff:

\[ \mathcal{K} \models \forall f \exists n (A(n) \land x \leq n). \]

Examples: \((\mathbb{Q}, +, \mathbb{N}, \leq)\) and \((\mathbb{R}, +, \mathbb{N}, \leq)\) are archimedean preinductive fields, while \((\mathbb{R}(X), +, \mathbb{N}, \leq)\) (where \(P(X) = a_pX^p + \cdots + a_0 \geq 0\) iff \(a_p \geq 0\)) is a preinductive field but is not archimedean. Of course, \((\mathbb{Q}, +, 2, \mathbb{N}, \leq)\) is a field which is not preinductive.

It is to be noted that Archimedean property is related to the natural numbers of the model, and not, as in the usual definition, to the standard natural numbers. This point is important because, for instance, we cannot define \( \mathbb{N} \) in a model of PA which is not standard (and we are concerned with conservative extensions of subtheories of PA); this will be the same for all the other notions involving \( \mathbb{N} \). Recall that \( \mathbb{N} \) is very seldom definable in a structure \((A, +, \cdot)\) where \(A\) is a set containing \( \mathbb{N} \). As an example, \( \mathbb{N} \) is not definable in \((\mathbb{R}, +, \cdot)\) (Tarski, 1948); a notable exception is the famous result of J. Robinson (1949), [12] stating that \( \mathbb{N} \) is definable in \((\mathbb{Q}, +, \cdot)\).

**Definition 2.4.** We call the *theory of \( \Sigma_p \)-inductive fields* \((p \geq 1)\) (resp. the *inductive fields*) the first-order theory with language \(L(\mathbb{IF})\) and with the following axioms:

1. Axioms of archimedean preinductive field.
2. For every \( \Sigma_p \)-formula (resp. every formula) \( \Phi \) of \(L(\mathbb{IF})\), we have the following inference rule:

\[\begin{align*}
\phi(0, \bar{y}) \quad \forall x \left( A'(x) \land \Phi(x, \bar{y}) \rightarrow \Phi(x+1, \bar{y}) \right) \\
\forall x \left( A'(x) \rightarrow \Phi(x, \bar{y}) \right)
\end{align*}\]

\((\mathbb{R}, \mathbb{N})\) is the standard model of this theory. Unfortunately, this theory is too powerful relative to our purpose: P. Cegielski proved that if we add to this theory axioms stating Cauchy rational sequences converge, then it is a (conservative) extension of second-order arithmetic \( \mathbb{Z}_2 \) (see [4]), thus it is not a conservative extension of PA. Hence we will focus on the weaker theory of “subinductive fields”:

**Definition 2.5.** We call the *theory of \( \Sigma_p \)-subinductive fields* \((p \geq 1)\) (resp. the *subinductive fields*) the first-order theory with language \(L(\mathbb{IF})\) and with the following axioms:

1. Axioms of archimedean preinductive field.
2. For every \( \Sigma_p \)-formula (resp. every formula) \( \Phi \) of \(L(\mathbb{IF})\) such that all the variables are relativised to \( A' \), we have the following inference rule:

\[\begin{align*}
\phi(0, \bar{y}) \land A'(\bar{y}) \quad \forall x \left( A'(x) \land A'(\bar{y}) \land \Phi(x, \bar{y}) \rightarrow \Phi(x+1, \bar{y}) \right) \\
\forall x \left( A'(x) \rightarrow \Phi(x, \bar{y}) \right)
\end{align*}\]

Of course, \((\mathbb{R}, +, \mathbb{N}, \leq)\) is a model of this theory; it is not so easy to display a subinductive field which is not inductive: in Section 3, we will construct such mod-
els. Moreover, the next section will provide $\Sigma_p$-inductive fields which are not $\Sigma_{p+1}$-inductive.

### 2.3. Theory of rational fields of $RI_{\Sigma_p}$

**Definition 2.6.** The theory of rational fields of $RI_{\Sigma_p}$ (resp. of PA) is the first-order theory $\Sigma_p$-$RAT$ (resp. PA-$RAT$) with the following axioms:

1. Axioms of $\Sigma_p$-inductive (resp. inductive) field.
2. Rational field: $\forall x \mathcal{A}(x)$.

As an example, $(\mathbb{Q}, \mathbb{N}) \models PA$-$RAT$, but of course not $(\mathbb{R}, \mathbb{N})$.

**Note:** The theory PA-$RAT$ is given in [3]; it is also the theory FA of Takeuti, if we disregard higher-type variables.

**Theorem 2.7.** $\Sigma_p$-$RAT$ (resp. PA-$RAT$) is a conservative extension of $RI_{\Sigma_p}$ (resp. of PA).

**Proof.** It is an extension of $RI_{\Sigma_p}$ since, by definition, $(\mathcal{N}, +, \cdot, \leq) \models RI_{\Sigma_p}$. It is conservative: from any model $\mathcal{N}$ of $RI_{\Sigma_p}$, we construct below a $\Sigma_p$-inductive field $\mathcal{K}$ such that $(\mathcal{N}, +, \cdot, \leq)$ is isomorphic to $(\mathcal{N}, +, \cdot, \leq)$:

**Lemma 2.8.** (1) Let $(\mathcal{N}, +, \cdot, \leq)$ be a model of $RI_{\Sigma_p}$. Then there exists a linearly ordered commutative unitary ring $(\mathcal{I}, +, \cdot, \leq)$, unique up to isomorphism, such that $(\mathcal{N}, +, \cdot, \leq)$ is isomorphic to $(\mathcal{I}, +, \cdot, \leq)$. Moreover, every element of $\mathcal{I}$ is written in a unique way $n;0$ or $-n$, with $n \in \mathcal{I}^+$.

(2) For every $\Sigma_p$-formula $F(x, \bar{y})$ of $L(\text{PA})$ the following induction rule holds in $(\mathcal{I}, +, \cdot, \leq)$:

$$\begin{align*}
F(0, \bar{y}) & \quad \forall x (F(x, \bar{y}) \rightarrow F(x+1, \bar{y})) \quad \forall x (x \geq 0 \rightarrow F(x, \bar{y})).
\end{align*}$$

**Proof.** The first point is absolutely the same as for the passage from $\mathbb{N}$ to $\mathbb{Z}$, since it is carried out using only elementary algebraic properties of $\mathbb{N}$ (regularity of the addition...), which were in fact proved in $RI_{\Sigma_1}$. For the second point, let us consider the one-to-one function $f$ of $\mathcal{I}$ onto $\mathcal{N}$ given by

$$f(k) = \begin{cases} 2k & \text{if } k \in \mathcal{I}^+, \\ 2|k|-1 & \text{if } k \in \mathcal{I}^-.
\end{cases}$$

We define the operations $+', \cdot'$ and the relation $\leq'$ in $\mathcal{N}$ by

$$\begin{align*}
m +' n &= f(f^{-1}(m) + f^{-1}(n)), \\
m \cdot' n &= f(f^{-1}(m) . f^{-1}(n)), \\
m \leq' n &\Leftrightarrow f^{-1}(n) \leq f^{-1}(n).
\end{align*}$$

They are $\Sigma_p$-definable; so that formulae about elements of $\mathcal{I}$ are reduced to formulae about elements of $\mathcal{N}$, on which we can make induction. □
Lemma 2.9.  (1) Let \( (\mathcal{N}, +, \leq) \) be a model of \( \text{RI}_p \). Then there exists a unique (up to isomorphism) linearly ordered commutative field with a unary relation \( \tilde{N} \), written as \( (\mathcal{N}, +, \leq, \tilde{N}) \), such that \( (\mathcal{N}, +, \leq) \) is isomorphic to \( (\tilde{N}, +, \leq) \) and such that every element \( r \) of \( \mathcal{N} \) is written in a unique way: \( r = -\frac{p}{q} \) or \( 0 \) or \( r = \frac{p}{q}; \) with \( p, q \in \mathbb{N} \), \( p \) and \( q \) being coprime (identifying \( \mathcal{N} \) and \( \tilde{N} \)).

(2) For every \( \Sigma_p \)-formula \( \Phi(x, \tilde{y}) \) of \( \text{L(PA)} \), we have the following induction rule

\[
\phi(0, \tilde{y}) \forall x[\tilde{N}(x) \land \Phi(x, \tilde{y}) \rightarrow \Phi(x + 1, \tilde{y})].
\]

Proof. As previously, the first part is known, the second is proved in the same way, considering the one-to-one function of \( \mathbb{Q} \) onto a part \( \mathbb{Q}' \) of \( \mathbb{N} \):

\[
f(r) = \begin{cases} 
1 & \text{if } r = 0, \\
2^p.3^q - 1 & \text{if } r = \frac{p}{q}, p \perp q, p, q \geq 0, \\
2^p.3^q - 1.5 & \text{if } r = -\frac{p}{q}, p \perp q, p, q \geq 0.
\end{cases}
\]

This completes the proof of the theorem. \( \Box \)

Note: As \( \text{RI}_p+1 \) is not a conservative extension of \( \text{RI}_p \), the theorem shows that there exists \( \Sigma_{p+1} \)-inductive fields which are not \( \Sigma_p \)-inductive.

Lemma 2.10. Let \( (\mathcal{K}, +, \mathcal{N}, \leq) \) be a \( \Sigma_p \)-subinductive (resp. subinductive) field, then setting \( \mathcal{P} := \{ x \in \mathcal{K}; P(x) \} \):

(a) \( (\mathcal{P}, +, \mathcal{N}, \leq) \) is a \( \Sigma_p \)-inductive (resp. inductive) field;

(b) \( \mathcal{P} \) is dense in \( \mathcal{K} : \forall x, y[x < y \rightarrow \exists r (\mathcal{P} \land x < r < y)] \).

Proof. (a) Thanks to the unicity in the previous proof.

(b) Let, by archimedean property, \( n \) be such that \( 1/n < y - x \); then the rational \( r := [x] + [n(x)]/n + 1/n \) is suitable, provided that the integer part \( [x] \) (and \( \{x\} := x - [x] \)) has been defined, that is, we can find \( k \in \mathcal{N} \) such that \( x \geq k \wedge k < x \). Suppose to get a contradiction, that \( \forall k \in \mathcal{N} \ (x \leq k \rightarrow x < k + 1) \); we cannot directly use an induction on this formula which contains a real parameter. We come through it by invoking Archimede: \( \exists l \in \mathcal{N} x < l \); whence \( \forall k \in \mathcal{N} \ (l \leq k \rightarrow l \leq k + 1) \), and this time we can make an induction on the formula \( l > k \). \( \Box \)

Proof of the point (b) of the previous proposition illustrates in a very simple case the necessity of being careful of fortuitous induction on formulae involving reals, which are not allowed.

Theorem 2.11. The theory of \( \Sigma_p \)-subinductive (resp. subinductive) fields is a conservative extension of \( \text{RI}_p \) (resp. of \( \text{PA} \)).

Proof. This is a subtheory of \( \Sigma_p \)-inductive fields (resp. of inductive fields), which is itself a subtheory of \( \Sigma_p \)-\( \text{RAT} \) (resp. of \( \text{PA-RAT} \)). \( \Box \)
3. Fields with sequences

Now, our aim is to construct a kind of real field, using Cauchy sequences in a rational field.

3.1. Language of algebras of sequences

We first need to introduce the notion of “sequences” of a field; but we will not define it as an application from the standard set of natural numbers \( \mathbb{N} \) into \( \mathcal{K} \): we would have difficulties to tell interesting things about such an object: a quantifier of the type \( \forall n \in \mathbb{N} \) is not easily handled in (a conservative extension of) PA: indeed we have already remarked that \( \mathbb{N} \) is, in general, not definable in any model of PA. Furthermore, our aim is to recover analysis from any model \( \mathcal{N} \), so that it will be necessary to replace everywhere \( \mathbb{N} \) by \( \mathcal{N} \).

Definition 3.1. Let \( (\mathcal{K},+,, \mathcal{N};,0) \) be a preinductive field. We call sequence of this field every element \( u \) of \( \mathcal{K}^\mathcal{N} \); \( u \) is written as \( (u_n) \).

We equip \( \mathcal{K}^\mathcal{N} \) with termwise operations (addition, multiplication by a scalar, multiplication). Then:

Lemma 3.2. \( (\mathcal{K}^\mathcal{N},+,, \times) \) is a commutative \( \mathcal{K} \)-algebra.

Of course, the set \( \mathcal{K}^\mathcal{N} \) is not definable in the structure \( (\mathcal{K},+,, \mathcal{N};,\leq) \), hence we cannot express such a property in the language of inductive fields. We thus introduce a language which overcomes this difficulty: it will allow to talk, when two sets \( K \) and \( N \) are given, about applications from \( N^i (i \in \mathbb{N}) \) into \( K \), with an evaluation map \( \text{eval} \), and a predicate symbol \( \mathcal{K} \) which distinguish the elements of \( K \) from the applications. When \( (K,N) \) is a preinductive field, these applications will be simple, double…, sequences of \( K \), i.e. members of \( K^N, K^{N^2}, \ldots \). The intended meaning of \( \text{eval} \) is that the evaluation of a triple sequence is a double one, and so on… \( (\bigcup_{i \in \mathbb{N}} K^{N^i}, \mathcal{K}, \mathcal{N}) \) where \( (K,N) \) is any preinductive field, will be the standard structure of this language.

Definition 3.3. The language of algebras of sequences is the language \( \text{L(AS)} := \{+,,, \mathcal{N}, \mathcal{K};,\leq,\text{eval} \} \) where \( \mathcal{K} \) is a unary predicate and \( \text{eval} \) is a binary function.

We keep the notations defined in \( \text{L(IF)} \). The following notions are definable in the standard structure \( (\bigcup_{i \in \mathbb{N}} \mathbb{R}^{N^i}, \mathbb{R}, \mathbb{N}) \): \( u \) is a simple sequence of elements of \( \mathcal{K} \), \( u \) is a sequence with \( i (\in \mathcal{N}) \) variables of elements of \( \mathcal{K} \), \( u \) is a rational sequence with \( i \) variables, respectively, given by

\[
\begin{align*}
    u \in \mathcal{F}_\mathcal{K} & \iff \forall x \forall n \in \mathcal{N} \ [\text{eval}(u,n) = x \rightarrow x \in \mathcal{K}] \\
    u \in \mathcal{F}_\mathcal{F} & \iff \forall x \forall n_1,\ldots,n_i \in \mathcal{N} \ [\text{eval}_i(u,n_1,\ldots,n_i) = x \rightarrow x \in \mathcal{K}] \\
    u \in \mathcal{F}_\mathcal{Q} & \iff \forall x \forall n_1,\ldots,n_i \in \mathcal{N} \ [\text{eval}_i(u,n_1,\ldots,n_i) = x \rightarrow x \in \mathcal{Q}] 
\end{align*}
\]
We will use \( x = u_n \) as a short form for \( x = \text{eval}(u, n) \land n \in \mathcal{N} \), as well as \( \text{eval}(u, n_1, \ldots, n_i) = x \) for \( \text{eval}(\ldots \text{eval}(u, n_1) \ldots, n_i) = x \). These notations make sense for any structure of \( \text{L(AS)} \), so that we shall use them as definitions. We define in the usual way: "\( u \) is a Cauchy sequence" and "\( (u_n) \) converges to a limit \( l \):

\[
\text{u is Cauchy} \iff \forall \varepsilon \in \mathcal{K}_+ \exists n_0 \in \mathcal{N} \forall n, p \in \mathcal{N} [n, p \geq n_0 \rightarrow |u_n - u_p| < \varepsilon] \\
\lim_{n} u_n = l \iff \forall \varepsilon \in \mathcal{K}_+ \exists n_0 \in \mathcal{N} \forall n \in \mathcal{N} [n \geq n_0 \rightarrow |u_n - l| < \varepsilon].
\]

### 3.2. Sequences in a preinductive (Field"

**Definition 3.4.** We call theory of preinductive (resp. \( \Sigma_p \)-subinductive, inductive) algebras of sequences the first-order theory with language \( \text{L(AS)} \) and with the following axioms:

1. \( (\mathcal{K}, +, \cdot, \ldots, \mathcal{N}, 6) \) is a preinductive (resp. \( \Sigma_p \)-subinductive, inductive) field.

2. Compatibility of the evaluation:

   - \( \forall u, v, n \left[ \text{eval}(u + v, n) = \text{eval}(u, n) + \text{eval}(v, n) \right] \)
   - \( u \leq v \iff \forall n \left[ \text{eval}(u, n) \leq \text{eval}(v, n) \right] \)
   - \( \forall x \in \mathcal{K} \forall n \left[ \text{eval}(x, n) = x \right] \)

**Lemma 3.5.** In the theory of preinductive algebras of sequences we have:

- (Unicity of the limit) \( \forall u, a, b \in \mathcal{K} \left[ \lim_{n} u_n = a \land \lim_{n} u_n = b \rightarrow a = b \right] \).

- Each convergent sequence is a Cauchy sequence.

- If \( (u_n) \) and \( (v_n) \) converge respectively to \( a \) and \( b \), and if \( k \in \mathcal{K} \), then \( (u_n + v_n) \), \( (u_n)(v_n) \) and \( k(v_n) \) converge to \( a + b, a \cdot b \) and \( k \cdot b \), respectively.

**Proof.** The classical proofs go through, but for the product; however, a minor modification gives the result: Let \( \varepsilon > 0 \), then

\[
\exists n_1 \forall n > n_1 |u_n - a| < \varepsilon, \exists n_2 \forall n > n_2 |v_n - b| < \varepsilon, \text{ and, with } \varepsilon = 1, \\
\exists n_3 \forall n > n_3 |v_n| \leq 1 + |b|, \text{ whence } n_0 := \max(n_1, n_2, n_3) \text{ is suitable, because }
\]

\[
\forall n > n_0 |u_n v_n - ab| \leq |u_n - a| |v_n| + |a| |v_n - b| \\
\leq \varepsilon(1 + |b|) + |a| \varepsilon. \quad \square
\]

We avoided any use of the boundedness of convergent sequences because of the following:

**Theorem 3.6.** The theory of inductive algebras of sequences does not prove that every initial segment of a sequence is bounded. That is to say there exists an inductive algebra of sequences \( \mathcal{S} \) such that

\[
\mathcal{S} \models \exists u \left[ \forall n \in \mathcal{N}, \exists M \in \mathcal{K}, \forall i \in \mathcal{N} (i \leq n \rightarrow \text{eval}(u, i) \leq M) \right].
\]
Proof. Let $\mathcal{N}$ be a denumerable non-standard model of PA. Recall that the order type of such a model is $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$, i.e. it is formed with $\mathbb{N}$ and $\mathbb{Q}$ copies of $\mathbb{Z}$, each copy is called a galaxy (see [8]). We choose an element $x_i$ in each galaxy (the set of which is denumerable, thus indexed by $\mathbb{N}$). Now, let the sequence $(u_n) \in \mathcal{N}^{-\mathbb{N}}$ be defined by

\[
\begin{align*}
  u_i &:= x_i \quad \text{if } i \in \mathbb{N}; \\
  u_i &:= 0 \quad \text{if } i > \mathbb{N}.
\end{align*}
\]

Take a non-standard $n$; for every $M \in \mathbb{N}$ there exists a galaxy greater than $M$ and let $i$ be its number: then $u_i = x_i > M$ and $(u_i)_{i \in n}$ is not bounded. If $\mathcal{Q}$ is the rational field of $\mathbb{N}$, then setting $S := \mathcal{Q}^\mathbb{N}$, we get the required model. 

**Theorem 3.7.** The theory of inductive field does not prove that a convergent sequence is bounded, that is to say that there exists an inductive algebra of sequences $\mathcal{S}$ such that

\[
\mathcal{S} \models \exists u \left[ \exists a \in \mathcal{K} \lim u = a \land \forall M \in \mathcal{K} \exists i \in \mathcal{N} \text{eval}(u,i) \geq M \right].
\]

Proof. We consider the previous sequence, which of course converges to 0. 

3.3. **Definable sequences in a $\Sigma_p$-subinductive field**

The previous phenomenon shows that our notion of “sequence” is too general to recover the classical results (as a convergent sequence is bounded). Fortunately, we will see that it suffices to restrict to “definable” sequences, to find again the results we will need. From now on, let $(\mathcal{S}, \mathcal{K}, +, \cdot, <, \ll)$ be a $\Sigma_p$-subinductive algebra of sequences. We first introduce a more restrictive notion of convergence, because the classical one has too many quantifiers: if a sequence $u$ is $\Sigma_p$-definable, telling “$u$ has a limit” uses a $\Pi_{p+3}$-formula.

**Definition 3.8.** A sequence $(u_n)$ is said to be rapid Cauchy if and only if

\[
\forall n \in \mathcal{N} \ |u_{n+1} - u_n| < 2^{-n}.
\]

We say that $(u_n)$ converges rapidly to $a$ if and only if $\forall n \in \mathcal{N} \ |u_n - a| < 2^{-n}$.

**Definition 3.9.** A rational sequence $(u_n) \in \mathcal{Q}^{-\mathbb{N}}$ is said to be $\Sigma_p$-definable if and only if there exists $\bar{b} \in \mathcal{Q}^{\mathbb{N}}$ and a $\Sigma_p$-formula $\Phi(n, r, \bar{x})$ of L(PA) such that

- $\forall n \in \mathcal{N}, \exists ! r \in \mathcal{Q} \Phi(n, r, \bar{b})$;
- $\forall n \in \mathcal{N} \Phi(n, u_n, \bar{b})$.

We write $\Sigma_p(\mathcal{Q}^{-\mathbb{N}})$ for the set of rational $\Sigma_p$-definable sequences. The double rational $\Sigma_p$-definable sequences are defined in the same way. As an example, in any non-standard model, the sequence $u_n = r^n$ where $r \in \mathcal{Q}$ is definable, but the sequence $u_n = 1$ if $n \in \mathbb{N}$, 0 otherwise, is not definable in a non-standard model.
Lemma 3.10. $\Sigma_p(\mathbb{R}^\mathbb{N})$ is a sub-algebra of $(\mathbb{R}^\mathbb{N}, +, \cdot)$. 

Definition 3.11. We define, if it exists, the maximum of a “finite” sequence by

$$x = \max_{k \leq m} u_k \iff (\forall k \leq m) x \geq u_k \land (\exists k \leq m) x = u_k.$$ 

Lemma 3.12. Let $(u_n) \in \Sigma_p(\mathbb{R}^\mathbb{N})$. Then for every $m \in \mathbb{N}$ $\max_{k \leq m} u_k$ exists.

Proof. We prove: $\exists k \leq m, \forall k' \leq m u_{k'} \leq u_k$ by induction on $m$. □

Corollary 3.13. Every rational $\Sigma_p$-definable Cauchy (or convergent) sequence is bounded.

Lemma 3.14. Let $(u_n) \in \Sigma_p(\mathbb{R}^\mathbb{N})$ be rapid Cauchy, then

1. $\forall n, k \in \mathbb{N} |u_{n+k} - u_n| \leq 2^{-n+1}.$
2. $\exists m \in \mathbb{N} (|u_0| \leq 2^m \land \forall n \in \mathbb{N} |u_n| \leq 2^{m+1}).$ Such an $m$ is called an upper modulus of $(u_n)$.
3. If $\forall k \in \mathbb{N} \exists n \in \mathbb{N} |u_n| \geq 2^{-k},$ then $\exists k \in \mathbb{N} \forall n \in \mathbb{N} |u_{k+n}| \geq 2^{-k+1}.$ Such a $k$ is called a lower modulus of $(u_n)$. 

Proof. (1) We prove it by induction on $k$ with the following $\Sigma_p$-formula:

$$|u_{n+k} - u_n| \leq 2^{-n+1}(1 - 2^{-k}).$$

(2) The existence of $m$ comes from the multiplicative version of the Archimedean property and the previous result.

(3) Taking $k = 2$ and using (1), we have

$$|u_{n+k}| \geq |u_n| - |u_{n+k} - u_k| \geq 2^2 - 2^{-n+1} = 2^{-n+1}.$$ □

3.4. Internal definability 

The previous section showed that the idea of definable sequence is a better one. But we used it as an external notion. Then it would be interesting to be able to say in $\text{L}(\text{AS})$ that a sequence is definable: for example to say “every definable rational Cauchy sequence is bounded”. But to write such a property, i.e. that there exists a formula which defines the sequence, a priori we would have to be able to quantify on the formulae: in the standard model, we can emulate this quantification, thanks to the existence of a universal $\Sigma_p$-formula. This formula associates a standard code to each $\Sigma_p$-formula. It is not clear whether we can generalize this trick to a non-standard model. This is not a real problem: we will generalize the notion of definable sequence: they will not be anymore defined by a $\Sigma_p$-formula, but by a code in this universal formula.

First, here is the universal $\Sigma_p$-formula. Its existence is got from Matiasevitch’s theorem ([9]; the theorem is provable in $\text{R}1\Sigma_1$: see [5]):
Theorem 3.15. There exists a $\Sigma_p$-formula of $L(\text{IF})$, written as $\Sigma_p$-$\text{Verif}(e,\bar{x})$, and called universal $\Sigma_p$-formula, such that for every $\Sigma_p$-formula $\Phi(\bar{x})$ of $L(\text{IF})$ such that all the variables are relativised to $\mathcal{N}$, there exists a standard natural number $\bar{c}$ such that for every $\Sigma_p$-subinductive field $(\mathcal{K},\mathcal{N})$ and $\bar{a} \in \mathcal{N}^m$, we have:

$$\mathcal{K} \models \Phi(\bar{a}) \text{ if and only if } \mathcal{K} \models \Sigma_p$-

We thus infer:

Lemma 3.16. There exists a $\Sigma_p$-formula of $L(\text{PA})$, written as $\Sigma_p$-$\text{Seq}(c(\Phi),n,r)$ such that for every $\Sigma_p$-subinductive field $(\mathcal{K},\mathcal{N})$ and every rational $\Sigma_p$-definable sequence $(u_n)$, there exists $c(\Phi) \in \mathbb{N}$ such that

- $\mathcal{K} \models \forall n \in \mathcal{N}, \exists! r \in 2^{\Sigma_p}$-$\text{Seq}(c(\Phi),n,r)$;
- $\mathcal{K} \models \forall n \in \mathcal{N} \forall^* (c(\Phi), n, u_n)$.

Proof. The desired formula is $\Sigma_p$-$\text{Verif}(x, \langle n, r \rangle)$, where $\langle n, r \rangle := ((n + r)(n + r + 1)/2) + n + 1$ is a coding function. If $(u_n) \in \Sigma_p(\mathcal{Z}^1)$, then let $\Phi(n,r)$ be the $\Sigma_p$-formula which defines it:

$$\forall n \in \mathcal{N}, \exists! r \Phi(n, r) \land \forall n \in \mathcal{N} \Phi(n, u_n)$$

and it remains to take, for $c(\Phi)$, the Gödel number of $\Phi$. □

Notations. In the theory of $\Sigma_p$-subinductive algebras of sequences, we can define the following predicates: (1) the sequence $(u_n)$ rapidly converges to $l$, (2) $\phi$ is the number of a formula defining a rational sequence, (3) $u$ is a rational $\Sigma_p$-definable sequence, (4) and (5) the same thing for a sequence with $i$ variables of $\mathcal{K}$:

1. rapid $\lim_{n} u_n = l \iff \forall n \in \mathcal{N} |u_n - l| < 2^{-n}$.
2. $\text{Def-seq}(\phi) \iff \forall n \in \mathcal{N} \exists x \in 2^{\Sigma_p}$-$\text{Seq}(\phi, n, x)$.
3. $u \in \Sigma_p(\mathcal{Z}^1) \iff \exists \phi[\text{Def-seq}(\phi) \land \forall x \forall n \in \mathcal{N} \Sigma_p$-$\text{Seq}(\phi, n, x) \leftrightarrow x = u_n]$.
4. and for every $i \in \mathbb{N}$:
5. $u \in \Sigma_p(\mathcal{Z}^{1+i}) \iff \exists \phi[\text{Def-seq}(\phi) \land \forall x \forall n_1, \ldots, n_i \in \mathcal{N} \Sigma_p$-$\text{Seq}(\phi, n_1, \ldots, n_i, x) \leftrightarrow x = u_{n_1, \ldots, n_i}$]$

Definition 3.17. Let $(\mathcal{S}, +, \text{eval}, \mathcal{K}, \mathcal{N}, \leq)$ be a $\Sigma_p$-subinductive algebra of sequences. A rational sequence $u \in \mathcal{S}_2$ is said to be (internally) $\Sigma_p$-definable if and only if

$$\mathcal{S} \models u \in \Sigma_p(\mathcal{Z}^1).$$

It remains to see that what we proved for the external notion of definability is still true for internal definability. The key aspect of this is to keep in mind that $\Sigma_p$-$\text{Seq}(\Phi, n, r)$ is a $\Sigma_p$-formula. So, instead of saying: “$u$ is externally definable, thus
there exists $\Phi(n, r)$ such that $\Phi(n, u_n)$ is a $\Sigma_p$-formula”, we now say: “$u$ is internally definable, thus there exists $\phi \in \mathcal{N}$ such that $\Sigma_p$-Seq($\Phi, n, u_n$) is a $\Sigma_p$-formula”.

As an example, let us prove that if $u$ is internally definable then $\max_{k\leq m} u_k$ exists. By hypothesis there exists $\phi \in \mathcal{N}$ such that $\forall n \in \mathcal{N} \Sigma_p$-Seq($\phi, n, u_n$). The formula $\exists k \leq m, \forall k' \leq m, \exists a, b \ (\Sigma_p$-Seq($\phi, k, a) \land \Sigma_p$-Seq($\phi, k', b) \land b \leq a$) is a $\Sigma_p$-formula, so that we can apply the induction rule to prove that this formula is valid.

Definition 3.18. We say that a sequence $u$ of $\mathcal{K}$ (with $i$ variables) is $\Sigma_p$-definable, if there exists a rational $\Sigma_p$-definable sequence (with $i + 1$ variables) which rapidly approaches it:

$$u \in \Sigma_p(\mathcal{K}^{i+1}) \iff \exists v \in \Sigma_p(\mathcal{A}^{i+1}) \text{ rapid } \lim_{n} v_n = u.$$  

3.5. $\Sigma_p$-real fields

As shown in Sections 3.2 and 3.3, the interesting notions of sequence and convergence are definable ones, so that we shall consider a “real” field which is “complete” relative to definable convergence and such that its corresponding algebra of sequences will contain only definable sequences:

Definition 3.19. We call theory of algebras of $\Sigma_p$-real sequences the first-order theory with language $L(AS)$ and with the following axioms:

1. Axioms of $\Sigma_p$-subinductive algebra of sequences.
2. Every real sequence is $\Sigma_p$-definable: for every standard natural number $i$, we have

$$\mathcal{S}^{\mathcal{K}^{i}} = \Sigma_p(\mathcal{K}^{i+1}).$$

3. “$\Sigma_p$-completeness”: every $\Sigma_p$-Cauchy sequence is $\Sigma_p$-convergent:

$$\forall u \in \mathcal{S}^{\mathcal{K}} \left\{ \exists e \in \Sigma_p(\mathcal{A}^{i+1}) \left( \lim_{n} e_n = 0 \land \forall n, k \in \mathcal{N} \ |u_n - u_k| < e_n \right) \right\} \rightarrow$$

$$\exists l \in \mathcal{K} \exists e' \in \Sigma_p(\mathcal{A}^{i+1}) \left( \lim_{n} e'_n = 0 \land \forall n \in \mathcal{N} \ |u_n - l| < e_n \right) \right\}.$$  

A model of this theory is called an algebra of $\Sigma_p$-real sequences and its field $\mathcal{K}$ is called a $\Sigma_p$-real field.

In the case $p = 1$, $(\text{Rec}(\mathbb{R}^{\mathbb{N}}), \text{Rec}(\mathbb{R}), +, \ldots, \leq)$, where $\text{Rec}(\mathbb{R})$ is the set of recursive reals (see [2] or [11]), is a model of this theory: this will be our “standard” model, thanks to which we will know that such or other results cannot be improved. On the contrary, $(\mathbb{Q}^{\mathbb{N}}, \mathbb{N})$ is not a model: axiom 2 requires that all sequences are definable; some sequences may exist which are not definable in the standard sense but which are in the sense of the model; however, it is easy to see that the cardinality of the set of definable sequences is less than or equal to the cardinality of the field. $(\mathbb{R}^{\mathbb{N}}, \mathbb{N})$ is neither a model because it would have $(\mathbb{Q}^{\mathbb{N}}, \mathbb{N})$ as a sub-model.
**Theorem 3.20.** The theory of algebras of $\Sigma_p$-real sequences is a conservative extension of $\Sigma_p$-RAT and thus, of $RI\Sigma_p$.

Note that this theory is therefore not complete.

**Proof.** We mimic the classical construction.

Let $(\mathcal{Q}, +, \cdot, \leq, \leq)$ be a model of $\Sigma_p$-RAT; let us construct an algebra of $\Sigma_p$-real sequences $\mathcal{S}$ such that $\mathcal{Q}$ is isomorphic to $\mathcal{S}$. Consider:

$$\text{CR}-\Sigma_p(\mathcal{Q}) := \{ u \in \mathcal{Q}; \ u \text{ is } \Sigma_p\text{-definable and } u \text{ is of rapid Cauchy} \}$$

We introduce on this set the relation defined by

$$u \equiv v \leftrightarrow \exists k \in \mathbb{N} \forall n \in \mathbb{N} \ |u_n - v_n| < 2^{-k+1}.$$

It is an equivalence relation.

Set $\Sigma_p\mathcal{R} := \text{CR}-\Sigma_p(\mathcal{Q})/\equiv$ and let us define the operations $+$ and $\cdot$ on the equivalence classes

$$[u_n] + [v_n] := [u_{n+1} + v_{n+1}],$$

$$[u_n] \cdot [v_n] := [u_{n+r+2}, v_{n+p+2}],$$

where $p$ and $r$ are respective upper moduli of $(u_n)$ and $(v_n)$. We check that these definitions have a sense and that these operations confer on $\Sigma_p\mathcal{R}$ a structure of commutative field. According to Lemma 3.14, we can set

$$[u_n] \geq 0 \iff [u_n] = 0 \lor \exists k \in \mathbb{N} \forall n \in \mathbb{N} \ u_k + n \geq 2^{-k+1}.$$  

$(\Sigma_p\mathcal{R}, +, \cdot, \leq)$ is then a linearly ordered commutative field.

The application from $\mathcal{Q}$ into $\Sigma_p\mathcal{R}$ which assigns to each $q$ the class of the constant sequence equal to $q$, is a monomorphism of ordered fields; thus we can consider that $\mathcal{Q}$ and $\mathcal{N}$ are included in $\Sigma_p\mathcal{R}$. Then every $\Sigma_p$-formula relativised to $\mathcal{N}$ is relativised to $\mathcal{Q}$, so that the induction scheme can operate on it.

Now let us prove

**Lemma 3.21.** Every element of $\Sigma_p\mathcal{R}$ is a rapid limit of an element of $\text{CR}-\Sigma_p(\mathcal{Q})$.

**Proof.** Let $x = [u_n] \in \Sigma_p\mathcal{R}$. We have: $\forall n, m \in \mathbb{N} \ |u_{n+2+m} - u_{n+2}| < 2^{-n-1}$. Choosing $k = n + 2$, we get $\forall n \in \mathbb{N} \ \exists k \in \mathbb{N} \forall m \in \mathbb{N} \ |u_{k+m} - u_{n+2}| < 2^{-n-1} < 2^{-n} - 2^{-k+1}$ that is to say $\forall n \in \mathbb{N} \ \exists k \in \mathbb{N} \forall m \in \mathbb{N} \ (u_{n+2} - u_{k+m} + 2^{-n} \geq 2^{-k} \land u_{k+m} - u_{n+2} + 2^{-n} \geq 2^{-k}$ whence $\forall n \in \mathbb{N} \ u_{n+2} - x + 2^{-n} > 0 \land x - u_{n+2} + 2^{-n} > 0$, i.e. $x = r \lim_{n} u_{n+2}$.

We infer from this lemma that $\Sigma_p\mathcal{R}$ is archimedean: indeed, let $x$ be a real and $(u_n)$ a rational sequence which rapidly converges to $x$; thus it is a rapid Cauchy sequence, and so, bounded by a rational $r$. Then $x$ is bounded by $[r] + 1$. 

Corollary 3.22. \((\mathcal{S}_p \mathbb{R}, +, \cdot, \leq)\) is a \(\Sigma_p\)-subinductive field.

Set now \(\mathcal{S} := \bigcup_{i \in \mathbb{N}} \mathcal{S}_p(\mathbb{R}^{I^*_i})\) with the natural interpretations for +, \cdot, \leq. By definition, every element of \(\mathcal{S}\) is \(\Sigma_p\)-definable. The following lemma will help us to show the property of \(\Sigma_p\)-completeness.

Lemma 3.23. Let \((u_n) \in \mathcal{S}_p(\mathbb{R}^{'})\) be a rapid Cauchy sequence. Then \((u_{n+5})\) rapidly converges.

Proof. Let \((r^p_n) \in \mathcal{S}_p(\mathbb{R}^{I^*_i})\) rapidly converging to \((u_n)\). We have

\[
\forall n, k \in \mathbb{N}|u_n - r^p_n| \leq 2^{-p} \quad \text{and} \quad \forall n \in \mathbb{N}|u_{n+1} - u_n| < 2^{-n}.
\]

Set \(s_n := r^p_{n+2}\) then \((s_n) \in \mathcal{S}_p(\mathbb{R}^{I^*_i})\) and is a rapid Cauchy sequence because

\[
|s_{n+1} - s_n| = |r^p_{n+3} - r^p_{n+2}|
\leq |r^p_{n+3} - u_{n+3}| + |u_{n+3} - u_{n+2}| + |u_{n+2} - r^p_{n+2}|
\leq 2^{-n-3} + 2^{-n-2} + 2^{-n-2}
\leq 2^{-n}.
\]

\((s_{n+2})\) rapidly converges to \(a := [s_n]\), whence

\[
|u_{n+5} - a| \leq |u_{n+5} - r^p_{n+5}| + |r^p_{n+5} - a|
\leq 2^{-n-5} + 2^{-n-1} \leq 2^{-n}.
\]

That is to say rapid \(\lim_{n} u_{n+5} = a\). \(\square\)

Furthermore, according to Lemma 3.25, every \(\Sigma_p\)-Cauchy sequence contains a \(\Sigma_p\)-definable rapid Cauchy sub-sequence. Thus let \((u_n) \in \mathcal{S}_p(\mathbb{R}^{'})\) be \(\Sigma_p\)-Cauchy; let \((\bar{e}_n) \in \mathcal{S}_p(\mathbb{R}^{'})\) a modulus of convergence and \(\sigma\) such that \((u_{\sigma(n)})\) is rapid Cauchy and so converging to a real \(a\). Then:

\[
|u_n - a| \leq |u_n - u_{\sigma(n+5)}| + |u_{\sigma(n+5)} - a| \leq \bar{e}_n + 2^{-n}.
\]

Let \(\lim_{n} u_n = a\).

This completes the proof of the theorem. \(\square\)

From now on, a \(\Sigma_p\)-real field will be written as \(\Sigma_p \mathbb{R}\).

3.6. Properties of sequences of a \(\Sigma_p\)-real field

In this section, we shall first show that we can mimic some classical notions about sequences, and then recover some elementary facts about them.
3.6.1. Convergence

Definition 3.24 (*Σ*-definable convergence). We say that the sequence \((u_n)\) *Σ*-definably converges to \(x\) if there exists \(\alpha \in \Sigma_p(\mathbb{N}^{\mathbb{N}})\) (called modulus of *Σ*-definable convergence) such that

\[
\forall n, k \in \mathbb{N} \quad (n \geq k) \rightarrow |u_n - x| < 2^{-k}.
\]

Lemma 3.25. The *Σ*-definable convergence is equivalent to the rapid convergence in the theory of algebras of *Σ*-real sequences.

Proof. If \((u_n)\) converges rapidly, \(\alpha(k) := k\) is the sought modulus. If \((u_n)\) *Σ*-definably converges with \(\alpha\) for modulus, the sub-sequence \((u_{\alpha(n)})\) which is *Σ*-definable, rapidly converges. □

3.6.2. Definitions by induction

Induction on a formula with real parameters is in general not allowed and it seems that we cannot directly define a real sequence by induction. Indeed we have:

(Meta-)Theorem 3.26. The theory of algebras of *Σ*-real sequences does not prove the induction scheme for *Σ*-formulae of L(AS) (or L(CI)).

Proof. Suppose that all algebras of *Σ*-real sequences satisfy the induction scheme for the *Σ*-formulae of L(AS), and thus of L(CI). They would therefore be inductive algebras. But the theory of (*Σ*-) inductive algebras of sequences is a conservative extension of the theory of archimedean rationally complete (*Σ*- inductive) fields, which in turn is a non-conservative extension of PA (P. Cegielski’s theorem, cf. [4]). But this is in contradiction to Theorem 3.20. □

Fortunately, for what we need, the device which consists in first defining a sequence on rationals and then taking the limit, will work. The following theorem put this method in a sufficiently general setting:

Theorem 3.27. Let \(\alpha \in \Sigma_p\mathbb{R}\) and \(f\) be a function from \(\Sigma_p\mathbb{R} \times \mathbb{N}\) into \(\Sigma_p\mathbb{R}\), which is \(k\)-lipschitzian \((k \in \mathbb{N})\) relative to the first variable and such that its restriction to \(\mathbb{Q} \times \mathbb{N}\) is *Σ*-definable; then there exists a unique *Σ*-definable sequence such that \(u_0 = \alpha\) and \(u_{n+1} = f(u_n, n)\).

Proof. Let \(r_{0,p} \in \Sigma_p(\mathbb{Q}^{\mathbb{N}})\) approaching rapidly \(\alpha\). For every \(p\), let us define by induction \(r_{n+1,p} = f(r_{n,p}, n)\), which is possible because we deal with a rational sequence, and \(r_{n,p} \in \Sigma_p(\mathbb{Q}^{\mathbb{N}})\). We shall prove that for all \(n \in \mathbb{N}\) \((r_{n,p})_p\) is *Σ*-Cauchy and thus converges:

\[
\forall n, p, q |r_{n,p+q} - r_{n,p}| = |f(r_{n,p+q}) - f(r_{n,p})| \leq k|r_{n,p+q} - r_{n,p}|
\]

\[
\leq k^n|r_{0,p+q} - r_{0,p}| \leq k^n2^{-p}
\]

\(u_n\) is then the limit of \((r_{n,p})_p\). □
**Corollary 3.28** (Maximum of a sequence). If \((u_n)\) ∈ \(\Sigma_p(\mathbb{R}^+)\) then \(\max_{i \leq n} u_i\) exists for every \(n\) and \((\max_{i \leq n} u_i)\) ∈ \(\Sigma_p(\mathbb{R}^+)\).

**Proof.** \(f(r, n) := \max(r; u_{n+1})\) and \(|f(r, n) - f(s, n)| \leq |r - s|\). Clearly, if \((u_n)\) ∈ \(\Sigma_p(\mathbb{R}^+)\), \(f\) is \(\Sigma_p\)-definable. \(\Box\)

**Corollary 3.29** (Definition of powers of reals). For every real \(a\), there exists a unique real \(\Sigma_p\)-definable sequence \((u_n)\) such that

\[ u_0 = 1, \quad \forall n \in \mathbb{N} \quad u_{n+1} = u_n \cdot a \]

**Proof.** \(f(r, n) := r \cdot a\) and \(|f(r, n) - f(s, n)| = |r - s|\). Clearly, if \(a \in \mathbb{R}\), \(f\) is \(\Sigma_p\)-definable. \(\Box\)

We want to define finite sums (in the sense of the model), i.e. \(\sum_{i=0}^{n} u_i\) for some possibly not standard \(n\).

**Corollary 3.30** (Existence of finite sums). If \((u_n)\) is a real \(\Sigma_p\)-definable sequence then there exists a unique real \(\Sigma_p\)-definable sequence \((S_n)\) such that: \(S_0 = u_0, \quad S_{n+1} = S_n + u_{n+1}\).

**Proof.** \(f(r, n) := r + u_n\) and \(|f(r, n) - f(s, n)| = |r - s|\). Moreover, if \((u_n)\) ∈ \(\Sigma_p(\mathbb{R}^+)\), \(f\) is \(\Sigma_p\)-definable. \(\Box\)

**Definition 3.31.** \(S_n\) is written as \(\sum_{i=0}^{n} u_i\) and we call it the partial sum of order \(n\) of the series \(\sum_{n \geq 0} u_n\).

Then we find, each time by taking rational approximations:

**Lemma 3.32.** (i) Linearity of finite sums: \(\forall c \in \mathbb{R}^+ \forall n \in \mathbb{N} : \sum_{i=0}^{n} a_i + c \cdot b_i = \sum_{i=0}^{n} a_i + c \cdot \sum_{i=0}^{n} b_i\).

(ii) If \(\forall n \in \mathbb{N} a_n \leq b_n\), then \(\forall n \in \mathbb{N} : \sum_{i=0}^{n} a_i \leq \sum_{i=0}^{n} b_i\).

The theorem on exchange of “finite” sums and limits works, providing the moduli of convergence form a definable sequence as well:

**Lemma 3.33.** If \((u^k_i)\) is a real \(\Sigma_p\)-definable sequence such that for every \(n\), \((u^k_i)\) converges to \(a_n\) with a modulus of convergence of the form \(A_n. (\varepsilon_k)\), \((\varepsilon_p)\) being a modulus of convergence and \((A_i)\) a real \(\Sigma_p\)-definable sequence, then for a given \(n\) \((\sum_{i=0}^{n} u^k_i)\) converges to \(\sum_{i=0}^{n} a_i\).

**Proof.** \(|\sum_{i=0}^{n} u^k_i - \sum_{i=0}^{n} a_i| \leq \sum_{i=0}^{n} |u^k_i - a_i| \leq (\sum_{i=0}^{n} A_i). (\varepsilon_k). \Box|
Lemma 3.34 (Newton’s formula).

\[ \forall x, y \in \mathbb{R}_p, \forall n \in \mathbb{N} \ (x + y)^n = \sum_{i=0}^{n} C_n^i x^i y^{n-i}. \]

Proof. Remark that if \((u_n)\) is a real sequence \(\Sigma_p\)-definable by the formula \(F(n,u)\) then \(v_{nj} = u_{n-i}\) if \(i \leq n\), 0 otherwise, it is \(\Sigma_p\)-defined by

\[ G(n,i,v) \equiv (v = 0 \land i > n) \lor (i \leq n \land \exists j \ (j = n - i \land F(j,v))). \]

3.6.3. Series
Definition 3.35. We say that a \(\Sigma_p\)-definable series \(\sum u_n\) is \(\Sigma_p\)-convergent if the sequence of partial sums \(\Sigma_p\)-converges.

We have, copying the usual proofs which use only the properties on the convergence in \(\mathbb{R}\) also valid in \(\mathbb{R}_p\):

Lemma 3.36 (Absolute convergence). Let \(\sum u_n\) be a \(\Sigma_p\)-definable series such that there exists a \(\Sigma_p\)-definable \(\Sigma_p\)-convergent series with positive terms \(\sum v_n\), and such that: \(\forall n \in \mathbb{N} |u_n| \leq v_n\). Then \(\sum u_n\) is a \(\Sigma_p\)-convergent series.

Lemma 3.37 (Alternate series). Let \(\sum u_n\) be a \(\Sigma_p\)-definable series, with \(u_n = (-1)^n v_n\) and \((v_n)\) be a positive decreasing sequence \(\Sigma_p\)-converging to 0. Then it \(\Sigma_p\)-converges to a real a such that

\[ \forall n \in \mathbb{N} \ (|a - \sum_{i=0}^{n} u_i| \leq |u_{n+1}|) \land (\sum_{i=0}^{2n+1} u_i \leq \sum_{i=0}^{2n} u_i). \]

Lemma 3.38 (Cauchy’s product). Let \(\sum u_n\) and \(\sum v_n\) be real \(\Sigma_p\)-definable series which absolutely \(\Sigma_p\)-converge, with respective sums a and b. Then the series \(\sum w_n\)

\[ \forall n \in \mathbb{N} \ w_n = \sum_{i=0}^{n} u_i v_{n-i} \]

is \(\Sigma_p\)-defined and (absolutely) \(\Sigma_p\)-converges, with sum \(a \cdot b\).

Then we can define the exponential of a real, and other expressions defined by power series:

Lemma 3.39 (and definition). For every \(\Sigma_p\)-definable real \(x\) the \(\Sigma_p\)-definable series

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

is \(\Sigma_p\)-convergent. We denote by \(e^x\) or \(\exp(x)\) the sum of this series.
Lemma 3.40. For every $\Sigma_{p}$-definable real $x$ the $\Sigma_{p}$-definable series:

\[
\sum (-1)^{n} \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \sum (-1)^{n} \frac{x^{2n}}{(2n)!}
\]

$\Sigma_{p}$-converge. We denote by $\sin x$ and $\cos x$ the respective sums of these series.

We have, thanks to Cauchy’s product and Newton’s formula:

Lemma 3.41. \(\forall x,y \in \Sigma_{p} \mathbb{R} \; e^{x+y} = e^{x} \cdot e^{y}\).

We thus get the trigonometric formulae.

If \((u_{n})\) is a real definable sequence, the sequence \((\lceil u_{n} \rceil)\) is not always definable (recall that in our standard model, i.e. the field of recursive reals, the function \(x \mapsto \lfloor x \rfloor\) is not recursive). Fortunately, the following weaker result will be sufficient:

Theorem 3.42. Let \((u_{n}) \in \Sigma_{p}(\mathbb{R}^{+})\). Then there exists a $\Sigma_{p}$-definable sequence of $\mathcal{N}$, written as $ES(u_{n})$, such that: $\forall i \; u_{i} \leq ES(u_{i})$.

Proof. Let $r_{i}^{p} \in \Sigma_{p}(\mathbb{R}^{+} \times \mathcal{V})$ rapidly converging to $(x_{i})$.

We have $\forall p \; \forall q (r_{i}^{p+q} - r_{i}^{p}) \leq 2^{-p+1}$. Clearly $|r_{i}^{p+q} - [r_{i}^{p+q}]| \leq 1$ and $|r_{i}^{p} - [r_{i}^{p}]| \leq 1$.

Whence, with $p = 2$, $\forall q |[r_{i}^{2+q}]| \leq 2 + \frac{1}{2}$.

We set $ES(x_{i}) := [r_{i}^{2}] + 4$.

We have $(ES(x_{i})) \in \Sigma_{p}(\mathcal{V})$ and $\forall q r_{i}^{q} \leq [r_{i}^{q}] + 1 \leq [r_{i}^{2}] + 2 + \frac{1}{2} + 1 < ES(x_{i})$. \(\square\)

3.7. A negative result

Contrary to what is known for the classical structure $(\mathbb{R}^{N}, \mathbb{R}, +, \cdot)$, the theory of algebras of $\Sigma_{p}$-real sequences does not allow to prove that an increasing bounded (even definable) sequence converges. Here is a fundamental difference from the analysis in PA [3, 4, 17], coming from the fact that the statement “to be the upper bound of a (rational) $\Sigma_{p}$-definable” part is $\Pi_{p+1}$, which is not a problem in PA, whereas it is crippling in $RI\Sigma_{p}$. Furthermore, we have the following theorem which is a generalisation of Specker’s theorem [16]:

Theorem 3.43. The following statement is provable in the theory of algebras of $\Sigma_{p}$-real sequences: “there exists a rational $\Sigma_{p}$-definable increasing and bounded sequence which does not converge”, formally:

\[
\exists u \in \Sigma_{p}(\mathbb{R}^{+}) \forall n \in \mathcal{N} u_{n} \leq u_{n+1} \land \exists M \in \mathcal{H} \forall n \in \mathcal{N} u_{n} \leq M \land \forall x \in \mathcal{H} \exists \epsilon > 0 \forall n \in \mathcal{N} \exists p > n |u_{n} - x| > \epsilon.
\]

To prove it, first recall the following fact:

Theorem 3.44. There exists a formula $\Psi(x) \equiv \exists \phi_{p}(e,x)$ which is $\Sigma_{p}$ but not $RI\Sigma_{p}$-equivalent to a $\Pi_{p}$-formula.
Proof. Let \( \phi_a(e,x) \) be a universal \( \Sigma_p \)-formula (see Section 3.15). Suppose that \( \exists \phi_a(e,x) \) is \( A_p \). Let \( f \) be the number of the \( \Sigma_p \)-formula \( \neg \exists \phi_a(e,x) \). Then we have:

\[
\phi_a(f,y) \leftrightarrow \neg \exists \phi_a(e,x), \text{ contradiction.}
\]

We will also need the following three lemmas (which are inspired by [11, p. 15]):

Lemma 3.45. Let \( \phi_a(m,x) \) be a \( \Delta_p \)-formula such that:
- \( \mathsf{R}\Sigma_p \vdash \forall m \exists! x \phi_a(m,x) \);
- \( \exists m \phi_a(m,x) \) is a \( \Sigma_p \) but not \( \mathsf{R}\Sigma_p \)-equivalent to a \( \Pi_p \)-formula in the standard model.

Set \( w(n) := \max \{ m; \phi_a(m,x) \land x \leq n \} \); then \( \neg (\exists u \in \Sigma_p(\varphi^{\omega}) \forall nw(n) \leq u_n) \).

Proof. Suppose the opposite and let \( \Psi(n) \) be the formula \( \exists m \phi_a(m,n) \).
If \( \exists m \leq u_n \phi_a(m,n) \) then \( \Psi(n) \). Otherwise, \( \forall m \leq u_n \forall x \phi_a(m,x) \rightarrow x \neq n \); but \( w(n) \leq u_n \) so \( \forall m \leq w(n) \forall x \phi_a(m,x) \rightarrow x \neq n \). Moreover, according to the definition of \( w \), if \( m > w(n) \) then \( \forall x \phi_a(m,x) \rightarrow x > n \), thus \( \neg \Psi(n) \). So, by writing \( \phi_a \) the formula \( \Sigma_p \)-defining \( u \):

\[
\Psi(n) \leftrightarrow \exists m \leq u_n \phi_a(m,n)
\]

\[
\leftrightarrow \exists y(\phi_a(n,y) \land \exists m \leq y \phi_a(m,n))
\]

\[
\leftrightarrow \forall y(\phi_a(n,y) \land \exists m \leq y \phi_a(m,n))
\]

and \( \Psi(n) \) is \( A_p \); a contradiction. \( \square \)

Lemma 3.46. With the same notations as those of the previous lemma, set

\[
x = a(m) \leftrightarrow \phi_a(m,x), \quad u_n := \sum_{i=0}^{n} 2^{-a(i)},
\]

and suppose that \( (u_n) \) \( \Sigma_p \)-converges to an element \( x \). Set

\[
\alpha_0(k) := \mu j n \leq j \rightarrow x - u_n \leq 2^{-k}
\]

(\( \alpha_0 \) is the lowest modulus of convergence). Then \( \alpha_0(k) = w(k) \).

Proof. First notice that \( (u_n) \in \Sigma_p(\varphi^{\omega}) \). Let us prove that \( w \) is the modulus of convergence \( \alpha_0 \):

If \( n < w(k) \) then \( x - u_n \) contains the term \( 2^{-a(w(k))} \); but \( a(w(k)) \leq k \), whence:

\[
x - u_n > 2^{-k}.
\]

If \( n \leq w(k) \) then, because \( m \leq n + 1 \rightarrow a(m) > k \), we have:

\[
x - u_n \leq \sum_{k+1}^{n} 2^{-i} = 2^{-k}. \quad \square
\]

Lemma 3.47. Consider an algebra of \( \Sigma_p \)-real sequences; let \( (u_n) \) be a \( \Sigma_p \)-definable increasing sequence; if \( (u_n) \) converges to \( x \), then \( (u_n) \) \( \Sigma_p \)-converges.
Proof. Let \((r_k) \in \Sigma_p(\mathbb{Q}^\mathbb{N})\) and \((r_{n,k}) \in \Sigma_p(\mathbb{Q}^\mathbb{N})\) rapidly converging respectively to \(x\) and \((u_n)\). That is: \(\forall k (|x - r_k| < 2^{-k})\) and \(\forall k \forall n (|u_n - r_{n,k}| < 2^{-k})\). We set \(z(k) := \mu n |r_{n,k} - r_k| < 2^{-k+1}\). Then \(z\) is well defined and \(z \in \Sigma_p(\mathbb{Q}^\mathbb{N})\). Then

\[ n = z(k) \rightarrow 0 \leq x - u_n < 2^{-k} + 2^{-k+1} + 2^{-k} = 2^{-k+2}. \]

But \(m \leq z(k) \rightarrow u_m \leq u_{z(k)}\) whence \(0 \leq x - u_m \leq x - u_n < 2^{-k+2}\) and \(z(k + 2)\) is a modulus of \(\Sigma_p\)-convergence. 

We can now prove the theorem:

Proof. Keeping the notations of the second lemma, the sequence \((u_n)\) is the sought sequence: if it converged, it would \(\Sigma_p\)-definably converge; let \(z\) be a modulus of convergence. Then \(\forall n \leq z(k) (|x - u_n| < 2^{-k})\). Thus \(\forall n \ z(n) \leq z_0(n) = w(n)\), which is impossible according to Lemma 3.46. 

4. Fields with functions

We now extend the language so that it will fit the structure \((\mathbb{R}^{\mathbb{N}}, \mathbb{N}, \mathbb{R}, +, , , <)\) and more generally, so that we will be able to talk about functions over a \(\Sigma_p\)-real field.

4.1. \(\Sigma_p\)-real algebras of functions

Definition 4.1. The language of algebras of functions is the language \(L(AF) := \{+, , , eval, \mathbb{F}, \mathcal{K}, \mathcal{N}, \leq\}\) where \(\mathbb{F}\) is a unary predicate.

We will write \(f(x) = y \leftrightarrow \neg \mathbb{F}(f) \land eval(f, x) = y\).

This language allows, being given a set \(K\), to give axioms for and to talk about some algebras of functions from \(K\) into \(K\), and (simple, double, . . .) sequences of functions. The standard structure of this language is \((\bigcup_{i \in \mathbb{N}} \mathbb{R}^{\mathbb{N}}, \mathbb{N}, \mathbb{R}, +, , , <)\). Note that we want to consider not only functions from \(\mathbb{R}\) into \(\mathbb{R}\), but also (simple, double, . . .) sequences of functions.

Definition 4.2. We call theory of \(\Sigma_p\)-real algebra of functions the theory with language \(L(AF)\) and with the following axioms:

1. \((\mathbb{F}, +, , , eval, \mathcal{K}, \mathcal{N}, \leq)\) is an algebra of \(\Sigma_p\)-real sequences.

2. Compatibility of the evaluation:

\[ \forall f, g, x, y [ eval(f + g, x) = eval(f, x) + eval(g, x) \land eval(f \cdot g, x) = eval(f, x) \cdot eval(g, x)]. \]

Theorem 4.3. The theory of \(\Sigma_p\)-real algebras of functions is a conservative extension of the theory of algebras of \(\Sigma_p\)-real sequences and thus of \(RI\Sigma_p\).
Lemma 4.7. If \( f \) and \( g \) be two sequentially \( \Sigma^p \)-definable functions. Then \( f, g, f \circ g, \) and, if \( f \neq 0, 1/f, \) are also sequentially \( \Sigma^p \)-definable functions. And so are the functions: constant and power of an integer.

Proof. Algebraic properties of \( \Sigma^p \)-definable sequences. \( \Box \)

Lemma 4.7. If \( f \) is a \( \Sigma^p \)-definable power series, converging on \([-R,R]\) or on \( \Sigma^p \mathcal{R} \), then \( f \) is sequentially \( \Sigma^p \)-definable.

Proof. Let \((u_k)\) be a \( \Sigma^p \)-bound on \([-R,R]\) and \( f(x) = \sum_{n=0}^{\infty} a_n x^n \).

\( \mathcal{N} \) be an algebra of \( \Sigma^p \)-real sequences. Consider the set \( \mathcal{F}' \) of functions of an interval \([a,b]\) of \( \mathcal{N} \), with the sequences (following \( \mathcal{F}' \)) of such functions. Set \( \mathcal{F} := \mathcal{F} \cup \mathcal{F}' \), and extend canonically to \( \mathcal{F} \) the operations +,.. and \( \text{eval} \). Then, by definition, \( \mathcal{F} \) is a \( \Sigma^p \)-real algebra of functions. \( \Box \)

Definition 4.4. A \( \Sigma^p \)-definable power series on an interval \( I \) is a function \( f \) such that there exists a \( \Sigma^p \)-definable real sequence \((a_i)\) such that the series \( \sum_{n=0}^{\infty} a_n x^n \) \( \Sigma^p \)-converges to \( f(x) \) for all \( x \) in \( I \).

4.1.1. Sequentially \( \Sigma^p \)-definable functions

Most of the results on functions make use of sequences of elements of the field: so, we must make sure that our functions conserve the definability of sequences.

Definition 4.5. A function \( f \) from \( \Sigma^p \mathcal{R} \) into \( \Sigma^p \mathcal{R} \) is said to be sequentially \( \Sigma^p \)-definable if and only if for every sequence \((u_n)\) of \( \Sigma^p \)-real sequences, \((f(u_n))_n\) is also a \( \Sigma^p \)-definable sequence. We thus define the following predicate:

\[ f \in \text{Seqdef}([a,b]) \leftrightarrow f \in \mathcal{N}^{[a,b]} \land \forall u \in \mathcal{F}(a \leq u \leq b \rightarrow \exists v \in \mathcal{F} v = \text{eval}(f,u)). \]

The function \( f(x) = \lambda x + \mu \) with \( \lambda, \mu \in \Sigma^p \mathcal{R} \) is sequentially \( \Sigma^p \)-definable. But the following function is not: \( g(x) = 1 \) if \( x \in \mathbb{N} \), \( g(x) = 0 \) otherwise; indeed, the definable sequence \( u_n = n \) is turned into a not definable one.

Lemma 4.6. Let \( f \) and \( g \) be two sequentially \( \Sigma^p \)-definable functions. Then \( f + g, f \cdot g, f \circ g \), and, if \( f \neq 0, 1/f, \) are also sequentially \( \Sigma^p \)-definable functions. And so are the functions: constant and power of an integer.

Proof. Algebraic properties of \( \Sigma^p \)-definable sequences. \( \Box \)

Lemma 4.7. If \( f \) is a \( \Sigma^p \)-definable power series, converging on \([-R,R]\) or on \( \Sigma^p \mathcal{R} \), then \( f \) is sequentially \( \Sigma^p \)-definable.

Proof. Let \((u_k)\) be a \( \Sigma^p \)-bound on \([-R,R]\) and \( f(x) = \sum_{n=0}^{\infty} a_n x^n \).

Set \( v_k := \sum_{n=0}^{\infty} a_n (u_k)^n \) and \( v_{k,j} := \sum_{n=0}^{\infty} a_n (u_k^n)^n \); \((v_{k,j})\) be \( \Sigma^p \)-real sequences. Consider the set \( \mathcal{F} \) of such functions. Set \( \mathcal{F} := \mathcal{F} \cup \mathcal{F}' \), and extend canonically to \( \mathcal{F} \) the operations +,.. and \( \text{eval} \). Then, by definition, \( \mathcal{F} \) is a \( \Sigma^p \)-real algebra of functions. \( \Box \)
Then $e(k, N) := \mu \epsilon_k(2^{2(k)}) < 2^{-N}$ is $\Sigma_p$-definable and $i \geq e(k, N) \rightarrow |v_k - v_k(i)| < 2^{-N}$. □

4.1.2. Classical notions of continuous, differentiable... functions

We can introduce the classical notions of continuous, differentiable... functions, with the same definitions, which are expressible in the language $L(AF)$. However, these definitions soon turn out to be inadequate: we cannot even prove that a continuous function has an upper bound on a bounded closed interval!

**Theorem 4.8.** There exists a $\Sigma_p$-real algebra $F$ of functions such that

$$F \models \exists a \in \Sigma_p \mathcal{R} \exists f \in \Sigma_p \mathcal{R}^{[0, a]} \{ [ \forall x \in [0, a] \forall e > 0 \exists \eta > 0 \forall y \in [0, a] |x - y| \rightarrow |f(x) - f(y)| < \epsilon ] \land \forall M \in \Sigma_p \mathcal{R} \exists x \in [0, a] f(x) > M \}. $$

**Proof.** We consider a $\Sigma_p$-real algebra of sequences constructed on a non-standard model $\mathcal{N}$ of PA, and put $F := \Sigma_p \mathcal{R}^{\mathcal{N}}$. We use the unbounded sequence $(u_n)$ defined in Theorem 3.6. We set: $\mathcal{R}_{st} := \{ x \in \Sigma_p \mathcal{R} ; \exists n \in \mathbb{N} |x| \leq n \}$ and $\mathcal{R}_{nst} := \Sigma_p \mathcal{R} \setminus \mathcal{R}_{st}$. Now let $f$ be the function defined by:

- $f(x) = \text{segment linking points } (n, u_n) \text{ and } (n + 1, u_{n+1}) \text{ if } x \in \mathcal{R}_{st} \text{ and } n \leq x \leq n + 1$;
- $f(x) = 1 \text{ otherwise}$.

$f$ is continuous because it is on $\mathcal{R}_{st}$ and $\mathcal{R}_{nst}$ which are open subsets. However, if $a \in \mathcal{R}_{nst}$, then $f$ is unbounded on $[0, a]$ because $f(n) = u_n$ for $n \in \mathbb{N}$. □

Nevertheless, we can prove:

**Lemma 4.9** (Intermediate value theorem). Let $f$ be a sequentially $\Sigma_p$-definable continuous function on $[a, b]$, such that: $f(a).f(b) < 0$. Then $\exists c \in ]a, b[f(c) = 0$.

**Proof.** We shall follow the classical proof, but, as we have problems with the equality of two reals (it may not be $\Sigma_p$-definable), we have to use the following trick from computable analysis:

- $\exists c \in ]a, b[ f(c) = 0$; nothing to prove.
- $\forall c \in ]a, b[ f(c) \neq 0$;

We can suppose that $a$ and $b$ are rationals. We define the sequences $(a_n), (b_n), (c_n)$ by

$$a_0 := a, \quad b_0 := b, \quad c_0 := \frac{a_n + b_n}{2}$$

$$a_{n+1} = \begin{cases} a_n & \text{if } f(c_n) < 0, \\ c_n & \text{if } f(c_n) > 0 \end{cases}, \quad b_{n+1} = \begin{cases} b_n & \text{if } f(c_n) > 0, \\ c_n & \text{if } f(c_n) < 0. \end{cases}$$

(By hypothesis, $f(c_n)$ is not zero.)
The two conditions being \( \Delta_p \), the sequences are well \( \Sigma_p \)-definable. They are rapid Cauchy and because \( |a_n - b_n| < 2^{-n} \), converge to the same real \( c \). But \( \forall n f(a_n) > 0 \) and \( f(b_n) < 0 \); by continuity \( f(c) = 0 \). □

### 4.2. \( \Sigma_p \)-definable continuity

A fundamental property of continuous functions is the uniform continuity on every closed interval. Moreover, if we want uniform continuity to be usable in the framework of \( \Sigma_p \)-real fields, we shall require the modulus of uniform continuity (i.e. the function \( \varepsilon \mapsto \eta \)) to be definable. But the recursive model provides an example of a sequentially \( \Sigma_1 \)-definable function which is continuous but not definably uniformly continuous (see [11, p. 67]). So that we shall set the following definitions:

**Definition 4.10.** (1) \( f : [a, b] \to \Sigma_p \mathcal{R} \) is said to be \( \Sigma_p \)-definably continuous (We write \( \Sigma_p C^0 \)) if:

(i) \( f \) is sequentially \( \Sigma_p \)-definable;

(ii) there exists a function \( \varepsilon \in \Sigma_p(\mathcal{N}^{-\varepsilon}) \), called modulus of continuity of \( f \) such that

\[
\forall n \in \mathcal{N} \ \forall x, y \in \mathcal{X} \cap [a, b] |x - y| < \frac{1}{\text{eval}(\varepsilon, n)} \rightarrow |f(x) - f(y)| < 2^{-n}.
\]

We thus define the following predicate:

\[
f \in \Sigma_p C^0([a, b]) \leftrightarrow \{ \forall u \in \mathcal{U} [a \leq u \leq b \rightarrow \exists v \in \mathcal{V} = \text{eval}(f, u)] \wedge \exists \varepsilon \in \Sigma_p(\mathcal{N}^{-\varepsilon}) \forall n \in \mathcal{N} \forall x, y \in \mathcal{X} \cap [a, b] \]
\[
|x - y| < \frac{1}{\text{eval}(\varepsilon, n)} \rightarrow |f(x) - f(y)| < 2^{-n} \}.
\]

(2) \( f : I \to \Sigma_p \mathcal{R} \), where \( I \) is a not closed interval, is said to be \( \Sigma_p \)-continuous on \( I \) if it is, on every interval \([a, b]\) contained in \( I \):

\[
f \in \Sigma_p C^0(I) \leftrightarrow \forall a, b \in I f \in \Sigma_p C^0([a, b])\).
\]

(3) \( f \) is said to be uniformly \( \Sigma_p \)-continuous on \( I \) if the modulus on continuity depends in a \( \Sigma_p \)-definable way on a parameter \( M \) such that (if, for example, \( I = [a, +\infty[)\):

\[
\forall n, M \in \mathcal{N} \ \forall x, y \in \left[a + \frac{1}{M}, M\right] |x - y| < \frac{1}{\varepsilon(n, M)} \rightarrow |f(x) - f(y)| < 2^{-n}.
\]

Note that this definition includes uniform continuity and we do not consider pointwise continuity.

There are functions which are \( \Sigma_p \)-continuous but not uniformly \( \Sigma_p \)-continuous on \([a, +\infty[\): first remark that if \( f \) is uniformly \( \Sigma_p \)-continuous then \( f \) is dominated by a function \( \beta \in \Sigma_p(\mathcal{N}^{-\beta}) \). Recall that there exists a \( \Sigma_p \)-definable sequence \( (b_{n,m}) \) which is a numbering for the simple \( \Sigma_p \)-definable sequences, i.e. for all \( \varepsilon \in \Sigma_p(\mathcal{N}^{-\varepsilon}) \), there exists \( n \in \mathcal{N} \) such that \( \forall m b_{n,m} = \varepsilon_m \). We now consider the function \( g \) formed with the lines
linking \((n, b_{n+1})\) to \((n + 1, b_{n+1}, n + 1)\). It is easy to see that \(g \in \Sigma_p C^0([a, +\infty])\), but \(g\) dominates every \(x \in \Sigma_p (\forall \neg^\ast\).

**Lemma 4.11.** If \(f \in \Sigma_p C^0([a, b])\) then \(f\) is bounded.

**Proof.** Let \(p \geq ((b - a)x(1) + 1)\); choose the subdivision \(x_i := a + i(b - a)/p\); \((x_i)\) is \(\Sigma_p\)-definable,

\[
\forall x \in [a, b] \exists i \in [x_i, x_{i+1}] \quad \text{i.e.} \quad |x_i - x| \leq \frac{b - a}{p} < \frac{1}{x(1)},
\]

whence \(|f(x_i) - f(x)| < 2^{-i}\). \(f\) being \(\Sigma_p\)-definable, \(\{|f(x_i)|\}\) is \(\Sigma_p\)-definable thus \(\max_{i=0\to p} |f(x_i)|\) exists. Whence

\[
|f(x)| \leq \max_{i=0\to p} |f(x_i)| + \frac{1}{2}. \quad \square
\]

**Lemma 4.12.** (a) The following functions are (uniformly) \(\Sigma_p C^0\) on their set of definition:

\[
x \mapsto a, \quad x \mapsto x^n (n \in \mathcal{N}), \quad x \mapsto \frac{1}{x},
\]

(b) If \(f\) and \(g\) are (uniformly) \(\Sigma_p C^0\), then \(f + g, fg\) and \(f \circ g\) are too.

**Proof.** The first condition has already been seen. Let us prove the second one.

(a) For \(x \mapsto x^n\) we can take \(x(M, p) := 2^p n(M + 1)^{n-1} + 1\), because \(|x^n - y^n| \leq n(|x| + 1)^{n-1}|x - y|\). For \(x \mapsto \frac{1}{x}\) we can take \(x(M, p) := 2^p M^2 + 1\), because \(|1/x - 1/y| \leq |x - y|/|xy| \leq M^2 |x - y|\) when \(x, y \in [\frac{1}{M}, M]\).

(b) \(f\) of modulus \(\alpha\), \(g\) of modulus \(\beta\):

- \(f + g; \gamma(p) := \max(\alpha(p + 1), \beta(p + 1))\) is a modulus.
- \(f \cdot g\): Let \(M\) be such that: \(\forall x \in [a, b] \quad |f(x)| \leq M\) and \(|g(x)| \leq M\) (\(M\) exists according to the previous lemma).

Let \(q\) be such that \(M \leq 2^q\), then \(\gamma(p) := \max(\alpha(p + 1 + q), \beta(p + 1 + q))\) is a modulus, because

\[
|f \cdot g(x) - f \cdot g(y)| \leq |f(x) - f(y)||g(x)| + |f(y)||g(x) - g(y)|
\]

\[
\leq 2^q (|f(x) - f(y)| + |g(x) - g(y)|)
\]

- \(f \circ g; \gamma(p) := \beta(\mu \alpha(p) \leq 2^k)\) (\(\Sigma_p\)-definable) is a modulus, because

\[
|x - y| \leq \frac{1}{\gamma(p)} \to |g(x) - g(y)| < 2^{-k}
\]

\[
\leq \frac{1}{\alpha(p)} \to |f((g(x)) - f(g(y))| < 2^{-p}. \quad \square
\]
Lemma 4.13. Let $f \in \Sigma_p C^0([a,b])$. Then $f$ possesses an upper and a lower bound.

NB: It cannot be proved that points $x$ such that $f(x) = \sup_{t \in [a,b]} f(t)$ exist, because it is not the case in $\text{Rec}(\mathbb{R})$ (see [16]).

Proof. It suffices to show that $f$ owns an upper bound.

Set $s_k := \max_{1 \leq j < k} \{f(a + (j/k)(b-a))\}; \ (f(a + (j/k)(b-a)))_{j,k}$ being $\Sigma_p$-definable, so is $(s_k)_k$. Besides

$$\forall x, y \in [a,b] |x - y| < \frac{1}{\alpha(K)} \to |f(x) - f(y)| < 2^{-K}.$$ 

Let $k > \alpha(K), [(b-a)]$ then $\forall x \exists j \in \{0 \ldots k\} |f(x) - f(a + (j/k)(b-a))| < 2^{-K}$ thus $\forall x f(x) < s_k + 2^{-K}$; in particular, $\forall q \in \mathcal{N} s_k < s_k + 2^{-K}$. Also, $\forall x f(x) < s_k + 2^{-K}$ whence $s_k \leq s_k + 2^{-K}$.

Thus, $\forall k > \alpha(K), [(b-a)] \forall q \in \mathcal{N} |s_k - s_k + q| < 2^{-K}$, i.e. $(s_k)$ is a $\Sigma_p$-Cauchy sequence thus $\Sigma_p$-converges to a real $s$. Let us prove that $\sup_{[a,b]} f(x)$:

On the one hand, $\forall k > \alpha(K), [(b-a)] \forall x f(x) < s_k + 2^{-K}$ whence $\forall x f(x) \leq s^\prime$;

On the other hand, if $s' \geq f(x)$ then $\forall k \forall j f(a + (j/k)(b-a)) \leq s'$ that is: $s_k \leq s'$ and $s \leq s'$.

4.3. $\Sigma_p$-definable differentiability

4.3.1. Definition

We now introduce a convenient notion of differentiability. First notice that, as $\Sigma_p$-continuity, it is not a pointwise definition. Furthermore, we require the derivative to be $\Sigma_p$-continuous, because we will need to know that it has a lower bound.

Definition 4.14. Let $(\mathcal{F}, +, \cdot, \mathcal{H}, \mathcal{N}, \leq)$ be a $\Sigma_p$-real algebra of functions; a function $f : [a,b] \to \Sigma_p \mathcal{H}$ will be called $\Sigma_p$-differentiable (we write $f \in \Sigma_p C^1([a,b])$) if and only if

(i) $f$ is $\Sigma_p C^0$;

(ii) there exists a $\Sigma_p C^0$ function written as $f'$ and $\beta \in \Sigma_p (\mathcal{N}^\times \mathcal{F}^{\times})$ such that

$$\forall x, y \in [a,b] |x - y| < \frac{1}{\alpha(N)} \to |f(y) - f(x) - f'(x)(y - x)| < 2^{-N} |y - x|.$$ 

$\beta$ is called the modulus of differentiability of $f$.

The predicate “$f$ is $\Sigma_p$-differentiable” is thus defined by

$$f \in \Sigma_p C^1([a,b]) \iff f \in \Sigma_p C^0([a,b]) \land \exists g \in \Sigma_p C^0([a,b])$$

$$\exists \beta \in \Sigma_p (\mathcal{N}^\times \mathcal{F}^{\times}) \forall x, y \in [a,b] \forall n \in \mathcal{N}$$

$$|x - y| < \frac{1}{\beta(n)} \to |f(y) - f(x) - g(x)(y - x)| < 2^{-n}.$$ 

If $f$ is defined on an open interval, the notions of $\Sigma_p$-differentiability and uniform $\Sigma_p$-differentiability are defined as for the continuity.
Lemma 4.15. (a) The following functions are uniformly \( \Sigma_p C^1 \): \( x \mapsto x^n \) for every \( n \in \mathbb{N}^* \), \( x \mapsto 1/x \).

(b) If \( f \) and \( g \) are \( \Sigma_p C^1 \) then \( f + g \), \( f \cdot g \), \( f \circ g \) are too.

Proof. (a) \( \forall x \in \mathbb{R} \), let \( n \) be a natural number.

\[ f(x) = x^n \text{ then } f'(x) = nx^{n-1} \text{ (in particular } f \text{ and } f' \text{ are } \Sigma_p C^0 \); indeed, we use the inequality, valid for \( n \geq 2 \):

\[ |x^n - y^n - nx^{n-1}(x - y)| \leq \frac{n(n-1)}{2}(x - y)^2(|y| + 1)^{n-2} \]

and for \( x, y \in [-M, M] \) we take \( z(k, M) := ((n(n-1)/2)(M+1)^n + 1)2^k \).

\( f(x) = 1/x \) then \( f'(x) = -1/x^2 \) (in particular \( f \) and \( f' \) are \( \Sigma_p C^0 \); indeed,

\[ \forall x, y \in \left[ \frac{1}{M}, M \right] \left| \frac{1}{y} - \frac{1}{x} + \frac{1}{x^2}(y - x) \right| = \frac{1}{|xy|} \left( |x - y| + \frac{y}{x}(y - x) \right) \]

\[ \leq \frac{1}{|xy|} \left( |x - y| \left| 1 - \frac{y}{x} \right| \right) \]

\[ \leq |x - y|^2 \frac{1}{|xy|} \frac{1}{|x|} \]

\[ \leq M^2|x - y||x - y| \]

then \( z(k, M) := (M^3 + 1).2^k \) is the sought modulus.

(b) Let us prove it for the product: Set \( h := f \cdot g \); then \( h' = f'g + g'f \) (in particular, \( h \) and \( h' \) are \( \Sigma_p C^0 \)), indeed let \( M \) be a common bound for \( f \), \( g \) and \( g' \) on \( [a, b] \)

\[ |f(y)g(y) - f(x)g(x) - (f'(x)g(x) + g'(x)f(x))(y - x)| \]

\[ \leq |f(y)||g(y) - g(x) - g'(x)(y - x)| + |f(y) - f(x)||g'(x)||y - x| \]

\[ + |g(x)||f(y) - f(x) - f'(x)(y - x)| \]

\[ \leq 3M2^{-N+1} |y - x| \quad \text{ if } |x - y| < \frac{1}{z(N)} \]

with \( z(N) := \max(z_f(N + n_0), z_g(N + n_0), \gamma_f(N + n_0)) \) where \( z_f, z_g \) are the moduli of continuity of \( f \) and \( g \), \( \gamma_f \) the modulus of differentiability of \( f \) and \( n_0 \) a number such that \( 3M \leq 2^{n_0} \). \( \square \)

4.3.2. Properties

Rolle’s theorem, in its classical statement, cannot be proved. Indeed, contrary to analysis within PA (see [3]), we cannot show that the derivative vanishes. This comes from the fact that points where it would vanish are points where it would reach its lower bound (for example); and we have already noticed that in the recursive model such points may not exist. Nevertheless, we have:

Lemma 4.16 (Approaching Rolle’s theorem). Let \( f \in \Sigma_p C^1([a, b]) \). If \( f(a) = f(b) \), then \( \forall \varepsilon > 0 \exists x \in [a, b] \) such that \( |f'(x)| \leq \varepsilon \).
Proof. Notice that $f'$ is supposed to be $\Sigma_p^{C^0}$.

- If $\exists \alpha, \beta \ f'(x), f'(\beta) < 0$: we apply the intermediate value theorem to $f'$, i.e. $\exists c \in ]a, b[ \ f'(c) = 0$.

- Otherwise, by example $\forall x f'(x) > 0$. Let $m := \in f_{[a,b]} f'$ (which exists because $f' \in \Sigma_p^{C^0}$).

  Suppose that $m > 0$ and let $k \in \mathbb{N}$ such that $2^{-k} \leq m/2$. Let $\lambda \in \Sigma_p^0(\mathbb{N}^\omega)$ the modulus of differentiability of $f$, and set $\gamma := 1/\lambda(k)$; then

  $$\forall x, y \in [a,b] |x - y| < \gamma \to |f(x) - f(y) - f'(x)(x - y)| \leq 2^k |x - y| < \frac{m}{2} |x - y|.$$ 

  Set $x_i := a + i\gamma$ for $i = 0 \ldots [(b - a)/\gamma]$ and $x_n := b$.

  $(x_i)$ is $\Sigma_p$-definable and $\forall i |x_{i+1} - x_i| \leq \gamma$. Whence

  

  $0 = f(b) - f(a)$

  $= \sum_{i=0}^{n} f(x_{i+1}) - f(x_i)$

  $= \sum_{i=0}^{n} f'(x_i)(x_{i+1} - x_i) + \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i) - f'(x_i)(x_{i+1} - x_i))$

  $\geq m \sum_{i=0}^{n} (x_{i+1} - x_i) - \frac{m}{2} \sum_{i=0}^{n} (x_{i+1} - x_i)$

  $= \frac{m}{2} (b - a) > 0$

  a contradiction; thus $m = 0$. \qed

Lemma 4.17 (Approaching mean value theorem). Let $f \in \Sigma_p^{\mathbb{C}^1([a,b])}$. Then

$$\forall \varepsilon > 0 \ \exists x \in [a,b] |f(b) - f(a) - f'(x)(b - a)| \leq \varepsilon.$$

Proof. Apply Rolle’s theorem to $h(x) := (x - a)(f(b) - f(a) - f(x)(b - a))$. \qed

Corollary 4.18. Let $f, g \in \Sigma_p^{\mathbb{C}^1([a,b])}$ such that $f' = g'$. Then $\exists c \in \Sigma_p \ f = g + c$.

Proof. We consider $h = f - g$ whence $h \in \Sigma_p^{\mathbb{C}^1([a,b])}$ and $h' = 0$. Let $z \in [a,b]; \forall \varepsilon > 0 \ \exists x \in [a,z] |h(z) - h(a)| \leq \varepsilon$ whence $h(z) = h(a)$. \qed

4.4. Integration

We now define the process of integration, using Riemann’s sums for $\Sigma_p$-definably continuous functions. This will be more than enough relative to our purposes. Note that these sums are finite in the sense of the model. The requirement of sequential $\Sigma_p$-definability for the function, is thus necessary.
Definition 4.19. Let \( f \) be a \( \Sigma_p \)-definably continuous function on \([a, b]\). Let \( x_i := a + i(b - a)/n, \ i = 0 \ldots n \). The \( \Sigma_p \)-definable sequence defined by \( S(n, f, a, b) := \sum_{k=0}^{n-1} f(x_k) \) \((b - a)/n\) is called sequence of Riemann’s sums.

Lemma 4.20. If \( f \in \Sigma_p C^0([a, b]) \) then \((S(n, f, a, b))_n \Sigma_p \)-converges.

Proof. Let \( \varepsilon \in \Sigma_p(\mathbb{N}, \mathbb{N}) \) be the modulus of continuity of \( f \):

\[
|x - y| < \frac{1}{\varepsilon(m)} \rightarrow |f(x) - f(y)| < 2^{-m}.
\]

Let \( n, p \in \mathbb{N} \) such that \( p \geq n > \varepsilon(m)([b - a] + 1) \)

\[
|S(n, f, a, b) - S(p, f, a, b)| = \frac{b - a}{np} \left| \sum_{i=0}^{n} p.f\left(a + i \frac{b - a}{n}\right) - \sum_{i=0}^{n} n.f\left(a + i \frac{b - a}{p}\right) \right|.
\]

Set for \( j = 0, 1, \ldots, np \):

\[
a_j := a + i \frac{b - a}{n} \quad \text{if} \quad j \in \{ip, ip + p - 1\},
\]

\[
b_j := a + i \frac{b - a}{p} \quad \text{if} \quad j \in \{in, in + n - 1\};
\]

\((a_j)\) and \((b_j)\) are \( \Sigma_p \)-definable: indeed, if \( \Phi(k, i, r) \) defines \((a + i(b - a)/n)\), then the formula \( \Psi(k, j, r) \equiv \exists i \left( \Phi(k, i, r) \land ip \leq j \leq ip + p - 1 \right) \) defines \((a_j)\). Whence we have

\[
|S(n, f, a, b) - S(p, f, a, b)| = \frac{b - a}{np} \left| \sum_{j=0}^{np} f(a_j) - f(b_j) \right| \leq (b - a)2^{-m}
\]

because, as \( j \in \{ip, ip + p - 1\} \cap \{in, in + n - 1\} \), we have \( ip - kn \leq n - 1 \) and \( kn - ip \leq p - 1 \), whence: \( |kn - ip| \leq \max(n, p) \) and

\[
|a_j - b_j| = (b - a) \left| \frac{i}{n} - \frac{k}{p} \right|
\]

\[
\leq (b - a) \frac{\max(n, p)}{\max(n, p) \min(n, p)}
\]

\[
\leq \frac{b - a}{\varepsilon([b - a] + 1)}
\]

\[
\leq \frac{1}{\varepsilon(m)}.
\]

\((S(n, f, a, b))_n \) is a \( \Sigma_p \)-Cauchy sequence, thus \( \Sigma_p \)-converges. \( \square \)

Definition 4.21. We write \( \int_a^b f := \lim_n S(n, f, a, b) \).
Lemma 4.22. The integral is a linear positive form, verifying the Chasles’s relation and $|\int_a^b f| \leq \int_a^b |f|$.

Proof. Properties of the limit process. □

Our notions of continuity, differentiability, allow to recover an analogue of the Fundamental Theorem of Analysis. The main difference is that we have to check the sequence $\int_{u_k}^a f$ is $\Sigma_p$-definable if $(u_k)$ is.

Lemma 4.23. If $f \in \Sigma_p C^0([a,b])$, then the function $F(x) := \int_a^x f$ is $\Sigma_p C^1$ on $[a,b]$, with $F' = f$.

Proof.
– Let us first see that $F \in \Sigma_p C^0([a,b])$:

$$|F(x) - F(y)| = \left| \int_y^x f \right| \leq \sup_{[a,b]} |f| |x - y|.$$  

The upper bound exists because $f \in \Sigma_p C^0([a,b])$.

Let $K$ be such that $\sup_{[a,b]} |f| \leq 2^K$; we then take $\beta(k) := \alpha(k + K)$ as a modulus of continuity, where $\alpha$ is the one of $f$. Besides, $F$ is sequentially $\Sigma_p$-definable: let $(u_k) \in \Sigma_p (A^+)$; set: $x'_k := a + i(u_k - a)/n$ then $(x'_k) \in \Sigma_p (A^+ \times A^+)$ and $S(n, f, a, u_k) := \sum_{k=0}^{n-1} f(x'_k)(u_k - a)/n$; clearly, $S(n, f, a, u_k) \in \Sigma_p (A^+ \times A^+)$, i.e. there exists $(r_{p,n,k}) \in \Sigma_p (A^+ A^+)$ which $\Sigma_p$-converges to $S(n, f, a, u_k)$:

$$\forall p, n, k \ |r_{p,n,k} - S(n, f, a, u_k)| < \varepsilon_p$$

then $(r_{p,n,k}) \in \Sigma_p$-converges to $F(u_k)$.

– By hypothesis $f \in \Sigma_p C^0([a,b])$.

– At least

$$|F(x) - F(y) - f(x)(x - y)| = \left| \int_y^x f(t) dt - \int_y^x f(x) dt \right|$$

$$= \left| \int_y^x (f(t) - f(x)) dt \right|$$

$$\leq 2^{-n} |y - x|$$

if $|x - y| < 1/\alpha(m)$. □

4.4.1. Logarithm

Definition 4.24. We set $x := \int_1^x \frac{dt}{t}$.

According to the theorems on the integral, we see that the function log is $\Sigma_p C^1$ on every interval $[a,b]$ with $0 < a$ and $(\log)' = 1/x$. Thus $\exp \circ \log$ is too, and $(\exp \circ \log)' =
1. According to the mean value theorem, \( \exp \circ \log(x) = x \). By this way, we find again the algebraic properties of \( \log \).

**Lemma 4.25.** \( \forall x \in ]-1,1[ \) \( \log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n/n \).

**Proof.** \( x \mapsto \log(1+x) \) is \( \Sigma_p^C \) on every interval \([a,b]\) with \(-1 < a < b < 1\). It is also the case for the series. But these two functions vanish at 0 and have the same derivative, thus they are equal. \( \square \)

### 4.5. Piecewise \( \Sigma_p^C^0 \) functions

**Definition 4.26.** A function \( f : [0, \infty] \rightarrow \Sigma_p^C^0 \) is said to be \((\Sigma_p^C\)-definably) piecewise \( \Sigma_p^C^0 \) if and only if there exists a \( \Sigma_p^C\)-definable sequence \( (x_i) \) such that

(i) \( \forall i \in \mathbb{N}^* \lim_{x \to x_i, x < x_i} f(x) \) and \( \lim_{x \to x_i, x > x_i} f(x) \) exist.

(ii) The functions \( f_i \) are \( \Sigma_p^C^0 \) on \([x_i, x_{i+1}]\) where \( f_i \) is defined by

\[
f_i(x) := \begin{cases} f(x) & \text{if } x \in [x_i, x_{i+1}] ; \\
\lim_{x \to x_i, x > x_i} f(x) & \text{if } x = x_i , \\
\lim_{x \to x_i+1, x < x_i+1} f(x) & \text{if } x = x_{i+1}.
\end{cases}
\]

(iii) For every sequence \( (u_{n,i}) \in \Sigma_p(\mathbb{R}^* \times \mathbb{R}^*) \) such that \( \forall n \ u_{n,i} \in [x_i, x_{i+1}] \), the sequence \( (f_i(u_{n,i}))_{n,i} \) is \( \Sigma_p^C\)-definable.

**Example.** Let \((a_n) \in \Sigma_p(\mathbb{R}^*)\). Then the function \( f(x) := \sum_{n \leq x} a_n \) is piecewise \( \Sigma_p^C^0 \). Indeed \( x_i = i; \ f_i(x) = \sum_{n=0}^{i} a_n =: b_i \) with \( (b_i) \in \Sigma_p^C(\mathbb{R}^*) \) and \( f_i(u_{n,i}) = b_i \).

**Definition 4.27.** If \( f \) is a piecewise \( \Sigma_p^C^0 \) function, we set

\[
\int_a^b f := \sum_{x_i \leq a} \int_{x_i}^{x_{i+1}} f_i + \int_{x_i}^{b} f_n.
\]

We must check that the sum is allowed: it is the case because the Riemann’s sum: \( S(n, f_i, x_i, x_{i+1}) = \sum_{k=0}^{n-1} f_i(u_{k,i})(x_i - x_{i+1})/n \), which forms a \( \Sigma_p\)-definable double sequence, \( \Sigma_p\)-converges to

\[
\int_{x_i}^{x_{i+1}} f_i,
\]

which is therefore a \( \Sigma_p\)-definable sequence.

**Lemma 4.28** (Abel’s formula). Let \((a_n) \in \Sigma_p(\mathbb{R}^*)\) and \( \phi \in \Sigma_p^C^1([0, +\infty]) \); we set \( A(x) := \sum_{n \leq x} a_n \); then

\[
\sum_{n \leq x} a_n \phi(n) = A(x)\phi(x) - \int_1^x a(u)\phi'(u) \, du.
\]

**Proof.** We notice that \((\phi(n))_n \in \Sigma_p(\mathbb{R}^*)\) and that \( A.\phi' \) is piecewise \( \Sigma_p^C^0 \).
\[
\sum_{n \leq x} a_n (\phi(x) - \phi(n)) = \sum_{n \leq x} a_n \int_n^x \phi'(u) \, du
\]
\[
= \int_1^x \sum_{n \leq u} a_n \phi'(u) \, du = \int_1^x a(u) \phi'(u) \, du.
\]

5. \(\Sigma_p\)-definable sequences of functions

5.1. Definitions

The language of algebras of functions allows to talk about sequences of functions. But, as usual, we shall restrict ourselves to definable sequences to obtain interesting results.

Definition 5.1. A sequence \((f_n)\) of functions from an interval \([a, b]\) into \(\Sigma_p R\) is said to be \(\Sigma_p\)-definable if:

(i) for every \(\Sigma_p\)-definable “real” sequence \((u_k), (f_n(u_k))_{n,k}\) is a double \(\Sigma_p\)-definable sequence;

(ii) there exists a modulus of continuity for each function of the sequence, which depends on \(n\) in a \(\Sigma_p\)-definable way.

More exactly, we set the following predicate:

\[
(f_n) \in \Sigma_p(\mathcal{R}^{[a,b]} \times \mathcal{A}_c) \iff (f_n) \in \mathcal{R}^{\mathcal{A}_c} \land \forall u \in \mathcal{S} [a \leq u \leq b \rightarrow \exists v \in \mathcal{S} \\
\forall n \in \mathcal{N} \, \text{eval}(v, n) = \text{eval}(f_n, u)] \land \\
\exists x \in \Sigma_p(\mathcal{N}^{\times 2}) \forall n, k \in \mathcal{N} \forall x, y \in [a, b] \\
|x - y| < \frac{1}{z(n, k)} \rightarrow |f_n(x) - f_n(y)| < 2^{-k}.
\]

We also need an adequate notion of convergence for this type of sequences:

Definition 5.2. A \(\Sigma_p\)-definable sequence \((f_n)\) of functions is said to be \(\Sigma_p\)-convergent to \(f\) on an interval \(I\) if there exists a \(\Sigma_p\)-modulus of convergence \(\varepsilon_n\) such that

\[
\forall x \in I \, |f_n(x) - f_n(x)| < \varepsilon_n.
\]

5.2. Links with continuity and differentiability

Now we can prove an analogue of the classical result: “the limit of a uniformly convergent sequence of continuous functions is continuous”. Taking constant functions shows that the requirement of \(\Sigma_p\)-convergence is necessary.
Lemma 5.3 (Closure of the $\Sigma_p$-continuity under the $\Sigma_p$-convergence). Let $(f_n)$ be a $\Sigma_p$-definable sequence of functions $\Sigma_p$-converging on $[a, b]$ to a function $f$; then $f \in \Sigma_p C^0([a, b])$.

Proof. (i) $f$ is sequentially $\Sigma_p$-definable: let $(u_k) \in \Sigma_p(\mathbb{R}^k)$; then $(f_n(u_k)) \in \Sigma_p(\mathbb{R}^k)$, i.e. there exists $(r_{n,k,l}) \in \Sigma_p(\mathbb{R}^{3})$ such that: $\forall n,k,l |r_{n,k,l} - f_n(u_k)| < 2^{-p}$.

Besides, if $(e_n)$ is the $\Sigma_p$-modulus of convergence: $\forall k |f_n(u_k) - f(u_k)| < e_n$. Set $a(l) := \mu ne_n < 2^{-l-1};$ then we have

$$|r_{a(l),k,l+1} - f(u_k)| \leq 2^{-(l+1)} + e_{a(l)} < 2^{-(l+1)} + 2^{-(l+1)} = 2^{-l}$$

and $s_{k,l} := r_{a(l),k,l+1}$ is $\Sigma_p$-definable.

(ii) Let $\alpha'$ be the modulus of continuity of the sequence $(f_n)$. We set $\beta(n) := \mu \alpha_k < 2^{-n/3}$ and $\alpha(n) := \alpha'(\beta(n), n+2);$ then

$$|x - y| < \frac{1}{\alpha(n)} \rightarrow |f(x) - f(y)| \leq |f(x) - f_{\beta(n)}(x)| + |f_{\beta(n)}(x) - f_{\beta(n)}(y)| + |f_{\beta(n)}(y) - f(y)|$$

$$\leq \epsilon_{\beta(n)} + 2^{-n+2} + \epsilon_{\beta(n)}$$

$$\leq 2 \cdot \frac{2^{-n}}{3} + \frac{2^{-n}}{4}$$

$$< 2^{-n}. \Box$$

Corollary 5.4. A $\Sigma_p$-definable power series is $\Sigma_p C^0$ on every closed disc of radius $r$ with $r < R$, where $R$ is the radius of $\Sigma_p$-convergence.

Proof. We set $f_n(x) := \sum_{i=0}^{n} a_i x^i$. $(f_n)$ is a $\Sigma_p$-definable sequence: (i) has already been seen; let us prove (ii):

$$\forall x, y \in [-r, r]|f_n(x) - f_n(y)| \leq \sum_{i=0}^{n} |a_i| |x^i - y^i| \leq \sum_{i=0}^{n} |a_i||x|^{i-1}|x - y|.$$ 

Then $\alpha(n, k) := 2^k \sum_{i=0}^{n} E(\alpha_i)(\alpha(|r|) + 2)^{i-1}$ is a modulus of continuity of $(f_n)$. At least, $\forall x \in [-r, r]|f_n(x) - f(x)| \leq \sum_{i=n+1}^{\infty} |a_i||x|^i \leq \sum_{i=n+1}^{\infty} |a_i|r^i$ the last series being $\Sigma_p$-convergent. $\Box$

Application. exp, sin, cos, arctan... are $\Sigma_p C^0$.

Lemma 5.5. Let $\sum_{n=0}^{\infty} a_nx^n$ be a $\Sigma_p$-definable power series which is $\Sigma_p$-convergent on $]-R, R[$. Then it is $\Sigma_p C^0$ on every interval $[-r, r]$ with $r < R$, its derivative being given by $\sum_{n=1}^{\infty} na_n x^{n-1}$.

Proof. Let $r < R$ and $K$ be such that $r + 2^{-K} < R$. We notice that $\sum_{i=0}^{\infty} i a_i x^i$ is sequentially $\Sigma_p$-definable on $]-R, R[$ too. Thus, let $(e_n)$ and $(e'_n)$ be the respective $\Sigma_p$-moduli of convergence of $\sum_{i=0}^{\infty} a_i i r^{i-1}$ and $\sum_{i=0}^{\infty} a_i i (r + 2^{-K})^{i-1}$.
Recall that we have
\[ \forall x, y \in [-r, r] |x - y| < 2^{-K} \rightarrow |y' - x'| \leq i(|x| + 2^{-K})^{-1}|y - x| \]
\[ \leq i(r + 2^{-K})^{-1}|y - x|. \]
Furthermore, notice that the modulus of differentiability of \( x \mapsto x' \) depends in a \( \Sigma_p \)-definable way on \( i \): indeed \( a(i, k):=((i(i - 1)/2)((r + 2)^i + 1)/2^k \) (see the previous lemma).

Let \( A = \left| \sum_{i=0}^{\infty} a_i y^i - \sum_{i=0}^{\infty} a_i x^i - \sum_{i=0}^{\infty} i a_i x^{i-1} (y - x) \right| \)
\[ \leq \sum_{i=0}^{\infty} a_i (y^i - x^i - ix^{i-1}(y - x)) + \sum_{i=n+1}^{\infty} a_i (y^i - x^i) + \sum_{i=n+1}^{\infty} i a_i x^{i-1} \cdot |y - x| \]
\[ \leq B + \sum_{i=n+1}^{\infty} a_i (r + 2^{-K})^i \cdot |y - x| + C \]
\[ |y - x| < \frac{1}{\max_{0 \leq a(i, k)} a(i, k)} \rightarrow B \leq \sum_{i=0}^{n} a_i \left( \frac{i(i - 1)}{2} \right) (r + 2^K)^{-2}) |y - x|^2. \]

Let \( K' \) be such that \( 2^K' \geq \sum_{i=0}^{\infty} (a(i(i - 1)/2)(r + 2^{-K})^{-2}). \)

Then the modulus of differentiability of the series will be: \( \beta(k):=\max(2^K, 2^{K'+k+1}, \gamma(k)) \) where \( \gamma(k):=\max(a(i, k); i = 0 \ldots \delta(k + 1)) \) with \( \delta(k):=\mu n(\epsilon_n + \epsilon_n < 2^{-k}) \). Then \( A \leq (2^K |y - x| + \epsilon_n + \epsilon_n) |y - x|. \)

5.3. Inversion of limits

In this section we shall consider the problem of inverting limits. Recall that a “finite” sum of \( \Sigma_p \)-reals is already a limit process: a formula involving such a finite sum cannot be proved by direct induction. However, dealing with moduli of \( \Sigma_p \)-convergence, we obtain strong enough results for our needs.

Lemma 5.6. Let \( (f_i) \) be a \( \Sigma_p \)-definable sequence of functions from \([a, b]\) into \( \Sigma_p \mathcal{R} \). Then \( \sum_{i=0}^{n} f_i \) is integrable and \( \int_{a}^{b} \sum_{i=0}^{n} f_i = \sum_{i=0}^{n} \int_{a}^{b} f_i. \)

Proof. Let \( S(k, f_i) \) be the Riemann’s sum associated with \( f_i \); then
\[ \sum_{i=0}^{n} S(k, f_i) = \sum_{i=0}^{n} \sum_{j=0}^{k} f_i(x_j) \frac{b-a}{k} = \sum_{i=0}^{k} \sum_{j=0}^{n} f_i(x_j) \frac{b-a}{k} = S \left( k, \sum_{i=0}^{n} f_i \right). \]

The sums are well defined because \((f_i(x_j))_{i,j} \in \Sigma_p(\mathcal{R}^{+}) \). We thus infer that
\[ \int_{a}^{b} \sum_{i=0}^{n} f_i = \lim_{k} S \left( k, \sum_{i=0}^{n} f_i \right) = \lim_{k} \sum_{i=0}^{n} S(k, f_i) \].
\[ \lim_{k \to \infty} S(k, f_i) = \sum_{i=0}^{n} \int_{a}^{b} f_i. \]

**Definition 5.7.**
- Let \( f \) be a \( \Sigma^0_p \) function on every interval \([a, x]\), \( x \in [a, \infty) \), we say that the infinite integral \( \int_{a}^{\infty} f \) \( \Sigma^0_p \)-converges if and only if there exists \( \beta \in \Sigma^0_p (A^{<\infty}) \) such that
\[ \forall n \, \forall x |x| > \beta(n) \to \left| \int_{x}^{\infty} f \right| < 2^{-n}. \]
- If \((f_i)\) is a \( \Sigma^0_p \)-definable sequence of functions, the same definition holds with, moreover, \( \beta \) depending on \( i \).

**Lemma 5.8.** Let \( f \) and \( g \) be two \( \Sigma^0_p \) functions on every interval \([a, x]\) such that \( 0 \leq f \leq g \). Then if \( \int_{a}^{\infty} g \) \( \Sigma^0_p \)-converges it is the same for \( \int_{a}^{\infty} f \).

**Proof.** The same function \( \beta \) is suitable.

**Theorem 5.9.** Let \((f_i)\) be a \( \Sigma^0_p \)-definable sequence of functions such that \((\int_{a}^{\infty} f_i)_i\) is a sequence of \( \Sigma^0_p \)-converging integrals; then
\[ \forall n \in A^{<\infty} \sum_{i=0}^{n} \int_{a}^{\infty} f_i = \int_{a}^{\infty} \sum_{i=0}^{n} f_i. \]

**Proof.** We shall first check that \((\int_{a}^{\infty} f_i)_i\) is a \( \Sigma^0_p \)-definable sequence of reals, to be able to talk about the sum. Set \( F_i(x) := \int_{a}^{x} f_i \); then \((F_i)\) is a \( \Sigma^0_p \)-definable sequence of functions on \([a, +\infty]\). In particular, \((F_i(k))_{i,k} \in \Sigma^0_p (A^{<\infty} \times A^{<\infty})\), i.e. there exists \((r_{i,k,l}) \in \Sigma^0_p (A^{<\infty} \times A^{<\infty})\) such that
\[ \forall i, k, l |F_i(k) - r_{i,k,l}| < 2^{-l}. \]

Besides, set \( b_i := \int_{a}^{\infty} f_i \); we have
\[ \forall i \, \forall m \, \forall k < \beta(m, i) \to |F_i(k) - b_i| < 2^{-m}. \]

Set \( s_{i,m} := r_{i,\beta(i,m+1),m+1}; \) \((s_{i,m})_{i,m} \in \Sigma^0_p (A^{<\infty} \times A^{<\infty})\).
\[ \forall i \, \forall m |s_{i,m} - b_i| \leq |s_{i,m} - F_i(\beta(i,m+1))| + |F_i(\beta(i,m+1)) - b_i| < 2^{-(m+1)} + 2^{-(m+1)} = 2^{-m}. \]

Thus \((b_i) \in \Sigma^0_p (A^{<\infty})\) and \( \sum_{i=0}^{n} b_i \) exists.

Notice that if the first limit exists, we have \( \lim_{m} \sum_{i=0}^{n} F_i(\beta(m,1,i)) = \lim_{k} \sum_{i=0}^{n} F_i(k) \) because if \( k \geq \beta(m+1,i) \) then
\[ \left| \sum_{i=0}^{n} F_i(k) - \sum_{i=0}^{n} F_i(\beta(m+1,i)) \right| \leq \sum_{i=0}^{n} |F_i(k) - b_i| + \sum_{i=0}^{n} b_i - F_i(\beta(m+1,i)) \]
\[ \forall m \left| \sum_{i=0}^{n} b_i - \sum_{i=0}^{n} F_i(\beta(m + 1)) \right| \leq \left| \sum_{i=0}^{n} b_i - \sum_{i=0}^{n} s_{i,m} \right| \\
+ \left| \sum_{i=0}^{n} s_{i,m} - \sum_{i=0}^{n} r_{i,\beta(m+1),m+1} \right| \\
+ \left| \sum_{i=0}^{n} r_{i,\beta(m+1),m+1} - \sum_{i=0}^{n} F_i(\beta(m + 1, i)) \right| \leq n.2^{-m} + 0 + n.2^{-(m+1)} \]

thus \( \lim k \sum_{i=0}^{n} F_i(k) = \lim m \sum_{i=0}^{n} F_i(\beta(m + 1, i)) = \sum_{i=0}^{n} b_i \) i.e. \( \lim k \sum_{i=0}^{n} \int_a^k f_i = \int_a^k \sum_{i=0}^{n} f_i \)

whence we get

\[ \int_a^\infty \sum_{i=0}^{n} f_i = \sum_{i=0}^{n} \int_a^\infty f_i. \]

Lemma 5.10. Let \((f_i)\) be a \(\Sigma_p\)-definable sequence of functions on \([a, \infty[\) such that

\( \forall i \forall t \in [a, \infty[ |f_i(t)| \leq \chi_i g(t), \)

where \( g \) is a function such that \( \int_a^\infty g \) \(\Sigma_p\)-converges and \((\chi_i)\) is the general term of a \(\Sigma_p\)-convergent series. Then \( \sum_{i=0}^{\infty} \int_a^\infty f_i = \int_a^\infty \sum_{i=0}^{\infty} f_i \).

Proof. \((\chi_i)\) \(\Sigma_p\)-converges to 0, thus \( \int_a^\infty |f_i| \) \(\Sigma_p\)-converges. Whence

\[ \sum_{i=0}^{n} \int_a^\infty f_i = \int_a^\infty \sum_{i=0}^{n} f_i. \]

Besides \( \sum_{i=n+1}^{\infty} |f_i(t)| \leq (\sum_{i=n+1}^{\infty} \chi_i) g(t) \) thus \( \int_a^\infty \sum_{i=n+1}^{\infty} |f_i| \) \(\Sigma_p\)-converges.

At least, \( \forall i \int_a^\infty |f_i| \leq \chi_i \int_a^\infty g \), thus \( \sum_{i=0}^{\infty} \int_a^\infty |f_i| \) is a \(\Sigma_p\)-convergent series. So

\[ \left| \int_a^\infty \sum_{i=0}^{n} f_i - \sum_{i=0}^{\infty} \int_a^\infty f_i \right| \leq \left| \int_a^\infty \sum_{i=n+1}^{\infty} f_i \right| + \left| \int_a^\infty \sum_{i=0}^{n} f_i - \sum_{i=0}^{\infty} \int_a^\infty f_i \right| \\
+ \left| \sum_{i=0}^{n} \int_a^\infty f_i - \sum_{i=0}^{\infty} \int_a^\infty f_i \right| \leq \left| \int_a^\infty \sum_{i=n+1}^{\infty} |f_i| \right| + 0 + \sum_{i=n+1}^{\infty} \int_a^\infty |f_i| \]
\[
\int_a^\infty g \rightarrow 0 \text{ when } n \to 0;
\]

thus \(\int_a^\infty \sum_{i=0}^\infty f_i = \sum_{i=0}^\infty \int_a^\infty f_i\).

6. Complex extensions

We can easily modify the previous theories so that to obtain analogues to the complex field \(\mathbb{C}\), just make suitable changes in the language.

6.1. \(\Sigma_p\)-subinductive complex fields

Definition 6.1. The language of inductive complex fields is the language \(L(\text{ICF}) := (i, +, \sigma, N)\) where \(\sigma\) is a unary function and \(i\) a constant.

The standard structure of this language is \((\mathbb{C}, i, +, \sigma, \mathbb{N})\), where \(\bar{x}\) is the conjugate of \(x\). We define the following predicates:

- \(x\) is a real \(\mathcal{R}(x) \leftrightarrow x = \sigma(x)\)
- order on \(\mathcal{R} \ x \leq y \leftrightarrow \mathcal{R}(x) \land \mathcal{R}(y) \land \exists t(\mathcal{R}(t) \land y = x + t)\).

The following theory is similar to the theory of \(\Sigma_p\)-subinductive fields, but its standard model is \((\mathbb{C}, \mathbb{N})\):

Definition 6.2. We call theory of \(\Sigma_p\)-subinductive complex fields the theory with language \(L(\text{ICF})\) and with the following axioms:
1. \((\mathcal{K}, +, \cdot)\) is a field;
2. \(i.i = -1\);
3. \(\sigma\) is an involutive automorphism:
   \[\forall x, y[\sigma(x + y) = \sigma(x) + \sigma(y) \land \sigma(x.y) = \sigma(x)\sigma(y) \land \sigma(\sigma(x)) = x]\]
4. \(\leq\) is a linear order on \(\mathcal{R}\);
5. \(-\mathcal{N}'(0)\)
   \[-\forall x (\mathcal{N}'(x) \to \mathcal{N}'(x + 1))\);
6. \(\mathcal{R}\) is archimedean: \(\forall x (\mathcal{R}(x) \to \exists n(\mathcal{N}'(n) \land x \leq n))\).
7. For every \(\Sigma_p\)-formula \(\Phi\) of \(L(\text{ICF})\) such that all the variables are relativised to \(\mathcal{N}'\):
   \[
   \phi(0, \bar{y}) \land \mathcal{N}'(\bar{y}) \quad \forall x [\mathcal{N}'(x) \land \mathcal{N}'(\bar{y}) \land \Phi(x, \bar{y}) \to \Phi(x + 1, \bar{y})]
   \]
   \[\forall x \mathcal{N}'(x) \to \Phi(x, \bar{y})\]

Note: \((\mathbb{Q}(i), \mathbb{N})\) is also a model of this theory.
Theorem 6.3. Let \((\mathcal{C}, i, +, \sigma, \mathcal{N})\) be a \(\Sigma_p\)-sub-inductive complex field; then
(a) \(\forall x \, \mathcal{N}(x) \rightarrow \mathcal{R}(x)\);
(b) \((\mathcal{R}, +, \leq, \mathcal{N})\) is a \(\Sigma_p\)-subinductive field.

Proof. \(\sigma\) being a morphism of fields, we see that \((\mathcal{R}, +, \cdot)\) is a field. We easily check that \(\leq\) is compatible with \(\cdot\) and \(+\). At least, in the formulae on which the schema of induction is allowed, all the variables being relativised to \(\mathcal{N}\), the function \(\sigma\) disappears and we deal in fact with formulae of the language L(IF).

Lemma 6.4. Let \(\mathcal{C}\) be a preinductive complex field. Then \(\sigma(i) = -i\) and \(\forall z \, \exists! x \, \exists! y (\mathcal{R}(x) \land \mathcal{R}(y) \land z = x + iy)\).

Proof. \(z^2 = -1 \iff z^2 - i^2 = 0 \iff (z - i)(z + i) = 0 \iff (z = i \lor z = -i)\). Besides, \((\sigma(i))^2 = \sigma(1) = -1\). But if \(i \in \mathcal{R}\), \(i^2 \geq 0\) and \(-1 < 0\), which is a contradiction; thus \(i \in \mathcal{R}\) and \(\sigma(i) = -i\). We have: \(\forall z \, z = z + \sigma(z)/2 + z - \sigma(z)/2\). But \(z + \sigma(z)/2 \in \mathcal{R}\) and \(z - \sigma(z)/2 \in \mathcal{R}\). At least, if \(z = x + iy\), then \(\sigma(z) = x - iy\) and \(x = z + \sigma(z)/2\), whence the unicity.

Theorem 6.5. The theory of \(\Sigma_p\)-subinductive complex fields is a conservative extension of the theory of \(\Sigma_p\)-subinductive fields and thus of \(\text{RI}_1\).

Proof. Let \((\mathcal{R}, +, \cdot, \mathcal{N})\) be a \(\Sigma_p\)-subinductive field. We consider \(\mathcal{C} := \mathcal{R}^2\) equipped with the operations:
- \((a, b) + (a', b') := (a + a', b + b')\)
- \((a, b) \cdot (a', b') := (a a' - b b', a' b + a b')\)
- \(\sigma((a, b)) := (a, -b)\)
- \(i := (0, 1)\).

We check that \((\mathcal{C}, i, +, \cdot, \mathcal{N})\) is a \(\Sigma_p\)-subinductive complex field.

6.2. \(\Sigma_p\)-complex algebras of sequences

We now consider the quadratic extension \((x^2 = -1)\) of a \(\Sigma_p\)-real field:

Definition 6.6. The language of complex algebras of sequences is the language \(L(\text{CAS}) := \{i, +, \cdot, \sigma, \mathcal{C}, \mathcal{N}\}\), where \(\mathcal{C}\) is a unary predicate.

We define the predicate “to be a real sequence” by: \(\mathcal{S}(u) := \sigma(u) = u\).

Definition 6.7. We call theory of algebras of \(\Sigma_p\)-complex sequences the first-order theory with language \(L(\text{CAS})\) and with the following axioms:
1. \((\mathcal{S}, +, \cdot, \sigma, \mathcal{C}, \mathcal{N})\) is an algebra of \(\Sigma_p\)-real sequences;
2. \(\forall u \exists v, w (\mathcal{S}(v) \land \mathcal{S}(w) \land u = v + i w)\).

If \((\mathcal{S}, +, \cdot, \sigma, \mathcal{C}, \mathcal{N})\) is a model of this theory, we call the field \((\mathcal{C}, i, +, \cdot)\) a \(\Sigma_p\)-complex field.
In the case $p = 1$, the standard model is $(\text{Rec}(\mathbb{C}^\mathbb{N}), \text{Rec}(\mathbb{C}), \mathbb{N})$, whereas $(\mathbb{C}^\mathbb{N}, \mathbb{C}, \mathbb{N})$ is not, for reasons of cardinality.

**Theorem 6.8.** The theory of algebras of $\Sigma_p$-complex sequences is a conservative extension of the theory of algebras of $\Sigma_p$-real sequences and thus of $\text{RI} \Sigma_p$.

**Proof.** If $(\mathcal{A}, \ast, +, \cdot, \text{eval}, \mathcal{C}, \mathcal{N})$ is an algebra of $\Sigma_p$-real sequences, we construct an algebra of $\Sigma_p$-complex sequences by setting: $\mathcal{A}_{\mathcal{C}} := \mathcal{A}^2$ which we equip with operations $+,, \sigma$ with the same definitions as to go from $\mathcal{A}$ to $\mathcal{C}$. □

### 6.3. Complex algebras of functions

**Definition 6.9.** The language of complex algebras of functions is the language $\text{L}(\text{CAF}) := \{i, +,, \cdot, \text{eval}, \text{eval}, \mathcal{C}, \mathcal{N}\}$ where $\mathcal{C}$ is a unary predicate.

**Definition 6.10.** We call theory of $\Sigma_p$-complex algebras of functions the first-order theory with language $\text{L}(\text{CAF})$ and with the following axioms:

1. $(\mathcal{A}, i, +,, \cdot, \text{eval}, \sigma, \mathcal{C}, \mathcal{N})$ is an algebra of $\Sigma_p$-complex sequences.
2. Compatibility of the evaluation:

\[
\forall f, g, x \quad [\text{eval}(f + g, x) = \text{eval}(f, x) + \text{eval}(g, x) \land \text{eval}(f \cdot g, x) = \text{eval}(f, x) \cdot \text{eval}(g, x) \land \text{eval}(\sigma(f), x) = \sigma(\text{eval}(f, x))].
\]

**Theorem 6.11.** The theory of $\Sigma_p$-complex algebras of functions is a conservative extension of the theory of algebras of $\Sigma_p$-complex sequences and thus of $\text{RI} \Sigma_p$.

**Proof.** The same as in the $\Sigma_p$-real case. □

We can define again the notion of $\Sigma_p$-continuity and find the same properties as in a $\Sigma_p$-real field:

**Definition 6.12.** A function is said to be $\Sigma_p$-continuous on a closed disc $\tilde{D}(a, R)$ iff:

\[
f \in \Sigma_p \text{C}^0(\tilde{D}(a, R)) \rightarrow \exists a, b \in \mathcal{C} \quad \{\forall u \in \mathcal{C} \quad [u \in \tilde{D}(a, R) \rightarrow \exists v \in \mathcal{C} \quad v = \text{eval}(f, u)] \land \exists z \in \Sigma_p(\mathcal{A}^{-1}) \forall n \in \mathcal{N} \forall x, y \in \mathcal{C} \cap \tilde{D}(a, R) \quad |x - y| < \frac{1}{\text{eval}(z, n)} \rightarrow |f(x) - f(y)| < 2^{-n}.
\]

### 6.4. Complex analysis

The notions of $\Sigma_p$-convergence, of $\Sigma_p$-definable continuity, . . . in the complex case are analogous to the real case. In particular, the series which $\Sigma_p$-converged on
It remains to develop a theory of analytical functions, which we handle in this section. Of course, we will set, as for differentiability, an adapted definition of holomorphic functions. But all the interesting functions will satisfy these axioms. Moreover, following quite closely the classical proofs, we are able to recover elementary results on analytical functions. However, Cauchy’s theorem will require further adaptions.

6.4.1. $\Sigma_p$-holomorphic functions

**Definition 6.13.** We call representable open set every subset $\mathcal{O}$ of $\mathbb{C}$ such that:

1. there exists a formula $\Phi(x)$ of $\mathcal{L}(\mathbb{C})$ such that $\Phi(x)$ if and only if $x \in \mathcal{O}$;
2. $\forall x(\Phi(x) \rightarrow \exists a \in \mathbb{C}, \exists R \in \mathbb{R}((\forall t \in \tilde{D}(a,R) \rightarrow \Phi(t))))$.

This notion is introduced to be able to talk about open subsets in the language of inductive fields; but when proving a statement involving such an open set, we in fact always consider a disc in it.

**Definition 6.14.** The predicate “to be $\Sigma_p$-holomorphic on $F$”, where $F$ is a closed disc or rectangle, is defined by

$$f \in \Sigma_p H(F) \leftrightarrow f \in \Sigma_p C^0(F) \land \exists g \in \Sigma_p C^0(F) \left[ \exists x \in \Sigma_p \mathcal{N} \forall n \in \mathbb{N}, \forall x,y \in F \mid x - y \mid < \frac{1}{2(a(n))} \rightarrow |f(x) - f(y) - g(y)(x - y)| < 2^{-n} \right].$$

The predicate “to be $\Sigma_p$-holomorphic on the representable open set $\Omega$” is defined by

$$f \in \Sigma_p H(\Omega) \leftrightarrow \forall a \in \mathbb{C} \forall R \in \mathcal{R}(\forall x \in \tilde{D}(a,R) \Phi_d(x)) \rightarrow f \in \Sigma_p H(\tilde{D}(a,R))).$$

**Lemma 6.15.** (a) The following functions are $\Sigma_p$-holomorphic on $\Sigma_p \mathcal{C}: z \mapsto z^n$ for every $n \in \mathbb{N}$; on $\Sigma_p \mathcal{C} \setminus \{0\}: z \mapsto 1/z$.

(b) If $f$ and $g$ are two $\Sigma_p$-holomorphic functions on $\Omega$, then $f + g$, $f \cdot g$, $f \circ g$ are too.

**Proof.** The same as for the $\Sigma_p C^1$ functions. □

**Lemma 6.16.** Let $f(z) := \sum_{n=0}^{\infty} a_n (z-a)^n$ be a $\Sigma_p$-definable power series which $\Sigma_p$-converges on the disc $D(a,R)$. Then $f \in \Sigma_p H(D)$ and $f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$, which is a series which $\Sigma_p$-converges on $D(a,R)$.

**Proof.** According to what we have seen on the series, $f$ is $\Sigma_p C^0$ on $D(a,R)$ as well the function $g(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$ (because it is also $\Sigma_p$-convergent on $D(a,R)$). It remains to prove that the derivative of $f$ is $g$. Let $r \in R$ and $z,w \in \tilde{D}(a,r)$.

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} a_n \left( \frac{z^n - w^n}{z - w} - n w^{n-1} \right).$$
Besides we have

$$\forall n \geq 1 \left| \frac{z^n - w^n}{z - w} - nw^{n-1} \right| \leq |z - w| \sum_{k=1}^{n-1} kr^{n-2} = |z - w| \frac{n(n - 1)}{2} r^{n-2}$$

$$\leq |z - w| n^2 r^{n-2}.$$  

$r < R$ thus $\sum_{n=2}^{\infty} |a_n| n^2 r^{n-2}$ $\Sigma_p$-converges; set $M := \left[ \sum_{n=2}^{\infty} |a_n| n^2 r^{n-2} \right] + 1.$

Taking $\alpha(n) := 2^n M$, we get

$$|z - w| < \frac{1}{\alpha(n)} \rightarrow \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} |a_n| n^2 r^{n-2} \leq 2^{-n}. \quad \Box$$

**Lemma 6.17.** Let $g$ and $h$ be two complex-valued $\Sigma_p \mathbb{C}^0$ functions on $[a, b]$, such that the intersection of the image of $h$ with $\Omega$ is empty, where $\Omega$ is an open set of $\Sigma_p \mathcal{E}$. Then the function $f$ defined by: $f(z) := \int_a^b g(t)/(h(t) - z) \, dt$ can be expanded in a power series: more exactly, for every disc $D(a, r) \subset \Omega$, there exists a $\Sigma_p$-definable $\Sigma_p$-convergent series such that: $\forall z \in D(a, r)$ $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$.

**Proof.** Let $\tilde{D}(a, R) \subset \Omega$. There exists $r' > r$ such that $\tilde{D}(a, r') \subset D(a, r) \subset \Omega$. Then $\forall t \in [a, b] \, |h(t) - a| > r'$ and $|(z - a)/(h(t) - a)| \leq r/r' < 1$ for $z \in \tilde{D}(a, r)$.

Thus the series

$$\sum_{n=0}^{\infty} \frac{(z - a)^n}{(h(t) - a)^{n+1}}$$

uniformly $\Sigma_p$-converges (relative to the variable $t$). But

$$\sum_{n=0}^{\infty} \frac{(z - a)^n}{(h(t) - a)^{n+1}} = \frac{1}{h(t) - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{h(t) - a} \right)^n = \frac{1}{h(t) - a} \frac{h(t) - a}{h(t) - a - (z - a)}$$

$$= \frac{1}{h(t) - z}.$$

Besides, $h$ being $\Sigma_p \mathbb{C}^0$ on $[a, b]$, so is $H(t) := (z - a)/(h(t) - a)$; thus there exists $\alpha \in \Sigma_p(\mathbb{N}^{<k})$ such that

$$|t - t'| < \frac{1}{\alpha(k)} \rightarrow |H(t) - H(t')| < 2^{-k}.$$  

Consider the sequence of functions: $H_n(t) := (H(t))^n$. Then

$$|H_n(t) - H_n(t')| = |(H(t))^n - (H(t'))^n| \leq |H(t) - H(t')| \cdot n \cdot (|H(t)| + 1)^{n-1}$$

$$< |H(t) - H(t')| \cdot n \cdot 2^{n-1} \text{ for } |H(t)| < 1.$$  

Let $\beta(n) := \mu k 2^k > n \cdot 2^{n-1}$; then $\gamma(n, k) := \alpha(\beta(n) + k)$ gives the modulus of continuity which proves that the sequence is $\Sigma_p$-definable. The two necessary conditions being
fulfilled to exchange the integral and the series, we get
\[ f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{with} \quad a_n = \int_a^b \frac{g(t)}{(h(t)-a)^{n+1}} \, dt. \]

6.4.2. Integration on definable paths

Our next step in the recovering of classical analytical tools is to define complex integration. Looking at the usual definition, we see that we have to impose obvious definability conditions on “paths”.

**Definition 6.18.** A definable path, parametrized by \([a, b]\) with \(a, b \in \Sigma_p \Re\), is an application \(\gamma: [a, b] \to \Sigma_p \Re^0\) on \([a, b]\) and piecewise \(\Sigma_p C^1\), i.e. there exists \((x_i) \in \Sigma_p (\Re^+)\) such that \(\forall i \in \Sigma_p C^1([x_i, x_{i+1}])\).

If \(f\) is a \(\Sigma_p C^0\) function on the image of \(\gamma\), we write
\[
\int_{\gamma} f := \int_a^b f(\gamma(t)) \gamma'(t) \, dt = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} f(\gamma(t)) \gamma'(t) \, dt.
\]

**Lemma 6.19.** If \(\gamma_1\) and \(\gamma_2\) are two definable paths, parametrized respectively by \([a, b]\) and \([c, d]\), such that \(\gamma(b) = \gamma(c)\); then \(\gamma := \gamma_1 \cup \gamma_2\) is a definable path and \(\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f\).

Following a classical proof, we have:

**Lemma 6.20** (and definition of the winding number). Let \(\gamma\) be a closed definable path, \(\Omega\) the complementary of its image. Let us define the winding number of \(z\) relative to \(\gamma\) as the function \(\text{Ind}_{\gamma}(z) := (1/2\pi i) \int_{\gamma} ds/(s-z)\) for \(z \in \Omega\). Then \(\text{Ind}_{\gamma}\) is an \(\mathcal{N}\)-valued function.

**Proof.** Let \((x_i)_{0 \leq i \leq n}\) the sequence of points of discontinuity of \(\gamma'\).

For every \(k \in \{0, \ldots, n\}\) we set \(h_k(t) := \exp(\int_a^t \gamma'(s)/(\gamma(s) - z) \, ds)\) for \(t \in [x_i, x_{i+1}]\); thus \(h_k \in \Sigma_p C^1([x_i, x_{i+1}])\).

We differentiate and get
\[
\frac{h'_k(t)}{h_k(t)} = \frac{\gamma'(t)}{\gamma(t) - z}.
\]

Let \(H_k(t) := h_k(t)/(\gamma(t) - z)\) for \(t \in [x_i, x_{i+1}]\); we have \(H_k \in \Sigma_p C^1([x_i, x_{i+1}])\).

\[
H'_k(t) = \frac{h'_k(t)}{\gamma(t) - z} - \frac{h_k(t) \gamma'(t)}{(\gamma(t) - z)^2} = 0 \quad \text{thus} \quad H_k = \text{constant}.
\]

But \(\forall k \in \{0, \ldots, n\}\) \(H_k(x_i) = H_{k+1}(x_i)\) thus the constant is the same.

Besides, \(H_0(a) = h_0(a)/(\gamma(a) - z) = 1/(\gamma(a) - z)\) thus \(\forall k, \forall t \in [x_i, x_{i+1}] H_k(t) = 1/(\gamma(a) - z)\).
The path $\gamma$ being closed, $\gamma(a) = \gamma(b)$; whence $H_t(b) = h_t(b)/(\gamma(b) - z) = h_t(b)/(\gamma(a) - z) = 1/(\gamma(a) - z)$. Thus $h_t(b) = 1$, and $\exp\left(\int_a^b \gamma'(s)/(\gamma(s) - z) \, ds\right) = 1$ whence we get $\int_a^b \gamma'(s)/(\gamma(s) - z) \, ds = 2i\pi$ with $p \in \mathcal{D}$.

**Lemma 6.21.** Let $F \in \Sigma_p H(\Omega)$. Then $\int_{\gamma} F'(z) \, dz = 0$ for every closed definable path in $\Omega$.

**Proof.**

\[
\int_{\gamma} F'(z) \, dz = \int_a^b F'(\gamma(t)) \, dt = \sum_{i=0}^n \int_{x_i}^{x_{i+1}} (F \circ \gamma)'(t) \, dt
\]
\[
= \sum_{i=0}^n F(\gamma(x_{i+1})) - F(\gamma(x_i))
\]
\[
= F(\gamma(b)) - F(\gamma(a)) = 0.
\]

The calculus is possible because $F \circ \gamma \in \Sigma_p C^1([x_i, x_{i+1}])$ and $F \circ \gamma \in \Sigma_p C^0([a, b])$ thus $(F(\gamma(x_i)))_i \in \Sigma_p (\mathcal{R}^1)$. 

**Corollary 6.22.** \forall n \in \mathcal{D} \setminus \{-1\} \int_{\gamma} z^n \, dz = 0 for every closed definable path if $n \geq 0$ and for the closed definable paths such that their image does not include 0 if $n \leq -2$.

**Proof.** $z^n$ is the derivative of $z^{n+1}/(n + 1)$ for $n \in \mathcal{D} \setminus \{-1\}$.

We are now in a position to prove an adaptation of Cauchy’s theorem. But to handle the main induction, we will have to use a “trick” from Aberth [1].

**Lemma 6.23** (Cauchy’s theorem for a triangle). Let $\Delta$ be a triangle included in a representable open set $\Omega$. Let $f \in \Sigma_p C^0(\Omega)$ such that $f \in \Sigma_p H(\Omega \setminus \{p\})$ where $p$ is a point of $\Omega$. Then $\int_{\partial \Delta} f = 0$.

**Proof.** Stage 1: reduction of the problem.

- If $p$ is a point of $\Delta$, we come down to the case where $p$ is one of the vertices by considering the three triangles: $\{a, b, p\}$, $\{b, c, p\}$, $\{c, a, p\}$.
- If $p$ is a vertex, for example $a$: Let $x \in [a, b]$ and $y \in [a, c]$; then

\[
\left| \int_{axy} f \right| \leq \int_0^1 |f((1 - t)a + tx)| \, dt \cdot |a - x| + \int_0^1 |f((1 - t)x + ty)| \, dt \cdot |x - y|
\]
\[
+ \int_0^1 |f((1 - t)y + a)| \, dt \cdot |y - a|
\]
\[
\leq M(|a - x| + |x - y| + |y - a|)
\]

where $M$ is a bound of $f$ (which exists because $f$ is $\Sigma_p C^0$).

Hence $\lim_{x \to a, y \to a} \int_{axy} f = 0$ and the result is known for the triangles $\{a, x, y\}$ and $\{x, b, y\}$. We can then suppose that $p$ does not belong to $\Delta$. 

If \((a_n), (b_n), (c_n)\) are \(\Sigma_p\)-definable sequences of \(\mathcal{P}[i]\) converging respectively to \(a, b, c\), then \(\int_{a_n} f \to \int_a f\). Indeed, let \(\alpha\) be the modulus of continuity of \(f\) and let \(K(n) := n_k \alpha(n) < 2^{-k}\), we then have

\[
\forall n \forall k > K(n) \forall t \in [0, 1] |(1 - t)(a_k - a) + t(b_k - b)| \leq |a_k - a| + |b_k - b| < 2^{-k + 2^{-k}} = 2^{-k - 1} < \frac{1}{\alpha(n)};
\]

thus

\[
\left| \int_{[a_n, b_n]} f - \int_{[a, b]} f \right| \leq \int_0^1 |f((1 - t)a_k + tb_k) - f((1 - t)a + tb)| dt(|b - a| + 1) \leq 2^{-n}(|b - a| + 1).
\]

**Stage 2**: We prove it for a triangle with rational vertex.

Let \(\{a, b, c\}\) be a triangle of \(\mathcal{P}[^\sqrt{-1}]\). We define the three double sequences \((a_n, i), (b_n, i), (c_n, i)\) by

\[
a_{0, i} := a,
\]

\[
\forall i \in [1, 4^{n+1}]a_{n+1, i} := \begin{cases} a_{n, j} & \text{if } i = 4(j - 1) + 1, \\
\frac{1}{2}(a_{n, j} + b_{n, j}) & \text{if } i = 4(j - 1) + 2, \\
\frac{1}{2}(b_{n, j} + c_{n, j}) & \text{if } i = 4(j - 1) + 3, \\
\frac{1}{2}(c_{n, j} + a_{n, j}) & \text{if } i = 4(j - 1) + 4.
\end{cases}
\]

The same holds for \((b_n, i)\) and \((c_n, i)\).

These are \(\Sigma_p\)-definable sequences (it is a definition by induction on rationals).

We write \(T_{n, i}\) the definable path \([a_{n, i}, b_{n, i}, c_{n, i}, a_{n, i}]\) and set

\[
J_{n, i} := \left| \int_{T_{n, i}} f \right|.
\]

Then \(J_{n, i} \in \Sigma_p(\mathcal{R}^{\times 4})\), i.e. there exists \((r_{n, i, l}) \in \Sigma_p(\mathcal{R}^{\times 4})\) such that \(|r_{n, i, l} - J_{n, i}| < 2^{-l}\).

Set

\[
e := \left| \int_{T_{0, 1}} f \right|, \quad e_n := e \cdot 4^{-n} (1 + 2^{-n}),
\]

\((e_n) \in \Sigma_p(\mathcal{R}^{\times 4})\) and let \((s_{n, i}) \in \Sigma_p(\mathcal{R}^{\times 4})\) such that \(|s_{n, i} - e_n| < 2^{-l}\). We have

\[
e_{n+1} < \frac{e_n}{4}.
\]

Suppose that \(e > 0\) (to get a contradiction).
Let \( \Phi(n,k,K) \) be the formula
\[
(k > 4^n \land K = 0) \lor \left\{ k \leq 4^n \land 4(k-1) + 1 \leq K \leq 4(k-1) + 4 \land J_{n,K} > e_{n+1} \land \forall j \leq 4(k-1) + 4j < K \rightarrow J_{n,j} < \frac{e_n}{4} \right\}.
\]

It is a \( \Sigma_p \)-formula including only rationals, because we can express the strict inequalities between reals by
\[
J_{n,K} > e_{n+1} \leftrightarrow \exists l,l'(r_{n,K,j} - 2^{-l} > s_{n,j'} + 2^{-l'}).
\]

We check that it is functional in \( n \) and \( k \), i.e. \( \forall n \forall k \exists ! K \Phi(n,k,K) \). We prove by induction on \( n \) that
\[
\forall n \forall k \leq 4^n \exists i \in A_k \left( \left( J_{n,i} > e_{n+1} \land \forall j \leq 4(k-1) + 4j < i \rightarrow J_{n,j} < \frac{e_n}{4} \right) \right)
\]
where we have set \( A_k := [4(k-1) + 1, 4(k-1) + 4] \).

The case \( n = 0 \) is obvious, suppose that it is true for \( n - 1 \).

Suppose that
\[
\exists k \leq 4^n \forall i \in A_k \left( J_{n,i} \leq e_{n+1} \lor \exists j \in A_k j < i \land J_{n,j} \geq \frac{e_n}{4} \right).
\]

Let \( k' - 1 \) and \( \varepsilon \) be the quotient and the rest of the euclidean division of \( k \) by 4; then \( k' \leq 4^{n-1} \) and we apply the hypothesis of induction which gives an \( i_0 \) such that \( J_{n-1,i_0} > e_n \).

But
\[
J_{n-1,j_0} \leq \sum_{j \in A_{k'}} J_{n,j} \leq 4e_{n+1} < \frac{4e_n}{4} = e_n
\]
which is a contradiction; hence there exists an \( i \) such that \( J_{n,i} \leq e_{n+1} \). Besides if \( \exists j < i \land J_{n,j} > \frac{4}{4}e_n > e_{n+1} \), the lowest \( j \) will be suitable (there are only four to try). The property is thus true for \( n \).

Hence, we can define by induction the sequence \( (k_n): k_0 := 1, \Phi(n,k_n,k_{n+1}) \). We can also \( \Sigma_p \)-define: \( a_n := a_{n,k_n}, b_n := b_{n,k_n}, c_n := c_{n,k_n} \).

Let \( \delta := [\max(|a - b|,|b - c|,|a - c|)] + 1 \) (it is a bound of the length of each side of \( A \)). We prove by induction: \( \forall n,m |a_n - a_m| \leq 2^{-\min(n,m)} \delta \).

The sequence \( (a_n) \) is thus of rapid Cauchy and converges to \( z_0 \in A \). Let \( r \) be such that \( \tilde{D}(z_0,r) \subset \Omega \). \( f \) being \( \Sigma_p \)-holomorphic, let \( \alpha \in \Sigma_p(\mathbb{N}^{\leq 4}) \) its modulus of differentiability on \( \tilde{D}(z_0,r) \), i.e.
\[
|z - z_0| < \frac{1}{\alpha(k)} \rightarrow |f(z) - f(z_0) - f'(z_0)(z - z_0)| < 2^{-k}|z - z_0|.
\]

Let \( T_n \subset \tilde{D}(z_0,r) \) then \( \int_{T_n} f = \int_{T_n} f(z) - f(z_0) - f'(z_0)(z - z_0) \, dz \) because \( \int_{T_n} dz = 0 \) and \( \int_{T_n} z \, dz = 0 \).

Thus \( |\int_{T_n} f| \leq 2^{-k}|z - z_0|2^{-n}L \) with \( L := |a - b| + |b - c| + |a - c| \).
That is
\[ e^{4^n (1 + \frac{1}{2}2^{-n})} = e_{n+1} \leq 2^{-k} \cdot 2^{-n} \cdot L. \]

Whence:
\[ e \leq 2^{-k} \cdot \delta L \cdot \frac{8}{1 + 2^{-n+1}} \]

and this, for every \( n \) and \( k \), thus \( e = 0 \). \( \square \)

**Lemma 6.24** (Cauchy’s theorem in a convex set). Let \( \Omega \) be a representable convex open set, \( p \) a point of \( \Omega \), \( f \in \Sigma_p \mathcal{C}^0(\Omega) \), and \( f \in \Sigma_p \mathcal{H}(\Omega \setminus \{p\}) \). We have \( \int_\gamma f = 0 \) for every closed definable path in \( \Omega \).

**Proof.** We fix \( a \in \Omega \); then \( \forall z \in \Omega[a, z] \subset \Omega \). We can thus set: \( F(z) := \int_{[a, z]} f F \in \Sigma_p \mathcal{C}^0(\Omega) \).

\( \forall z, \omega \in \Omega \) the triangle \( \{a, z, \omega\} \) is included in \( \Omega \). We can apply the previous theorem to find
\[ \frac{F(z) - F(\omega)}{z - \omega} - f(\omega) = \int_{[\omega, z]} [f(s) - f(\omega)] \, ds \cdot \frac{1}{z - \omega}. \]

Let \( \alpha \in \Sigma_p (\mathcal{N}^{-1}) \) be the modulus of continuity of \( f \):
\[ \forall s, \omega |s - \omega| < \frac{1}{\alpha(N)} \rightarrow |f(s) - f(\omega)| < 2^{-n} \]

whence \( \forall z, \omega |z - \omega| < 1/2(N) \rightarrow |F(z) - F(\omega) - f(\omega)(z - \omega)| < 2^{-n} |\omega - z| \); thus \( F \in \Sigma_p \mathcal{H}(\Omega) \) with \( F' = f \). But we have seen that \( \int_\gamma F' = 0 \) for every closed definable path \( \gamma \). \( \square \)

**Definition 6.25.** (a) Let \( p \) be a point of \( \Omega \), and \( f \in \Sigma_p \mathcal{H}(\Omega \setminus \{p\}) \); if there exists a function \( \tilde{f} \in \Sigma_p \mathcal{H}(\Omega) \) equal to \( f \) on \( \Omega \setminus \{p\} \) we say that \( f \) has an artificial singularity in \( p \).

(b) If there exist \( (u_n) \in \Sigma_p(\mathcal{H}^{-1}) \) and \( m \in \mathcal{N} \) with \( u_m \neq 0 \), such that
\[ f(z) = \sum_{k=1}^{m} \frac{u_k}{(z - p)^k} \]

has an artificial singularity in \( p \), we say that \( f \) has a pole of order \( m \) in \( p \). \( \text{Res}(f; p) := u_1 \) is called the residue of \( f \) in \( p \).

**Lemma 6.26** (Theorem of residues). Let \( \Omega \) be a representable convex open set, \( (a_k)_{k \leq n} \in \Sigma_p(\mathcal{H}^{-1}) \) of points of \( \Omega \) and \( f \in \Sigma_p \mathcal{H}(\Omega \setminus \{a_1, \ldots, a_n\}) \) having a pole in each point \( a_k \). If \( \gamma \) is a closed definable path in \( \Omega \) not including the \( a_k \), we have
\[ \frac{1}{2\pi i} \int_\gamma f(z) \, dz = \sum_{k=1}^{n} \text{Res}(f; a_k) \cdot \text{Ind}_\gamma(a_k). \]
Proof. Let $Q_k$ be the principal part of $f$ in $a_k$; this gives a $\Sigma_p$-definable sequence of functions (since $(a_k) \in \Sigma_p(\mathcal{A}^v)$).

$$f - (Q_1 + \cdots + Q_n)$$

having only artificial singularities in $\Omega$, we can apply to it the Cauchy theorem:

$$\int_{\gamma} f = \int_{\gamma} \sum_{k=1}^{n} Q_k.$$ 

We can invert the sum and the integral and find the result. □

6.4.3. Functions defined by an integral

We state the following theorem in order to obtain a very useful corollary for number theory.

Theorem 6.27. Let $f : \Sigma_p(\mathcal{E}) \times \Sigma_p(\mathcal{A}) \rightarrow \Sigma_p(\mathcal{E})$ be sequentially $\Sigma_p$-definable such that for every $z \in \Omega$, the integral $F(z) := \int_{0}^{\infty} f(z,t) dt$ $\Sigma_p$-definably converges.

(1) If for every disc $D(a,r)$, there exists $\alpha \in \Sigma_p(\mathcal{A}^v)$ and $g$ an integrable function such that

$$\forall n \forall z,z' \in D |z - z'| < \frac{1}{\alpha(n)} \rightarrow \forall t |f(z,t) - f(z',t)| < 2^{-n}g(t)$$

then $F \in \Sigma_p H(\Omega)$.

(2) If, furthermore,

(i) the same properties are true for $G(z) := \int_{1}^{\infty} \hat{f}(z,t) dt$;

(ii) for every disc $D(a,r)$, there exists $\beta \in \Sigma_p(\mathcal{A}^v)$ and $h$ an integrable function such that

$$\forall n \forall z, \omega \in D |z - \omega| < \frac{1}{\alpha(n)} \rightarrow \forall t \left| f(z,t) - f(\omega,t) - (z - \omega) \frac{\partial f}{\partial \omega}(\omega,t) \right| \leq 2^{-n}|z - \omega|h(t).$$

Then $F \in \Sigma_p H(\Omega)$.

Proof. (1) (a) the modulus of continuity of $F$ will be: $\alpha'(n) := \alpha(n + k)$ where $k$ is such that $\lceil \int_{1}^{\infty} g \rceil \leq 2^k$.

(b) $F$ is sequentially $\Sigma_p$-definable: If we let $(z_k) \in \Sigma_p(\mathcal{E}^v)$, then we have

$$\int_{1}^{\infty} f(z_k,t) dt = \lim_{B \rightarrow \infty} \int_{1}^{B} f(z_k,t) dt = \lim_{B \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(z_k,x_i) \frac{B - 1}{n}.$$ 

Let $(r_k,B,n,j) \in \Sigma_p(\mathcal{A}^{v^3})$ approaching the sum with $2^{-j}$ as error. Then

$$\forall B \forall j |r_k,B,j + 2,j + 2 - \int_{1}^{B} f(z_k,t) dt| < 2^{-j+2} + 2^{-j+2} = 2^{-j+1}.$$
If $\beta$ is the modulus of $\Sigma_p$-definable convergence of the integral, then:

$$\left| r_k, \beta(j+1), j+2, j+2 - \int_1^\infty f(z_k, t) \, dt \right| < 2^{-j}.$$ 

(2) From property (i), we get $G \in \Sigma_p C^0(\Omega)$. Thanks to (ii), we have the modulus of differentiability $\beta'(n) := \beta(n + k)$ where $k$ is such that $[\int_1^\infty h] \leq 2^k$.

**Corollary 6.28.** Let $u$ be a piecewise $\Sigma_p C^0$ function, such that $u(t) = O(1)$; then the function $F: \Sigma_p$-holomorphic on the half-plane $\Re(z) > 1$.

**Proof.** Set $f(z, t) := u(t), t^{-z} = u(t)e^{-z \log t}$. $f$ is sequentially $\Sigma_p$-definable.

Let $\tilde{D}(a, r) \subset \{\Re(z) > 1\}$, i.e. such that $b := \Re(a) - r > 1$; let $K$ be such that $1 + 2^{-K} < b$ and $\varepsilon := 2^{-K}$. Then $\forall t \in [1, \infty] \forall z, z' |z - z'| < \varepsilon$,

$$|f(z, t) - f(z', t)| \leq |u(t)||e^{z \log t}||e^{-(z-z') \log t} - 1|$$

$$\leq M t^{-\Re(z')} \sum_{n=1}^{\infty} \frac{(\log t)^n}{n!} |z - z'|^n$$

$$\leq M t^{-b} \frac{|z - z'|}{\varepsilon} \sum_{n=1}^{\infty} \frac{(\log t)^n}{n!} e^n$$

$$\leq M t^{-b} \frac{|z - z'|}{\varepsilon} (e^t - 1).$$

Let $g(t) := (M/\varepsilon)(t^{-b} - t^{-b})$: $g$ is integrable because it is $\Sigma_p C^0$ and, because $b > 1$, $b - \varepsilon = b - 2^{-K} > 1$; which proves properties (1) and (2)(i) because $\partial f / \partial z = (- \log t). t^{-z}$ and $g(t). \log t$ remains integrable. Property (2)(ii) is proved in the same way.

7. The prime number theorem

7.1. Arithmetical functions

From Section 4.5, we see that the following arithmetical functions are piecewise $\Sigma_1 C^0$:

$$\pi(x) = \sum_{p \leq x} 1,$$

$$\Pi(x) = \sum_{p \leq x} \frac{1}{\log p},$$

$$\theta(x) = \sum_{p \leq x} \log p,$$

$$\psi(x) = \sum_{n \leq x} A(n).$$
With Abel’s formula, we can show elementary results (say, up to Tchebychev’s theorems) about these functions. We have also that (the proof goes through PRA without any problem), if one of the limits exists, then

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = \lim_{x \to \infty} \frac{\psi(x)}{x}.$$  

7.2. An elementary statement equivalent to the prime number theorem

To get it, we will need the following analytical result, which is easily adapted in the theory of $\Sigma_p$-real algebras of functions:

**Lemma 7.1.**

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+1},$$

$$\forall x > 0 \quad \left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}.$$  

We now introduce the following arithmetical formula, due to Matiassevitch [9], and which represents a kind of logarithmic function:

$$\explog(a, b) \leftrightarrow \exists x \left[ x > b + 1 \land (1 + x)^b x \leq (a + 1)x^{b+1} < 4(1 + x)^{xb+1} \right].$$

More exactly, we show within the theory of $\Sigma_p$-real algebras of functions:

**Lemma 7.2.** (1) $\explog(a, b) \rightarrow |b - \log(a + 1)| < 2$;
(2) $\forall a \exists b \explog(a, b)$.

**Proof.** Raising the inequality of the proposition to the power of $b$, we get

$$\left(1 + \frac{1}{x}\right)^{xb} < e^b < \left(1 + \frac{1}{x}\right)^{(x+1)b}$$

whence

$$\frac{e^b}{e^b} = \left(1 + \frac{1}{x}\right)^{(x+1)b} \leq \left(1 + \frac{1}{x}\right)^{xb} \leq a + 1 < 4 \left(1 + \frac{1}{x}\right)^{xb} \leq 4e^b$$

and, taking the logarithm, we have the result.

Besides, choosing $b := \lfloor \log(a + 1) \rfloor$, and $x$ great enough for $(1 + 1/x)^{x+1}$ and $(1 + 1/x)^{x+1}$ to be sufficiently close to $e$, we have the second formula. □

The following theorem gives a purely arithmetic statement equivalent to the prime number theorem.
Theorem 7.3. The theory of \( \Sigma_p \)-real algebras of functions proves the following equivalence:

\[
\psi(n) \sim n \iff \neg \{ \exists k \forall N \exists n > N \exists m \exists b \left[ m = \text{pcm}(1, \ldots, n) \land \explog(m - 1, b) \land k^2(b - n)^2 > (n - 2k)^2 \right] \}.
\]

Proof. We call APNT the right side formula:

\[
\begin{align*}
\text{APNT} & \rightarrow \neg \left\{ \exists k \forall N \exists n > N \exists m \exists b \left[ m = \text{pcm}(1, \ldots, n) \land \explog(m - 1, b) \land |b - n| > \frac{n}{k} - 2 \right] \} \\
& \rightarrow \neg \left\{ \exists k \forall N \exists n > N \exists m \exists b \left[ m = \text{pcm}(1, \ldots, n) \land |b - \log m| > -2 \land |b - n| > \frac{n}{k} - 2 \right] \} \\
& \quad \rightarrow \neg \left\{ \exists k \forall N \exists n > N \exists m \exists b \left[ m = \text{pcm}(1, \ldots, n) \land |n - \log m| > \frac{n}{k} - 4 \right] \} \\
& \quad \quad \rightarrow \neg \left\{ \exists k \forall N \exists n > N \left[ |n - \psi(n)| > \frac{n}{k} - 4 \right] \} \\
& \quad \quad \quad \rightarrow \forall k \forall N \forall n > N \left( \left| \frac{\psi(n)}{n} - 1 \right| < \frac{1}{k} - \frac{4}{n} < \frac{1}{k} \right) \\
& \quad \quad \quad \quad \rightarrow \forall k \forall N \forall n > N |\psi(n) - n| < \frac{n}{k} \\
& \quad \quad \quad \quad \quad \rightarrow \psi(n) \sim n.
\end{align*}
\]

On the other hand, and using again the first implications:

\[
\neg \text{APNT} \rightarrow \exists k \forall N \exists n > N \left[ |n - \psi(n)| > \frac{n}{k} - 4 \right] \\
\rightarrow \exists k \forall N \exists n > N \left[ \left| 1 - \frac{\psi(n)}{n} \right| > \frac{1}{k} - \frac{4}{n} > \frac{1}{2k} \right] \\
\rightarrow \neg \text{PNT}
\]

with \( n \) large enough. □
7.3. Proof of the prime number theorem

We shall follow very closely the classical proof in the version of [6] (pp. 42–55), and translate it in the framework of “$\Sigma_\varphi$-real algebras of functions”, which gives the result since this theory is a conservative extension of PRA. In fact, we shall only point out differences in the statement of the lemmas and the proof of a few of them.

7.3.1. The Riemann $\zeta$ function

We write $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$; the sequence $(n^{-s})_n$ is $\Sigma_\varphi$-definable and the series $\Sigma_\varphi$-converges for $|s| > 1$.

Lemma 7.4. There exists a function $f$ $\Sigma_\varphi$-holomorphic on the half-plane $\text{Re}(s)>0$, but at $s=1$, such that $\forall s \ |s| > 1 f(s) = \zeta(s)$. The point $s=1$ is a simple pole, with residue 1.

Proof. For $\text{Re}(s)>1$, we have, by Abel’s formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = s \int_1^{\infty} \frac{[u]}{u^{s+1}} \, du$$

(with $a_n = 1$, $\phi(x) = 1/x^2$: $\phi$ is $\Sigma_\varphi C^0$ on $[1, \infty[).$ We infer that

$$\zeta(s) = s \int_1^{\infty} \frac{u - \{u\}}{u^{s+1}} \, du = \frac{s}{s + 1} - s \int_1^{\infty} \frac{\{u\}}{u^{s+1}} \, du =: f(s).$$

According to Corollary 6.28, the integral defines a $\Sigma_\varphi$-holomorphic function on $\text{Re}(s+1)>1$ that is $\text{Re}(s)>0$. The singularities of $f$ are the same as $1/(s - 1)$. □

NB: $f$ is still written as $\zeta$.

Lemma 7.5. $\zeta$ has no zero on $\text{Re}(s) \geq 1$.

Proof. If $\text{Re}(s)>1$ then $\zeta(s) = \prod_p (1 - p^{-s})^{-1} \neq 0$.

For $\sigma>1$ $\log \zeta(\sigma + it) = - \sum_p \log(1 - p^{-\sigma - it})$

$$= \sum_p \sum_{n=1}^{\infty} \frac{1}{n} p^{-n(\sigma + it)}.$$

Taking the real part:

$$\log |\zeta(\sigma + it)| = \sum_p \sum_{n=1}^{\infty} \frac{1}{n} p^{-n\sigma} \cos(nt \log p).$$

We see that $3 + 4 \cos \phi + \cos 2\phi = 2(1 + \cos \phi)^2 \geq 0$. Whence $3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| \geq 0$ Hence, $|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$
7.3.2. Bounds for $\zeta, \zeta'$, $Z$

We shall state several results which are needed for the final proofs, but we shall prove with details only one of them.

**Lemma 7.6.** Let $0 \in ]0, 1[$. For every $\varepsilon > 0$ and every $t$ such that $|t| \geq 1$, we have

(a) $|\zeta(\sigma + it)| \leq \frac{2}{\sqrt{3}} |t|^{1-\theta} / \theta (1-\theta)$

(b) $|\zeta'(\sigma + it)| \leq |t|^{1-\theta} / \theta (1-\theta) (\log |t| + 1/\theta + \frac{5}{4})$.

**Proof.** (a) We can suppose $t > 0$ because $|\zeta(\sigma + it)| = |\zeta(\sigma - it)|$.

By Abel’s formula, we have

$$\sum_{n \leq x} n^{-s} = [x]x^{-s} + s \int_{1}^{x} [u]u^{-s-1} du$$

$$= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} - \{x\}x^{-s} - s \int_{1}^{x} \{u\}u^{-1-s} du.$$  

From the formula which gives analytical continuation of $\zeta$, we get

$$\zeta(s) - \sum_{n \leq x} n^{-s} = -\frac{x^{1-s}}{1-s} + \{x\}x^{-s} - s \int_{x}^{\infty} \{u\}u^{-1-s} du.$$  

In particular, if $s = \theta + it$ where $\alpha > 0$ and $t \geq 1$, we have

$$|\zeta(s)| \leq \sum_{n \leq x} |n^{-s}| + \left| \frac{x^{1-s}}{1-s} \right| + \left| \{x\} \right| x^{\theta} + |s| \int_{x}^{\infty} \left| \frac{\{u\}}{u^{\theta+1}} \right| du$$  

whence

$$|\zeta(s)| \leq \sum_{n \leq x} n^{-\theta} + \frac{x^{1-\theta}}{t} + x^{-\theta} + |s| \int_{x}^{\infty} \frac{du}{u^{\theta+1}}.$$  

We notice that $|s|/\sigma \leq (\sigma + t)/\sigma = 1 + t/\sigma \leq 1 + t/\theta = (\theta + t)/\theta$, and clearly with $s = \theta$ we have

$$\sum_{n \leq x} n^{-\theta} \leq \frac{x^{1-\theta}}{1-\theta}$$  

whence

$$|\zeta(s)| \leq \frac{x^{1-\theta}}{1-\theta} + \frac{x^{1-\theta}}{t} + x^{-\theta} + \left( 1 + \frac{t}{\theta} \right) x^{-\theta}.$$
Thus if $x = t \geq 1$,
\[ |\zeta(s)| \leq \frac{t^{1-\theta}}{\theta(1-\theta)} \left( 1 + \frac{3\theta(1-\theta)}{t} \right) \leq \frac{7}{4} \frac{t^{1-\theta}}{\theta(1-\theta)}. \]

(b) The same as (a). □

**Lemma 7.7.** There exist $c_1, c_2, c_3, c_4 \in \Sigma_\rho \Re$ such that for all $\sigma \geq 1$ and all $t$ such that $|t| \geq 8$, we have
(a) $|\zeta(\sigma + it)| \leq c_1 \log |t|$; (b) $|\zeta'(\sigma + it)| \leq c_2 (\log |t|)^2$;
(c) $|\zeta(\sigma + it)^{-1}| \leq c_3 (\log |t|)^7$; (d) $Z(\sigma + it) \leq c_4 (\log |t|)^9$.

### 7.3.3. Proof of the theorem

Set
\[ E(x) := \begin{cases} 0 & \text{if } x \leq 1, \\ x - 1 & \text{if } x > 1. \end{cases} \]

This is a piecewise $\Sigma_\rho C^0$ function.

**Lemma 7.8.** (a) For $\Re(s) > 0$, we have
\[ \frac{1}{s(s+1)} = \int_0^\infty \frac{E(x)}{x^{s+2}} \, dx. \]

(b) If $c > 0$ is a fixed real number, then for every $x \geq 0$, we have
\[ E(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \, ds. \]

**Proof.** The first equality is obvious. For the second one, we distinguish two cases:

**Case 1:** $x \geq 1$. The singularities of the function $f(s) := x^{s+1}/s(s+1)$ are simple poles at $s = 0$ and $s = -1$ with respective residues $x$ and $-1$. We consider the closed definable path made up of the arc of the circle $\gamma$ centred in 0 with radius $\sqrt{c^2 + T^2}$ ($T > 1$), and of the segment $[c - iT, c + iT]$. The theorem of residues gives
\[ \int_{c-i\infty}^{c+i\infty} \frac{x^{1+s}}{s(s+1)} \, ds + \int_{\gamma} \frac{x^{1+s}}{s(s+1)} \, ds = 2\pi i (x - 1). \]

When $T$ tends towards infinity, the second integral tends towards 0 because
\[ \left| \int_{\gamma} \frac{x^{s+1}}{s(s+1)} \, ds \right| \leq \frac{2\pi R x^{1+c}}{R(R-1)} \to 0 \]

hence $x - 1 = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} x^{1+s}/s(s+1) \, ds$.

**Case 2:** $0 \leq x < 1$. Just consider the complementary definable path of the previous one. □
Lemma 7.9. (a) In the half-plane \( \text{Re}(s) > 1 \), we have

\[
Z(s) = s(s + 1) \int_1^\infty \frac{\psi_1(x)}{x^{s+2}} \, dx.
\]

(b) For every \( x \geq 1 \) and for every \( \Sigma_p \)-real number \( c > 1 \) we have

\[
\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{Z(s)}{s(s+1)} x^{s+1} \, ds.
\]

Proof. Let us prove (b): let \( c > 1 \) be a fixed \( \Sigma_p \)-real. By definition,

\[
\int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^{1+s}}{s(s+1)} \, ds = \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} A(n)n^{-s} \frac{x^{s+1}}{s(s+1)} \, ds.
\]

Set \( f_n(t) := A(n)n^{-c-it}x^{1+it}/(c+it)(c+1+it) \); then \( \forall t \in \Sigma_p \mathbb{R} | f_n(t) | \leq (A(n)/n^c)(x^{c+1}/c(c+1)) \) the right member being the general term of a \( \Sigma_p \)-definable \( \Sigma_p \)-convergent series. We can thus exchange the integral and the sum

\[
\int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^{1+s}}{s(s+1)} \, ds = \sum_{n=1}^{\infty} nA(n) \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^{s+1}}{s(s+1)} \, ds
\]

\[
= 2\pi i \sum_{n=1}^{\infty} nA(n)E \left( \frac{X}{n} \right)
\]

\[
= 2\pi i \sum_{n \leq x} A(n)(x-n).
\]

But by Abel’s formula, we have \( \sum_{n \leq x} A(n)(x-n) = \psi_1(x) \). \( \square \)

Lemma 7.10. When \( x \) goes to the infinite in \( \Sigma_p \mathbb{R} \), we have \( \psi_1(x) \sim \frac{1}{2} x^2 \).

Theorem 7.11 (Prime number theorem). \( \psi(x) \sim x \) with \( x \in \Sigma_p \mathbb{R} \).

Proof. Set \( \Psi(x) := \varepsilon x \) with \( \varepsilon \in [0, \frac{1}{2}] \)

\[
\Psi \text{ being increasing, we have }
\]

\[
\frac{1}{h} \int_{x-h}^{x} \psi(u) \, du \leq \psi(x) \leq \frac{1}{h} \int_{x}^{x+h} \psi(u) \, du
\]

whence

\[
\frac{\psi_1(x) - \psi_1(x-h)}{h} \leq \psi(x) \leq \frac{\psi_1(x+h) - \psi_1(x)}{h}.
\]

According to the previous theorem

\[
\forall \gamma \exists x_0 > x_0 \rightarrow \left| \psi_1(x) - \frac{x^2}{2} \right| < \gamma \frac{x^2}{2}
\]

\[
\left| \psi_1(x-h) - \frac{(x-h)^2}{2} \right| < \gamma \frac{(x-h)^2}{2}.
\]
Adding these two inequalities, replacing \( h \) by its value and choosing \( \gamma := \varepsilon^2 \), we get

\[
\forall \varepsilon \exists x : x > 1 - \varepsilon - \varepsilon^3 \leq \frac{\psi(x)}{x} \leq 1 + \frac{\varepsilon}{2} + \varepsilon^2 (2 + \varepsilon) + \varepsilon. \quad \square
\]

8. Conclusion

The framework developed here shows how to prove in PRA a lot of classical results from analytical number theory. However, it is very likely that there are some which are not (recall that PRA is not complete) although for the moment no such example is known. Perhaps algebraic number theory could provide such an unprovable theorem because of the use of very powerful algebraic facts.

References