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Spaces of operators with the Riesz separation property*

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1. INTRODUCTION

The space of *regular* operators between two Banach lattices E and F, $\mathcal{L}_r(E,F)$, is the linear span of the cone of positive operators ordered by that cone. It is rare that this space is a lattice – one of the few cases known dating back to [7], being when F is Dedekind complete. Little attention has been paid in the literature to weaker order theoretic properties of $\mathcal{L}_r(E,F)$, which it might be hoped would be true rather more often.

One property worthy of consideration is the *Riesz separation property* (we abbreviate this to RSP in future) which states that if $x_1, x_2 \leq z_1, z_2$ then there is y with $x_1, x_2 \leq y \leq z_1, z_2$. A simple induction argument shows that if a space has the RSP then it also satisfies the corresponding property for n- and m-tuples: if $x_1, \ldots, x_n \leq z_1, \ldots, z_m$, then there is y with $x_1, \ldots, x_n \leq y \leq z_1, \ldots, z_m$. The Riesz separation property is sometimes called the finite interpolation property. It is clearly satisfied by lattices, but also by some other ordered vector spaces. It is well-known that an ordered vector space E satisfies the RSP if and only if E has the Riesz decomposition property: if $0 \leq x_i, y_j \in E$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ and $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$ then there exist $0 \leq z_{ij} \in E$ such that

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$$x_i = \sum_{k=1}^{m} z_{ik}, \qquad y_j = \sum_{k=1}^{n} z_{kj}$$

for all i,j. The RSP is of substantial interest because of a theorem of Ando, [4], which states that an ordered Banach space has the RSP if and only if its dual has. If that dual is positively generated then it is but a small step to deduce that these conditions are also equivalent to the dual being a lattice. In fact Davies in [5] shows that if E is a Banach space ordered by a closed cone then E' is a Banach lattice (under the usual dual norm and order) if and only if

(i) E has the RSP.

(ii) If $x, y \in E$ and $-x \leq y \leq x$ then $\|y\| \leq \|x\|$.

(iii) If $x \in E$ and $\varepsilon > 0$ then there is $y \in E$ with $y \geq x, -x$ and $\|y\| \leq \|x\| + \varepsilon$.

The question of when a space $\mathcal{L}_r(E,F)$ has the RSP does not appear to have been addressed in the literature. In [1], in a slightly more general context, we gave an example of a (non-Banach) vector lattice with the property that the space of all regular operators from it into itself fails to have the RSP, but in the Banach lattice setting such an example does not seem to have been published previously.

In this paper we commence the study of the RSP in spaces of regular operators between Banach lattices by showing among other things that it is possible for such a space to have the RSP without being a lattice, and that not all spaces of regular operators have the RSP. Moreover, in Theorem 3.1 we give a characterization of Banach lattices E, for which the space $\mathcal{L}_r(c, E)$ has the RSP.

For any undefined terms in the theory of Banach lattices or positive operators we refer the reader to [3] or [8]. The author would like to thank Professor J.J. Grobler for posing the problem or whether of not the space of all regular operators between two Banach lattices could have the RSP without actually being a lattice.

2. SOME PRELIMINARY RESULTS

In this section we prove some technical results that will be required in the main result. The first of these will also be used in another forthcoming paper.

Proposition 2.1. *Let X be a compact Hausdorff space and A be an open F_σ-subset of X. There are two sequences in the space $C(X)$, of all continuous real-valued functions on X, (f_n) and (g_n), with the following properties:*

(i) $0 \leq f_n(x), g_n(x) \leq 1$ *for all* $x \in X$.

(ii) *If* $m \neq n$ *then* $f_m \wedge f_n = g_m \wedge g_n = 0$.

(iii) $f_n \wedge g_n = 0$ *for all* $n \in \mathbb{N}$.

(iv) $\bigcup_{n=1}^{\infty} (f_n^{-1}(1) \cup g_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{x\colon f_n(x) > 0\} \cup \{x\colon g_n(x) > 0\} = A$.

Proof. Suppose that $A = \bigcup_{n=1}^{\infty} H_n$ where each H_n is closed. By the Tietze extension theorem, for each $n \in \mathbb{N}$ there is $p_n \in C(X)$ with

(α) $0 \leq p_n(x) \leq 1$ for all $x \in X$,

(β) $p_n(x) = 1$ for all $x \in H_n$, and

(γ) $p_n(x) = 0$ for all $x \notin A$.

If we define $p = \sum_{n=1}^{\infty} p_n/2^{-n}$ then this series is uniformly convergent so that $p \in C(X)$ and it is clear that $0 \leq p(x) \leq 1$ for all $x \in X$ and that $f(x) > 0$ if and only if $x \in A$. Let (α_n) be a strictly decreasing sequence of positive reals converging to zero, with $\alpha_1 = \sup p(X)$. Let $F_n = p^{-1}([\alpha_{2n}, \alpha_{2n-1}])$ for each $n \in \mathbb{N}$, so that each F_n is closed and $F_n \cap F_m = \emptyset$ if $m \neq n$. Let also $U_n = p^{-1}(((\alpha_{2n} + \alpha_{2n+1})/2), ((\alpha_{2n-1} + \alpha_{2n-2})/2))$ (setting $\alpha_0 = \alpha_1 + 1$, for example), so that each U_n is open, $F_n \subseteq U_n$ and we still have $U_m \cap U_n = \emptyset$ if $m \neq n$. We may use Urysohn's lemma to produce $f_n \in C(X)$ with

(a) $0 \leq f_n(x) \leq 1$ for all $x \in X$,

(b) $f_n(x) = 1$ for all $x \in F_n$, and

(c) $f_n(x) = 0$ for all $x \notin U_n$.

The sequence (f_n) certainly does all that is claimed in (i) and (ii) of the statement of the theorem. Similarly working with the closed sets $G_n = p^{-1}([\alpha_{2n+1}, \alpha_{2n}])$ and the open sets $V_n = p^{-1}(((\alpha_{2n+1} + \alpha_{2n+2})/2), ((\alpha_{2n} + \alpha_{2n-1})/2))$ we may find $g_n \in C(X)$ with

(a′) $0 \leq g_n(x) \leq 1$ for all $x \in X$,

(b′) $g_n(x) = 1$ for all $x \in G_n$, and

(c′) $g_n(x) = 0$ for all $x \notin V_n$.

Again the sequence (g_n) does all that is claimed in (i) and (ii).

Claim (iv) in the theorem is true because

$$
\begin{aligned}
A &\supseteq \bigcup_{n=1}^{\infty} \{x\colon f_n(x) > 0\} \cup \{x\colon g_n(x) > 0\} \\
&\supseteq \bigcup_{n=1}^{\infty} (f_n^{-1}(1) \cup g_n^{-1}(1)) \\
&\supseteq \bigcup_{n=1}^{\infty} (F_n \cup G_n) \\
&\supseteq p^{-1}((0, \infty)) \quad (\text{as } \alpha_n \downarrow 0) \\
&= A.
\end{aligned}
$$

This only leaves claim (iii) to be verified. Note that $V_n \cap U_m = \emptyset$, and hence $f_m \wedge g_n = 0$, unless $m = n$ or $m = n + 1$. Adding two zero functions at the start of the sequence (g_m) does not alter what we have already established and ensures that f_n will be disjoint from g_n. □

The main result in the next section deals with operators from the space c of all convergent real sequences (with the usual norm and order) into a Banach lattice. We extract here some results in this setting that may be of independent interest. We use e_n to denote the sequence with n'th entry equal to 1 and all others zero and also denote by **1** the constantly one sequence.

Proposition 2.2. *If T is a bounded linear operator from c into a Banach lattice E then T is positive if and only if*

(i) $Te_n \geq 0$ *for all* $n \in \mathbb{N}$

and

(ii) $T(\mathbf{1} - \sum_{k=1}^{n} e_k) \geq 0$ *for all* $n \in \mathbb{N}$.

Proof. The proof of 'only if' is clear as $e_n \geq 0$ and $\mathbf{1} - \sum_{k=1}^{n} e_k$ is in c and is positive.

Suppose that (i) and (ii) are satisfied and that $\boldsymbol{x} = (x_n) \in c_+$. Let $\lim_{n\to\infty} x_n = x_\infty$ and put

$$\boldsymbol{x}_n = (x_1, x_2, \ldots, x_n, x_\infty, x_\infty, \ldots)$$

so that $\|\boldsymbol{x}_n - \boldsymbol{x}\|_\infty \to 0$ as $n \to \infty$. If we show that $T\boldsymbol{x}_n \geq 0$ then the continuity of T (and the fact that Banach lattices have closed positive cones) will guarantee that $T\boldsymbol{x} \geq 0$.

We may write

$$\boldsymbol{x}_n = x_\infty \cdot \mathbf{1} + (x_1 - x_\infty, x_2 - x_\infty, \ldots, x_n - x_\infty, 0, 0, \ldots).$$

As $\boldsymbol{x} \geq 0$ we have $x_n \geq 0$ for all $n \in \mathbb{N}$ and hence $x_\infty \geq 0$. Thus for each $n \in \mathbb{N}$ we have $-x_\infty \leq x_n - x_\infty$ and hence (by (i)) $-x_\infty Te_n \leq (x_n - x_\infty)Te_n$. Thus we have

$$\begin{aligned} T\boldsymbol{x}_n &= T(x_\infty \cdot \mathbf{1}) + \sum_{k=1}^{n} (x_k - x_\infty)Te_k \\ &\geq x_\infty T\mathbf{1} - \sum_{k=1}^{n} x_\infty Te_k \\ &= x_\infty T\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \\ &\geq 0 \end{aligned}$$

by (b) and the fact that $x_\infty \geq 0$. □

Proposition 2.3. *Let E be a Banach lattice and let S, T be continuous linear operators from c into E. Let (y_n) be a sequence in E with $Se_n \leq y_n \leq Te_n$ for all $n \in \mathbb{N}$ and let $y \in E$ be arbitrary. There is a continuous linear operator R from c into E with $Re_n = y_n$ for all $n \in \mathbb{N}$ and $R\mathbf{1} = y$.*

Proof. It clearly suffices to consider only the case that $S = 0$. We only need to verify that if $\alpha_n \to 0$ (for some real sequence (α_n)) then $\sum \alpha_n y_n$ converges in E. It also clearly suffices to prove this in the case that $\alpha_n \geq 0$ for all $n \in \mathbb{N}$. If $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $m > n > n_0$ implies that

$$\left\| \sum_{k=n}^{m} \alpha_k e_k \right\| < \varepsilon$$

and hence

$$\left\| \sum_{k=n}^{m} \alpha_k Te_k \right\| < \|T\|\varepsilon.$$

But

$$0 \leq \sum_{k=n}^{m} \alpha_k y_k \leq \sum_{k=n}^{m} \alpha_k T e_k$$

so that

$$\left\| \sum_{k=n}^{m} \alpha_k y_k \right\| \leq \left\| \sum_{k=n}^{m} \alpha_k T e_k \right\| < \|T\|\varepsilon$$

and hence $\sum \alpha_k y_k$ is Cauchy and therefore convergent. □

3. THE MAIN THEOREM

The main result in this section characterises those Banach lattices E such that $\mathcal{L}_r(c, E)$ has the RSP as being those with the *Cantor property, countable interpolation property* or *σ-interpolation property.* This means that whenever we have *sequences* (x_n) and (z_n) in E with $x_n \uparrow$, $z_n \downarrow$ and $x_m \leq z_n$ for all $m, n \in \mathbb{N}$ then there is $y \in E$ with $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$. Unlike the RSP, vector lattices need not have the countable interpolation property. For example, Seever has shown in [9] that $C(X)$ has the countable interpolation property if and only if X is an F-space, i.e. any pair of disjoint open F_σ subsets of X have disjoint closures. Moreover it was shown by C.B. Huijsmans and B. de Pagter in [6] that an Archimedean vector lattice E has the countable interpolation property if and only if E is uniformly complete and normal (the latter meaning that $E = \{x^+\}^d + \{x^-\}^d$ for all $x \in E$).

We introduce now the related notion of the *strong countable interpolation property.* This asserts that for any sequences (x_n) and (z_n) with $x_m \leq z_n$ for all $m, n \in \mathbb{N}$ there exists $y \in E$ with $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$. For vector lattices the countable interpolation property is clearly equivalent to the strong countable interpolation property. For more general ordered vector spaces the two notions are not equivalent. For example finite dimensional vector spaces ordered by closed cones certainly have the countable interpolation property (use compactness of order intervals) but need not have the strong countable interpolation property as, for example, that implies the RSP.

Although our interest was initially to find examples of spaces of regular operators which had the RSP but were not lattices, it turns out the class of examples that we construct with the RSP will automatically have the strong countable interpolation property.

Theorem 3.1. *The following conditions on a Banach lattice E are equivalent:*

(i) *E has the countable interpolation property.*
(ii) *$\mathcal{L}_r(c, E)$ has the strong countable interpolation property.*
(iii) *$\mathcal{L}_r(c, E)$ has the Riesz separation property.*

Proof. Our first step in the proof will be to show that if E has the countable interpolation property then $\mathcal{L}_r(c, E)$ has the strong countable interpolation property. To this end, let (S_p), (U_p) be two sequences in $\mathcal{L}_r(c, E)$ with

(a) $\qquad S_p \leq U_q \quad \forall p, q \in \mathbb{N}$

and recall ([8], Proposition I.3.5) that these operators are all norm bounded. We wish to find a bounded linear operator R from c into E with

(b) $\qquad S_p \leq R \leq U_p \quad \forall p \in \mathbb{N}$

and it will certainly be the case then that R is regular. By Proposition 2.2, (a) is equivalent to

(a_1) $\qquad S_p e_n \leq U_q e_n \quad \forall n, p, q \in \mathbb{N}$

and

(a_2) $\qquad S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \leq U_q\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \quad \forall n, p, q \in \mathbb{N}.$

Similarly the operator R that we wish to construct must satisfy the following two conditions

(b_1) $\qquad S_p e_n \leq R e_n \leq U_p e_n \quad \forall n, p \in \mathbb{N}$

and

(b_2) $\qquad S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \leq R\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \quad \forall n, p \in \mathbb{N}.$

The countable interpolation property for E certainly tells us that for each $n \in \mathbb{N}$ we can find elements of E lying above $S_p e_n$ and below $U_p e_n$ for each $p \in \mathbb{N}$. Pick one such element and call it Re_n. The sequence (Re_n) will be norm bounded and for any choice of $y = R\mathbf{1} \in E$ there will be a (unique) continuous linear extension of R to the whole of c by Proposition 2.3. Thus we need only worry about choosing $R\mathbf{1}$ so that (b_2) holds. This is equivalent to asking that

(b_3) $$\begin{cases} S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \leq R\mathbf{1} \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \\ \forall n, p \in \mathbb{N} \end{cases}$$

which we will be able to do, using the countable interpolation property, if we can show that

(b_4) $\qquad S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \quad \forall n, p \in \mathbb{N}.$

Let us now introduce a further sequence of elements V_m of $\mathcal{L}_r(c, E)$ defined by requiring that

$$V_m(e_n) = \begin{cases} S_1(e_n) & \text{if } m \neq n \\ Re_n & \text{if } m = n \end{cases}$$

$$V_m\mathbf{1} = S_1\mathbf{1}.$$

By Proposition 2.3 bounded linear operators, V_m, with such properties do exist. Note that if $m \neq n$ then

$$V_m e_n = S_1 e_n \leq U_q e_n$$

whilst if $m = n$ then

$$V_m e_n = Re_n \leq U_q e_n$$

so that we always have $V_m e_n \leq U_q e_n$. Also we have, if $m \leq n$,

$$(U_q - V_m)\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) = (U_q - S_1)\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + (Re_m - S_1 e_m) \geq 0$$

since $U_q \geq S_1$ and $Re_m \geq S_1 e_m$, whilst if $m > n$ then certainly

$$(U_q - V_m)\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) = (U_q - S_1)\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \geq 0.$$

By Proposition 2.2 we thus have $U_q \geq V_m$ for all $m, q \in \mathbb{N}$.

Consider the set $\mathcal{F} = \{S_p\colon p \in \mathbb{N}\} \cup \{V_m\colon m \in \mathbb{N}\}$ and the set $\mathcal{G}$ of all finite suprema, *calculated in* $\mathcal{L}_r(c, E'')$, from $\mathcal{F}$. If $T_1, T_2 \in \mathcal{F}$ then for each $n \in \mathbb{N}$ we have, using the Riesz–Kantorovich formula ([8], Corollary 1.3.4), which we may use as E'' is Dedekind complete, and using the fact that each e_n is an atom,

$$\begin{aligned}(T_1 \vee T_2)(e_n) &= \sup\{\lambda T_1 e_n + (1-\lambda) T_2 e_n\colon 0 \leq \lambda \leq 1\} \\ &= T_1 e_n \vee T_2 e_n.\end{aligned}$$

From the definition of, first, Re_n and then of V_m we certainly have $Te_n \leq Re_n$ for all $T \in \mathcal{F}$ and hence for all $T \in \mathcal{G}$. If Q denotes the supremum of the family $\mathcal{G}$ in $\mathcal{L}_r(c, E'')$ then, as $\mathcal{G}$ is upward directed, for each $x \in c_+$ we have $Qx = \bigvee\{Tx\colon T \in \mathcal{G}\}$, so in particular we have

$$Qe_n = \bigvee\{Te_n\colon T \in \mathcal{G}\} \leq Re_n.$$

But we also must have $Qe_n \geq V_n e_n = Re_n$ so that we actually have

(c) $\qquad Qe_n = Re_n \quad \forall n \in \mathbb{N}.$

We thus have found $Q \in \mathcal{L}_r(c, E'')$ with $S_p \leq Q \leq U_p$ for all $p \in \mathbb{N}$ and also with $Qe_n = Re_n \in E$ for all $n \in \mathbb{N}$. If it weren't for the fact that $Q\mathbf{1}$ need not lie in E then we would be finished.

The fact that $S_p \leq Q \leq U_p$ tells us that

(d₁) $$S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \leq Q\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \quad \forall n, p \in \mathbb{N}$$

holds and hence, noting (c), that

(d₂) $$\begin{cases} S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \leq Q\mathbf{1} \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \\ \forall n, p \in \mathbb{N}.\end{cases}$$

In particular this shows that we do have

(d₃) $$S_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \leq U_p\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) + \sum_{k=1}^{n} Re_k \quad \forall n, p \in \mathbb{N}.$$

As the order in E'' extends that in E, this is precisely (b_4) so the proof of the first implication is complete.

The strong countable interpolation property clearly implies the RSP, so let us now suppose that $\mathcal{L}_r(c,E)$ has the RSP. Clearly, $\mathcal{L}_r(c,I)$ has the RSP for any principal ideal I in E. In order to prove that E has the Cantor property, it suffices to prove that every principal ideal in E has the Cantor property. Thus (using Kakutani's representation theorem, [8], Theorem 2.1.3) we may restrict our attention to the case that $E = C(X)$ for some compact Hausdorff space X. In view of the theorem of Seever quoted above, we must prove that X is an F-space.

Let A and B be disjoint open F_σ's in X. We must prove that they have disjoint closures. By Proposition 2.1 we can find disjoint non-negative sequences (s_n), (t_n), (u_n) and (v_n) in $C(X)$, lying under the constantly one function, with

$$\bigcup_{n=1}^{\infty} (s_n^{-1}(1) \cup t_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{x\colon s_n(x) > 0\} \cup \{x\colon t_n(x) > 0\} = A$$

and

$$\bigcup_{n=1}^{\infty} (u_n^{-1}(1) \cup v_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{x\colon u_n(x) > 0\} \cup \{x\colon v_n(x) > 0\} = B$$

and $s_n \wedge t_n = u_n \wedge v_n = 0$ for all $n \in \mathbb{N}$.

Define operators $S, T : c \to C(X)$ with $Se_n = s_n$, $S\mathbf{1} = 0$, $Te_n = t_n$ and $T\mathbf{1} = 0$. The disjointness of the sequences (s_n) and (t_n) guarantees the existence of such operators. For example if $\alpha_n \to 0$ and m, n are such that $\|\sum_{k=n}^{m} \alpha_k e_k\| < \varepsilon$ then each $|\alpha_k| < \varepsilon$ and hence

$$\left\|\sum_{k=n}^{m} \alpha_k s_k\right\| = \left\|\bigvee_{k=n}^{m} \alpha_k s_k\right\| = \bigvee_{k=n}^{m} \|\alpha_k s_k\| = \bigvee_{k=n}^{m} |\alpha_k| \cdot \|s_k\| < \varepsilon.$$

Define also $U, V : c \to C(X)$ with $Ue_n = s_n + t_n + u_n$, $Ve_n = s_n + t_n + v_n$, and $U\mathbf{1} = V\mathbf{1} = \mathbf{1}_X$, where $\mathbf{1}_X$ denotes the constantly one function in $C(X)$. This time the fact that the operators U and V extend to the whole of c is proved by writing them as $S + T$ plus another operator for which the image of the (e_n) is a disjoint sequence. Clearly $Se_n, Te_n \le Ue_n, Ve_n$ for all $n \in \mathbb{N}$. We also have, for each $n \in \mathbb{N}$,

$$\begin{aligned}(U - S)\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) &= U\mathbf{1} - S\mathbf{1} + \sum_{k=1}^{n} Se_k - \sum_{k=1}^{n} Ue_k \\ &= \mathbf{1}_X + 0 + \sum_{k=1}^{n} s_k - \sum_{k=1}^{n} (s_k + t_k + v_k) \\ &= \mathbf{1}_X - \sum_{k=1}^{n} (t_k + v_k) \\ &\ge 0\end{aligned}$$

as all the functions t_k and v_k for $1 \le k \le n$ are disjoint, non-negative and lie below $\mathbf{1}_X$. Thus $S \le U$ and similarly we can show that $S, T \le U, V$. By hypothesis there is a linear operator $R : c \to C(X)$ with $S, T \le R \le U, V$. Note first that

$$Se_n, Te_n \le Re_n \le Ue_n, Ve_n \quad \forall n \in \mathbb{N}$$

so that

$$s_n, t_n \le Re_n \le s_n + t_n + v_n, s_n + t_n + v_n$$

and hence

$$s_n \vee t_n = s_n + t_n \le Re_n \le s_n + t_n + (u_n \wedge v_n) = s_n + t_n$$

showing that $Re_n = s_n + t_n$. At this stage, let us ask what we know about $R\mathbf{1}$. For each $n \in \mathbb{N}$

$$S\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right), T\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right) \le R\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right)$$
$$= R\mathbf{1} - \sum_{k=1}^{n} (s_n + t_n)$$
$$\le U\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right), V\left(\mathbf{1} - \sum_{k=1}^{n} e_k\right).$$

That is,

$$-\sum_{k=1}^{n} s_k, -\sum_{k=1}^{n} t_k \le R\mathbf{1} - \sum_{k=1}^{n} (s_k + t_k)$$
$$\le \mathbf{1}_X - \sum_{k=1}^{n} (s_k + t_k + u_k), \mathbf{1}_X - \sum_{k=1}^{n} (s_k + t_k + v_k)$$

so that

$$\sum_{k=1}^{n} s_k, \sum_{k=1}^{n} t_k \le R\mathbf{1} \le \mathbf{1}_X - \sum_{k=1}^{n} u_k, \mathbf{1}_X - \sum_{k=1}^{n} v_k.$$

If, for some $n \in \mathbb{N}$ and some $x \in X$ we have $s_n(x) = 1$ or $t_n(x) = 1$ then this shows us that $R\mathbf{1}(x) = 1$ as if, for example, $s_n(x) = 1$ then

$$1 \le \sum_{k=1}^{n} s_k(x) \le R\mathbf{1}(x) \le \mathbf{1}_X(x) = 1,$$

and similarly $R\mathbf{1}(x) = 0$ if $u_n(x) = 1$ or $v_n(x) = 1$. That is, the function $R\mathbf{1}$, which is an element of $C(X)$, takes the value 1 on A and 0 on B. This certainly suffices to prove that $\bar{A} \cap \bar{B} = \emptyset$ and hence that X is an F-space. □

Example 3.2. If X is an F-space which is not quasi-Stonean (such as $\beta(\mathbb{N})\backslash\mathbb{N}$) then $C(X)$ has the Cantor property but is not Dedekind σ-complete (see [8] Proposition 2.1.5). By Theorem 3.1, $\mathcal{L}_r(c, C(X))$ has the RSP. On the other hand, by Theorem 3.10 of [2], $\mathcal{L}_r(c, C(X))$ is not a lattice. Thus we can have spaces of operators between Banach lattices which are not lattices but which do have the RSP.

Example 3.3. If E does not have the Cantor property, e.g. if $E = c$, then $\mathcal{L}_r(c, E)$ does not have the RSP.

In Corollary 3.4 of [2] we show that if S is a compact metric space and K is an F-space then for every bounded linear operator $T : C(S) \to C(K)$ there is a positive operator $U : C(S) \to C(K)$ with $U \ge T, -T$ and $\|U\| = \|T\|$ (and hence all bounded operators are regular). In the light of this, of Davies' characterisation of preduals of Banach lattices cited in the introduction, and of Theorem 3.1 the following result is immediate.

Corollary 3.4. *If K is a compact Hausdorff space then $\mathcal{L}(c, C(K))'$ is a Banach lattice if and only if K is an F-space.*

It is possible to prove rather more than has been shown above. The techniques used in the proof of Theorem 3.1 of [2] may be modified to prove that $\mathcal{L}_r(C(S), E)$ has the RSP provided that E has the Cantor property and S is a compact metric space. This leads us propose to the following conjecture: $\mathcal{L}_r(H, E)$ will always have the RSP when E has the Cantor property and H is a separable Banach lattice.

The assumption of separability in the above conjecture is essential. If we allow the domain of our spaces of operators to be non-separable, then we obtain yet another characterisation of Dedekind complete Banach lattices.

Theorem 3.5. *Let E be a Banach lattice, then the folllowing are equivalent:*
(i) *E is Dedekind complete.*
(ii) *For all Banach lattices X, $\mathcal{L}_r(X, E)$ is a Dedekind complete vector lattice.*
(iii) *For all Banach lattices X, $\mathcal{L}_r(X, E)$ has the Riesz separation property.*

Proof. That (i) implies (ii) dates back to [7], whilst it is clear that (ii) implies (iii). In order to show that (iii) implies (i) it suffices to prove that each principal ideal in E is Dedekind complete and hence, once again, we need only consider the case that $E = C(K)$ for some compact Hausdorff space K. In order to prove that $C(K)$ is Dedekind complete, we must prove ([8], Proposition 2.1.4) that any two disjoint open subsets of K have disjoint closures. The proof is very similar to the last part of the proof of Theorem 3.1, so we will omit many of the details.

Let A and B be two such disjoint open subsets of K and let Γ be an index set sufficiently large that there are disjoint collections of open F_σ's, $\{A_\gamma\colon \gamma \in \Gamma\}$ and $\{B_\gamma\colon \gamma \in \Gamma\}$ with $A_\gamma \subseteq A$ and $B_\gamma \subseteq B$ for each $\gamma \in \Gamma$ and $A \subseteq \overline{\bigcup_{\gamma \in \Gamma} A_\gamma}$ and $B \subseteq \overline{\bigcup_{\gamma \in \Gamma} B_\gamma}$. In order to accomplish this it may be necessary to take some of the A_γ and B_γ to be empty. In that case the corresponding sequences of functions that we will define will all be zero but that will not affect the proof at all. For each $\gamma \in \Gamma$, construct sequences $(s_{n\gamma})$, $(t_{n\gamma})$, $(u_{n\gamma})$ and $(v_{n\gamma})$ as in the proof of Theorem 3.1. Let X denote the Banach lattice of all real-valued functions on the discrete topological space $\mathbb{N} \times \Gamma$ which tend to a limit at infinity. Note that analogues of Propositions 2.2 and 2.3 are valid in this context. If we now use $e_{\gamma n}$ to denote the function on $\mathbb{N} \times \Gamma$ that is zero except at $\langle n, \gamma \rangle$, where it takes the value 1, and $\mathbf{1}$ to denote the constantly one function on $\mathbb{N} \times \Gamma$ then we may again construct operators $S, T, U, V : X \to C(K)$ with

$$Se_{\gamma n} = s_{\gamma n} \quad S\mathbf{1} = 0$$

$$Te_{\gamma n} = t_{\gamma n} \quad T\mathbf{1} = 0$$

$$Ue_{\gamma n} = s_{\gamma n} + t_{\gamma n} + u_{\gamma n} \quad U\mathbf{1} = 0$$

$$Ve_{\gamma n} = s_{\gamma n} + t_{\gamma n} + v_{\gamma n} \quad V\mathbf{1} = 0$$

and then verify that $S, T \leq U, V$. If there is $R : X \to C(K)$ with $S, T \leq R \leq U, V$ then we will again have $Re_{\gamma n} = s_{\gamma n} + t_{\gamma n}$ and we will again be able to see that $R\mathbf{1}$ will be constantly 1 on each A_γ and 0 on each B_γ. Continuity will then tell us that $R\mathbf{1}$ is constantly 1 on A and 0 on B, showing that A and B do indeed have disjoint closures. □

REFERENCES

[1] Abramovich, Y.A. and A.W. Wickstead – Regular operators from and into a small Riesz space. Indag. Math. N.S. **2**, 257–274 (1991).

[2] Abramovich, Y.A. and A.W. Wickstead – The regularity of order bounded operators into $C(K)$ II. Quart. J. Math. Oxford (2) **44**, 257–270 (1993).

[3] Aliprantis, C.D. and O. Burkinshaw – Positive Operators. Academic Press, New York and London (1985).

[4] Andô, T. – On fundamental properties of a Banach space with a cone. Pacific J. Math. **12**, 1163–1169 (1962).

[5] Davies, E.B. – On the Banach duals of certain spaces with the Riesz decomposition property. Quart. J. Math. Oxford (2) **18**, 109–111 (1967).

[6] Huijsmans, C.B. and B. de Pagter – On z-ideals and d-ideals in Riesz spaces II. Indag. Math. **42**, 391–408 (1980).

[7] Kantorovich, L.V. – Concerning the general theory of operators in semi-ordered spaces. Dokl. Akad. Nauk USSR **1**, 271–274 (1936).

[8] Meyer-Nieberg, P. – Banach Lattices. Springer-Verlag, Berlin, Heidelberg, New York (1991).

[9] Seever, G.L. – Measures on F-spaces. Trans. Amer. Math. Soc. **133**, 267–280 (1968).