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Artinian Gorenstein algebras of embedding dimension four: components of $\mathbb{P}\mathrm{Gor}(H)$ for $H=(1,4,7,\dots,1)$

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

A Gorenstein sequence H is a sequence of nonnegative integers $H=(1,h_1,\dots,h_j=1)$ symmetric about $j/2$ that occurs as the Hilbert function in degrees less or equal j of a standard graded Artinian Gorenstein algebra $A=R/I$, where R is a polynomial ring in r variables and I is a graded ideal. The scheme $\mathbb{P}\mathrm{Gor}(H)$ parametrizes all such Gorenstein algebra quotients of R having Hilbert function H and it is known to be smooth when the embedding dimension satisfies $h_1\leqslant 3$. The authors give a structure theorem for such Gorenstein algebras of Hilbert function $H=(1,4,7,\dots)$ when $R=K[w,x,y,z]$ and $I_2\cong\langle wx,wy,wz\rangle$ (Theorems 3.7 and 3.9). They also show that any Gorenstein sequence $H=(1,4,a,\dots)$, $a\leqslant 7$ satisfies the condition $\Delta H_{\leqslant j/2}$ is an O-sequence (Theorems 4.2 and 4.4). Using these results, they show that if $H=(1,4,7,h,b,\dots,1)$ is a Gorenstein sequence satisfying $3h-b-17\geqslant 0$, then the Zariski closure $\overline{\mathfrak{C}(H)}$ of the subscheme $\mathfrak{C}(H)\subset\mathbb{P}\mathrm{Gor}(H)$ parametrizing Artinian Gorenstein quotients $A=R/I$ with $I_2\cong\langle wx,wy,wz\rangle$ is a generically smooth component of $\mathbb{P}\mathrm{Gor}(H)$ (Theorem 4.6).

They show that if in addition $8\leqslant h\leqslant 10$, then such $\mathbb{P}\mathrm{Gor}(H)$ have several irreducible components (Theorem 4.9). M. Boij and others had given previous examples of certain $\mathbb{P}\mathrm{Gor}(H)$ having several components in embedding dimension four or more (Pacific J. Math. 187(1) (1999) 1–11).

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The proofs use properties of minimal resolutions, the smoothness of $\mathbb{P}\mathrm{Gor}(H')$ for embedding dimension three (J.O. Kleppe, J. Algebra 200 (1998) 606–628), and the Gotzmann Hilbert scheme theorems (Math. Z. 158(1) (1978) 61–70).

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1. Introduction

Let R be the polynomial ring $R = K[x_1, \ldots, x_r]$ over an algebraically closed field K, and denote by $M = (x_1, x_2, \ldots, x_r)$ its maximal ideal. When $r = 4$, we let $R = K[w, x, y, z]$ and regard it as the coordinate ring of the projective space $\mathbb{P}^3$. Let $A = R/I$ be a standard graded Artinian Gorenstein (GA) algebra, quotient of R. We will denote by $\mathrm{Soc}(A) = (0 : M)$ the socle of A, the one-dimensional subvector space of A annihilated by multiplication by M. It is the minimal nonzero ideal of A. Its degree is the *socle degree* $j(A) : \ j(A) = \max\{i \mid A_i \neq 0\}$. A sequence $H = (h_0, \ldots, h_j) = (1, r, \ldots, r, 1)$ of positive integers symmetric about $j/2$ is called a *Gorenstein sequence* of socle degree j, if it occurs as the Hilbert function of some graded Artinian Gorenstein (GA) algebra $A = R/I$. We let $\Delta H_i = h_i - h_{i-1}$, and denote by $H_{\leqslant d}$ the subsequence $(1, h_1, \ldots, h_d)$. The graded Betti numbers of an algebra are the dimensions of the various graded pieces that occur in the minimal graded R-resolution of A.

When $r = 2$, Macaulay had shown [25] that an Artinian Gorenstein quotient of R is a complete intersection quotient $A = R/(f, g)$; thus, for A graded, the Gorenstein sequence must have the form $H(A) = H(s) = (1, 2, \ldots, s-1, s, s, \ldots, 2, 1)$. Also, when $r = 2$ the family $\mathbb{P}\mathrm{Gor}(H(s))$ parametrizing such Artinian quotients is smooth; its closure $\overline{\mathbb{P}\mathrm{Gor}(H(s))} = \bigcup_{t \leqslant s} \mathbb{P}\mathrm{Gor}(H(t))$ is naturally isomorphic to the secant variety of a rational normal curve, so is well understood (see, for example [20, Section 1.3]).

For Artinian Gorenstein algebras A of embedding dimension three ($r = 3$), the Gorenstein sequences $H(A)$, and the possible sequences β of graded Betti numbers for A given the Hilbert function $H(A)$ had been known for some time [9,31,12,17,18], see also [20, Chapter 4]. More recently, the irreducibility and smoothness of the family $\mathbb{P}\mathrm{Gor}(H)$ parametrizing such GA quotients having Hilbert function H was shown by Diesel and Kleppe, respectively [12,22]. When $r = 3$, there are also several dimension formulas for the family $\mathbb{P}\mathrm{Gor}(H)$, due to Conca and Valla, Kleppe, Cho and Jung [11,22,10] (see also [20, Section 4.4] for a survey); also, M. Boij has found the dimension of the subfamily $\mathbb{P}\mathrm{Gor}(H, \beta)$ parametrizing A with a given sequence β of graded Betti numbers [5]. The closure $\overline{\mathbb{P}\mathrm{Gor}(H)}$ is in general less well understood when $r = 3$, but see [20, Theorem 5.71, Sections 7.1–7.2].

For embedding dimensions five or greater, it is known that a Gorenstein sequence may be nonunimodal: that is, it may have several maxima separated by a smaller local minimum [2,6].

When the embedding dimension is four, it is not known whether Gorenstein sequences must satisfy the condition that the first difference $\Delta H_{\leqslant j/2}$ is an O-sequence—a sequence admissible for the Hilbert function of some ideal of embedding dimension three (see Definition 2.4). Nor do we know whether height four Gorenstein sequences are unimodal, a weaker restriction. Little was known about the parameter scheme $\mathbb{P}\mathrm{Gor}(H)$ when $r = 4$, except that for suitable Gorenstein sequences H, it may have several irreducible components [4,21, Example C.38]. We had the following questions, that guided this portion of our study.

- Can we find insight into the open problem of whether height four Gorenstein sequences H must satisfy the condition, $\Delta H_{\leqslant j/2}$ is an O-sequence?
- Do most schemes $\mathbb{P}\mathrm{Gor}(H)$ when $r = 4$ have several irreducible components, or is this a rare phenomenon?

We now outline our main results. We consider Hilbert sequences $H = (1, 4, 7, \dots 1)$. Thus, I is always a graded height four Gorenstein ideal in $K[w, x, y, z]$ whose minimal sets of generators include exactly three quadrics. First, in Theorem 3.7, we obtain a structure theorem for Artinian Gorenstein quotients $A = R/I$ with Hilbert function $H(A) = H$ and with $I_2 \cong \langle wx, wy, wz\rangle$. The proof relies on the connection between I and the intersection $J = I \cap K[x, y, z]$, which is a height three Gorenstein ideal. We also construct the minimal resolution of A in Theorem 3.9. This allows us to determine the tangent space $\mathrm{Hom}_0(I, R/I)$ to A on $\mathbb{P}\mathrm{Gor}(H)$, and to show that under a simple condition on H, if such an algebra A is general enough, then A is parametrized by a smooth point of $\mathbb{P}\mathrm{Gor}(H)$ (Theorem 3.11).

We then study the intriguing case $A = R/I$ where $I_2 \cong \langle w^2, wx, wy\rangle$ and exhibit a subtle connection between A and a height three Gorenstein algebra. We determine in Theorem 3.20 that the possible Hilbert functions $H = H(A)$ for such Artinian algebras A satisfy

$$H = H' + (0, 1, \dots, 1, 0), \tag{1.1}$$

where H' is a height three Gorenstein sequence.

Our result pertaining to the first question is

Theorem (*Theorem 4.2, Corollary 4.3, Proposition 4.4*). *All Gorenstein sequences of the form $H = (1, 4, a, \dots)$, $a \leqslant 7$ must satisfy the condition that $\Delta H_{\leqslant j/2}$ is an O-sequence.*

To show this we eliminate potential sequences not satisfying the condition by frequently using the symmetry of the minimal resolution of a graded Artinian Gorenstein algebra A, the Macaulay bounds on the Hilbert function, and the Gotzmann Persistence and Hilbert scheme theorems (Theorem 2.3). However, these methods do not extend to all height four Gorenstein sequences, and we conjecture that not all will satisfy the condition that $\Delta H_{\leqslant j/2}$ is an O-sequence (see Remark 4.5).

We then combine these results with a well known construction of Gorenstein ideals from sets of points to obtain our theorem concerning irreducible components of $\mathbb{P}\mathrm{Gor}(H)$

Theorem (*Theorem 4.9i*). *Let $H = (1, 4, 7, h, b, \dots, 1)$ be a Gorenstein sequence satisfying $8 \leqslant h \leqslant 10$ and $3h - b - 17 \geqslant 0$. Then $\mathbb{P}\mathrm{Gor}(H)$ has at least two components. The first is the Zariski closure of the subscheme $\mathfrak{C}(H)$ of $\mathbb{P}\mathrm{Gor}(H)$ parametrizing Artinian Gorenstein quotients $A = R/I$ for which I_2 is* $\mathrm{Pgl}(3)$-*isomorphic to $\langle wx, wy, wz\rangle$. The second component parametrizes quotients of the coordinate rings of certain punctual schemes in $\mathbb{P}^3$.*

2. Notation and basic results

In this section, we give definitions and some basic results that we will need. Recall that $R = K[w, x, y, z]$ is the polynomial ring with the standard grading over an algebraically closed field, and that we consider only graded ideals I.

Let $V \subset R_v$ be a vector subspace. For $u \leqslant v$ we let $V : R_u = \langle f \in R_{v-u} \mid R_u \cdot f \subset V \rangle$. We state as a lemma a result of Macaulay [25, Section 60ff] that we will use frequently.

Lemma 2.1 (*F.H.S. Macaulay [25]*). *Let* char $K = 0$ *or* char $K > j$. *There is a one-to-one correspondence between graded Artinian Gorenstein algebra quotients $A = R/I$ of R having socle degree j, on the one hand, and on the other hand, elements $F \in \mathscr{R}_j$ modulo K^*-action where $\mathscr{R} = K[W, X, Y, Z]$, the dual polynomial ring. The correspondence is given by*

$$I = \operatorname{Ann} F = \{h \in R \mid h \circ F = h(\partial/\partial W, \dots, \partial/\partial Z) \circ F = 0\},$$
$$F = (I_j)^{\perp} \in \mathscr{R}_j \ \bmod K^*. \tag{2.1}$$

Here F is also the generator of the R-submodule $I^{\perp} \subset \mathscr{R}$, $I^{\perp} = \{G \in \mathscr{R} \mid h \circ G = 0$ for all $h \in I\}$. The Hilbert function $H(R/I)$ satisfies

$$H(R/I)_i = \dim_K (R \circ F)_i = H(R/I)_{j-i}. \tag{2.2}$$

Furthermore, for $i \leqslant j$, I_i is determined by I_j or by F as follows:

$$I_i = I_j : R_{j-i} = \{h \in R_i \mid h \cdot R_{j-i} \subset I_j\} = \{h \in R_i \mid h \circ (R_{j-i} \circ F) = 0\}. \tag{2.3}$$

When char $K = p > j$ *the statements are analogous, but we must replace $K[W, X, Y, Z]$ by the ring of divided powers $\mathscr{D}$, and the action of R on $\mathscr{D}$ by the contraction action* (*see below*).

Proof. For a modern proof see [20, Lemmas 2.15 and 2.17]. For a discussion of the use of the divided power ring when char $K = p$ see also [20, Appendix A]. □

Corollary 2.2. *Let $A = R/I$ be a graded Artinian Gorenstein algebra of socle degree j. Let $J = I_{\mathfrak{Z}}$ be a saturated ideal defining a scheme $\mathfrak{Z} \subset \mathbb{P}^3$, such that for some i, $2 \leqslant i \leqslant j$, $\mathfrak{Z} =$ Proj $(R/(J_i))$, with $J_i \subset I_i$. Then for $0 \leqslant u \leqslant i$ we have $J_u \subset I_u$. If also $J_i = I_i$, then for such u, $J_u = I_u$.*

Proof. Let $0 \leqslant u \leqslant i$. Since J is its own saturation, we have $J_u = J_k : R_{k-u}$ for large k, so we have

$$J_u = J_k : R_{k-u} = \{J_k : R_{k-i}\} : R_{i-u} = J_i : R_{i-u}.$$

Now (2.3) implies that for $0 \leqslant u \leqslant i$

$$I_u = I_j : R_{j-u} = \{I_j : R_{j-i}\} : R_{i-u} = I_i : R_{i-u}.$$

This completes the proof of the relation between $I_{\mathfrak{Z}}$ and I. □

Note that [19, Example 3.8], due to Berman, shows that one cannot conclude that $J \subset I$ in Corollary 2.2. For let $I = (x^3, y^3, z^3)$, and let J be the saturated ideal $J = (x^2y^3, y^2z^3, x^3z^2, x^2y^2z^2)$, a local complete intersection of degree 18 defining a punctual scheme concentrated at the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Then we have $J_5 \subset I_5$ but $x^2y^2z^2 \in J$ so $J \not\subseteq I$.

We suppose $R = K[w, x, y, z]$. Let $\mathcal{D} = K_{\mathrm{DP}}[W, X, Y, Z]$ denote the divided power algebra associated to R: the basis of $\mathcal{D}_j$ is $\{W^{[j_1]} \cdot X^{[j_2]} \cdot Y^{[j_3]} \cdot Z^{[j_4]}, \sum j_i = j\}$. We let $x^i \circ X^{[j]} = X^{[j-i]}$ when $j \geqslant i$ and zero otherwise; this action extends in a natural way to the contraction action of R on $\mathcal{D}$. Multiplication in $\mathcal{D}$ is determined by $X^{[u]} \cdot X^{[v]} = \binom{u+v}{v} X^{[u+v]}$. By $(\alpha X + Y)^{[u]}$, $\alpha \in K$ we mean $\sum_{0 \leqslant i \leqslant u} \alpha^i X^{[i]} \cdot Y^{[u-i]}$: this is $(\alpha X + Y)^u / u!$ when the latter makes sense. When char $K = 0$, or char $K > j$ we may replace $\mathcal{D}$ by the polynomial ring $\mathcal{R} = K[W, X, Y, Z]$ with R acting on $\mathcal{R}$ as partial differential operators (2.1), and we replace all $X^{[u]}$ by X^u, and $(\alpha X + Y)^{[u]}$ by $(\alpha X + Y)^u$.

The inverse system $I^\perp \subset \mathcal{D}$ of the ideal $I \subset R$ satisfies

$$I^\perp = \{G \in K_{\mathrm{DP}}[W, X, Y, Z], h \circ G = 0 \textit{ for all } h \in I\} \tag{2.4}$$

and it is an R-submodule of $\mathcal{D}$ isomorphic to the dual module of $A = R/I$. When $A = R/I$ is graded Gorenstein of socle degree j, then by Macaulay's Lemma 2.1 the inverse system is principal, generated by $F \in \mathcal{D}_j$: we call F the *dual generator* of A or for I. Thus, we may parametrize the algebra A by the class of F mod nonzero K^*-multiple, an element of the projective space $\mathbb{P}^{N-1}$, $N = \binom{j+3}{j}$. Given a Gorenstein sequence H of socle degree j (so $H_j \neq 0$, $H_{j+1} = 0$) we let $\mathbb{P}\mathrm{Gor}(H) \subset \mathbb{P}^{N-1}$ denote the scheme parametrizing the family of all GA quotients $A = R/I$ having Hilbert function H. Here, we use the scheme structure given by the catalecticants, and described in [20, Definition 1.10]. A "geometric point" p_A of $\mathbb{P}\mathrm{Gor}(H)$ parametrizes a Artinian Gorenstein quotient $A = R/I$ of R having Hilbert function H.

We now state Macaulay's theorem characterizing Hilbert functions or O-sequences, and the version of the Persistence and Hilbert Scheme theorems of Gotzmann that we will use [15].

Let d be a positive integer. The dth Macaulay coefficients of a positive integer c are the unique decreasing sequence of nonnegative integers $k(d), \ldots, k(1)$ satisfying

$$c = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1}.$$

We denote by $c^{(d)}$ the integer

$$c^{(d)} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \cdots + \binom{k(1)+1}{2}. \tag{2.5}$$

Then, the Hilbert polynomial $p_{c,d}(t)$ for quotients B of the polynomial ring R, such that B is regular in degree d and $H(B)_d = c$ satisfies

$$p_{c,d}(t) = \binom{k(d)+t-d}{t} + \binom{k(d-1)+t-d}{t-1} + \cdots + \binom{k(1)+t-d}{t-d}. \tag{2.6}$$

The length of the dth Macaulay expansion of c, or of the Macaulay expansion of the polynomial $p_{c,d}$, is the number of $\{k(i) \mid k(i) \geqslant i\}$, equivalently, the number of nonzero binomial coefficients in the Macaulay expansion, and this is well known to be the Gotzmann regularity degree of $p_{c,d}$ [7, Theorem 4.3.2].

Theorem 2.3. *Suppose that* $1 \leqslant c \leqslant \dim_k R_d$, *and* I *is a graded ideal of* $R = K[x_1, \ldots, x_r]$.

(i) [26] *If* $H(R/I)_d = c$, *then* $H(R/I)_{d+1} \leqslant c^{(d)}$ *(Macaulay's inequality).*
(ii) [15] *If* $H(R/I)_d = c$ *and* $H(R/I)_{d+1} = c^{(d)}$, *then* Proj $(R/(I_d))$ *is a projective scheme in* $\mathbb{P}^{r-1}$ *of Hilbert polynomial* $p_{c,d}(t)$.

In particular $H(R/(I_d))_k = p_{c,d}(k)$ *for* $k \geqslant d$, *and* $H' = H(R/(I_d))$ *has extremal growth* $(h'_{k+1} = {h'_k}^{(k)})$ *in each degree* k *to* $k+1$, $k \geqslant d$.

Proof. For a proof of Theorem 2.3(i) see [7, Theorem 4.2.10]. For a proof of the persistence (second) part of Theorem 2.3(ii) see [7, Theorem 4.3.3]; for the Gotzmann–Hilbert scheme theorem see [15], or the discussion of [21, Theorem C.29]. □

Definition 2.4. A sequence of nonnegative integers $H = (1, h_1, \ldots, h_d, \ldots)$, is said to be an *O-sequence*, or to be *admissible* if it satisfies Macaulay's inequality of Theorem 2.3(i) for each integer $d \geqslant 1$.

Recall that the regularity degree $\sigma(p)$ of a Hilbert polynomial $p = p(t)$ is the smallest degree for which all projective schemes $\mathfrak{Z}$ of Hilbert polynomial p are Castelnuovo–Mumford regular in degree less or equal $\sigma(p)$. Gotzmann and Bayer showed that this bound is the length $\sigma(p)$ of the Macaulay expression for p [15,1]: for an exposition and proof see [7, Theorem 4.3.2]; also see [21, Definition C.12 and Proposition C.24], which includes some historical remarks. As an easy consequence we have

Corollary 2.5. *The regularity degree of the polynomial* $p(t) = at + 1 - \binom{a-1}{2} + b$ *where* $a > 0, b \geqslant 0$ *satisfies* $\sigma(p) = a + b$. *These Hilbert polynomials cannot occur with* $b < 0$. *In particular we have, the regularity degree of the polynomial* $p(t) = 3t + b, b \geqslant 0$ *is* $3 + b$, *of* $p(t) = 2t + 1 + b, b \geqslant 0$ *is* $b + 2$, *and of* $p(t) = t + 1 + b, b \geqslant 0$ *is* $b + 1$. *The regularity of the constant polynomial* $p(t) = b$ *is* b.

Proof. One has for $p(t) = at + 1 - \binom{a-1}{2} + b$, the following sum, equivalent to a Macaulay expansion as in (2.6) of length $a + b$,

$$p(t) = \binom{t+1}{1} + \binom{t+1-1}{1} + \binom{t+1-2}{1} + \cdots + \binom{t+1-(a-1)}{1} + \binom{t-a}{0} + \binom{t-(a+1)}{0} + \cdots + \binom{t-(a+b-1)}{0}.$$

Corollary 2.6. *Let* H *be a Gorenstein sequence of socle degree* j, *and suppose for that some* $d < j$, $h_{d+1} = (h_d)^{(d)}$ *is extremal in the sense of Theorem* 2.3(i). *Then* $\Delta H_{\leqslant d+1}$ *is an* O*-sequence.*

Proof. Theorem 2.3(ii) and Corollary 2.2 show the existence of a scheme $\mathfrak{Z} \subset \mathbb{P}^{r-1}$ satisfying $h_u = H(R/I_{\mathfrak{Z}})_u$ for $u \leqslant d+1$. Since $I_{\mathfrak{Z}}$ is saturated and thus $R/I_{\mathfrak{Z}}$ has depth at least one, there is a homogeneous degree one nonzero divisor, implying that the first difference $\Delta(H(R/I_{\mathfrak{Z}}))$ is an *O*-sequence. □

Remark 2.7. The assertion of Corollary 2.6 as well as those of Corollary 2.2 are valid more generally for graded Artinian algebras having socle only in degree *j* (level algebras), or those having socle only in degrees greater or equal *j*.

As an example of the application of Theorem 2.3, we determine below the Gorenstein sequences $H = (1, 4, 7, h, 7, 4, 1)$ that occur, having socle degree 6.

Corollary 2.8. *The sequence* $H = (1, 4, 7, h, 7, 4, 1)$ *is a Gorenstein sequence if and only if* $7 \leqslant h \leqslant 11$.

Proof. From Macaulay's extremality Theorem 2.3(i) we have $H(3) = h \leqslant H(2)^{(2)} = 7^{(2)} = 11$, and $H(4) = 7 \leqslant h^{(3)}$ which implies $h \geqslant 6$. Now $H = (1, 4, 7, 6, 7, 4, 1)$ implies that the growth of $H_3 = 6$ to $H_4 = 7$ is maximum, since $6 = \binom{4}{3} + \binom{2}{2} + \binom{1}{1}$, while $7 = 6^{(3)} = \binom{5}{4} + \binom{3}{3} + \binom{2}{2}$. Corollary 2.6 shows this is impossible. □

For a subscheme $\mathfrak{Z} \subset \mathbb{P}^3$ we will denote by $H_{\mathfrak{Z}} = H(R/I_{\mathfrak{Z}})$ its Hilbert function, sometimes called its postulation; here $I_{\mathfrak{Z}} \subset R$ is the saturated ideal defining $\mathfrak{Z}$. Inequalities among Hilbert functions are termwise. The following result is well known and easy to show, since a degree-d punctual scheme can cut out at most d conditions in a given degree.

Lemma 2.9. *Let* $\mathfrak{Z} = W \cup \mathfrak{Z}_1 \subset \mathbb{P}^3$, *be a subscheme of* $\mathbb{P}^3$, *where W is a degree d punctual scheme. Then for all i*, $(H_{\mathfrak{Z}})_i \leqslant (H_{\mathfrak{Z}_1})_i + d$.

Proof. We have (the first inequality is from Maroscia's result [27], see [20, Theorem 5.1A])

$$\begin{aligned} d \geqslant (H_W)_i &= \dim R_i - \dim(I_W)_i \geqslant \dim(I_{\mathfrak{Z}_1})_i - \dim(I_W \cap I_{\mathfrak{Z}_1})_i \\ &= H(R/(I_W \cap I_{\mathfrak{Z}_1}))_i - H(R/I_{\mathfrak{Z}_1})_i \geqslant (H_{\mathfrak{Z}})_i - (H_{\mathfrak{Z}_1})_i. \qquad \square \end{aligned} \tag{2.7}$$

3. Nets of quadrics in $\mathbb{P}^3$, and Gorenstein ideals

In Section 3.1 we give preparatory material on nets of quadrics, and on the Hilbert schemes of low degree curves in $\mathbb{P}^3$. In Section 3.2, we prove a structure theorem for Artinian Gorenstein algebras $A = R/I$ of Hilbert function $H(A) = (1, 4, 7, \ldots)$ for which the net of quadrics I_2 has a common factor and is isomorphic after a change of variables to $\langle wx, wy, wz\rangle$ (Theorem 3.7). We then determine the dimension of the tangent space to $\mathbb{P}\mathrm{Gor}(H)$ at a point parametrizing such an ideal; we also show that when H has socle degree 6, the subfamily parametrizing such Gorenstein algebras is an irreducible component of $\mathbb{P}\mathrm{Gor}(H)$ (Theorem 3.11), a result which we will later generalize to arbitrary socle degree

(Theorem 4.6). In Section 3.3 we determine the possible Hilbert functions $H(A)$, $A = R/I$ when $I_2 = \langle w^2, wx, wy\rangle$ (Theorem 3.20).

3.1. Nets of quadrics

Three homogeneous quadratic polynomials f, g, h in $R = K[w, x, y, z]$ form a family $\alpha_1 f + \alpha_2 g + \alpha_3 h$, $\alpha_i \in K$, comprising a net of quadrics in $\mathbb{P}^3$. Here we will use the term net also for the vector space span $V = \langle f, g, h\rangle$. We divide these families according to the number of linear relations among the three quadrics. We now show that they can have at most 3 linear relations. Let $(I_2) = (f, g, h)$ be the ideal generated by a net of quadrics $I_2 = \langle f, g, h\rangle$. Then $H(R/(I_2)) = (1, 4, 7, h, \ldots)$, where $h \leqslant 11 = 7^{(3)}$ by Macaulay's growth condition. When there are no relations $H(R/(I_2))_3 = 20 - 12 = 8$, so the number of linear relations on the net of quadrics $\langle f, g, h\rangle$ is no greater than $11 - 8 = 3$, as claimed.

Nets of quadrics in $\mathbb{P}^3$ have been extensively studied geometrically, earlier by W. L. Edge and others, more recently by C.T.C. Wall and others for their connections with mapping germs, and instantons. I. Vainsecher and also G. Ellingsrud, R. Peine, and S.A. Strømme have showed that the Hilbert scheme of twisted cubics in $\mathbb{P}^3$ is a blow-up of the family $\mathfrak{F}_{\mathrm{RNC}}$ of nets of quadrics arising as minors of a 2×3 matrix (Definition 3.1) along the sublocus of those nets having a common factor. Nets of quadrics are parametrized by the Grassmanian $\mathbb{G} = \mathrm{Grass}(3, R_2) \cong \mathrm{Grass}(3, 10)$, of dimension 21. It is easy to see that up to isomorphism under the natural Pgl(3) action, an open dense subset of the vector spaces $V = \langle f, g, h\rangle \subset R_2$ have a six-dimensional family of orbits, as dim Grass(3, 10) − dim Pgl(3) = 21 − 15 = 6, and the stabilizer of a general enough net is finite. In this section, we determine the irreducible components of the subfamily $\mathfrak{F}$ of nets having at least one linear relation (Lemma 3.3), and also the possible graded Betti numbers for the algebras $R/(V)$, for nets $V \in \mathfrak{F}$ (Lemma 3.4).

Definition 3.1. We denote by $\mathfrak{F} \subset \mathbb{G} = \mathrm{Grass}(3, R_2)$ the subfamily of nets of quadrics, vector spaces $V = \langle f, g, h\rangle \subset R_2$, for which f, g, h have at least one linear relation

$$\alpha_1 f + \alpha_2 g + \alpha_3 h = 0, \quad \exists \alpha_i \in R_1 = \langle w, x, y, z\rangle. \tag{3.1}$$

We denote by $\mathfrak{F}_i \subset \mathbb{G} = \mathrm{Grass}(3, R_2)$ the subfamily of $\mathfrak{F}$ consisting of those nets that have exactly i linear relations, $i = 1, 2, 3$. We denote by $\mathfrak{F}_{\mathrm{RNC}} \subset \mathfrak{F}_2$ the subset of nets defining twisted cubic curves, and by $\mathfrak{F}_{\mathrm{sp}}$ the subset of nets Pgl(3) isomorphic to $\langle w^2, wx, wy\rangle$.

Lemma 3.2. *The family $\mathfrak{F}_1$ comprises those nets that can be written $V = \langle \ell \cdot U, h\rangle$, where $\ell \in R_1$ is a linear form, $U \subset R_1$ is a two-dimensional subspace of linear forms, and h is not divisible by either ℓ or by any element of U.*

Up to isomorphism $V \in \mathfrak{F}_1$ may be written either $V = \langle xw, yw, h\rangle$ for some quadric h divisible neither by w nor by any element of $\langle x, y\rangle$, or $V = \langle w^2, wx, h\rangle$ with h divisible by no element of $\langle w, x\rangle$.

Proof. First consider nets $V = \langle f, g, h\rangle$ having no two-dimensional subspace with a common factor: we show that V cannot be in $\mathfrak{F}_1$. When the coefficients of a relation as in (3.1) form an R-sequence, a simple argument given in the proof of Lemma 3.3 shows that $V \in \mathfrak{F}_2$, and is determinantal (see Eq. (3.2)).

Now assume that V has a relation as in (3.1) such that $\dim_K \langle \alpha_1, \alpha_2, \alpha_3 \rangle = 2$; after a change of basis in R we may suppose that $xf + yg + (x+y)h = 0$ where h may be zero. Replacing f by $f+h$, and g by $g+h$, we obtain $xf = -yg$. Thus V may be written $V = \langle U\ell, h\rangle$ with $\ell = f/y$ and $U = \langle x, y\rangle$, and, evidently if $V \in \mathfrak{F}_1$ then h is not divisible by ℓ nor by any element of U. We have shown the first claim of the lemma. The second follows. □

As we shall see below, $\mathfrak{F}_2$ has $\mathfrak{F}_{\mathrm{RNC}}$ as open dense subset. Evidently, the family $\mathfrak{F}_3$ of nets V having a common factor, contains as open dense subset the Pgl(3) orbit of $V = \langle wx, wy, wz\rangle$; the family also contains $\mathfrak{F}_{\mathrm{sp}}$, the orbit of $\langle w^2, wx, wy\rangle$.

The dimension calculations of the following lemmas are elementary; recall that dim $\mathbb{G} = 21$. The results about closures also involve standard methods but are more subtle: for example to identify $\mathfrak{F}_{\mathrm{sp}}$ with $\overline{\mathfrak{F}_2} \cap \mathfrak{F}_3$ we rely on previous work on the closure of the family of rational normal curves, such as [24,29,30,32].

Lemma 3.3 (*Components of* $\mathfrak{F}$). *The subfamily* $\mathfrak{F} \subset \mathbb{G} = \mathrm{Grass}(3, R_2)$ *parametrizing quadrics having at least one linear relation, has two irreducible components,* $\overline{\mathfrak{F}_1}$ *and* $\overline{\mathfrak{F}_2} = \overline{\mathfrak{F}_{\mathrm{RNC}}}$, *of codimensions* 7 *and* 9, *respectively, in* G. *They satisfy*

(i) *The intersection* $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2$, *has an open dense subset parametrizing nets isomorphic to* $\langle wx, wy, xz\rangle$; *this intersection has codimension* 11 *in* G.
(ii) *We have* $\overline{\mathfrak{F}_1} - \mathfrak{F}_1 = (\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2) \cup \mathfrak{F}_3$. *Each element of* $\mathfrak{F}_2$ *has a basis consisting of minors of a* 2×3 *matrix of linear forms.*
(iii) *The locus* $\mathfrak{F}_3 \subset \overline{\mathfrak{F}_1}$ *has codimension* 15 *in* G; $\mathfrak{F}_3 - \mathfrak{F}_{\mathrm{sp}}$ *consists of nets isomorphic to* $\langle wx, wy, wz\rangle$. *The locus* $\mathfrak{F}_{\mathrm{sp}} = \overline{\mathfrak{F}_2} \cap \mathfrak{F}_3$, *and is a subfamily of codimension* 16 *in* G.

Proof. We first calculate dim $\mathfrak{F}_1$. By Lemma 3.2 $V \in \mathfrak{F}_1$ may be written as $\langle \ell \cdot U, h\rangle$, where $\ell \in R_1$ and $U \subset R_1$ is a two-dimensional subspace, and h is not divisible by ℓ nor by any element of U. Since there is a single linear relation, V determines both ℓ and U uniquely. Thus, there is a surjective morphism

$$\pi_1 : \mathfrak{F}_1 \to \mathbb{P}^3 \times \mathrm{Grass}(2, R_1) : \pi_1(V) = (\ell, U),$$

The fibre of π_1 over the pair (ℓ, U) corresponds to the choice of h; given V, h is unique up to constant multiple, mod an element of $\ell \cdot U$. Thus, the fibre of π_1 is parametrized by an open dense subset of the projective space $\mathbb{P}(R_2/\langle \ell \cdot U\rangle)$, of dimension 7. Thus, $\mathfrak{F}_1$ has dimension 14, and codimension 7 in G.

We next show that $\mathfrak{F}_2$ contains $\mathfrak{F}_{\mathrm{RNC}}$ as dense open subset. When there is a linear relation for V as in (3.1) whose coefficients α_i are a length 3 regular sequence we may suppose after a coordinate change that $xf + yg + zh = 0$; letting $f = uz + f_1$, $g = vz + g_1$, with f_1, g_1 relatively prime to z, we obtain $h = -(ux + vy)$, and $xf_1 = -yg_1$, whence there is a linear form $\beta \in R_1$ with $f = uz + y\beta$, $g = vz - x\beta$, and (f, g, h) is the ideal of 2×2 minors of

$$\begin{pmatrix} u & v & \beta \\ -y & x & z \end{pmatrix}. \tag{3.2}$$

When also (f, g, h) has height two, then V is an element of $\mathfrak{F}_2$ determining a twisted cubic in $\mathbb{P}^3$; for a dense open subset of such elements of $\mathfrak{F}_2$ one may up to isomorphism choose in (3.2) the triple $(u, v, \beta) = (x, z, w)$. Otherwise, if f, g, h is not Cohen–Macaulay of height two, V has a common linear factor, and it is well known that then $V \in \mathfrak{F}_{\mathrm{sp}} = \overline{\mathfrak{F}_2} \cap \mathfrak{F}_3$ [24,30,32].

We now consider those nets $V \subset \mathfrak{F}_2$ for which there is no linear relation as in (3.1) whose coefficients form a length three R-sequence. By the proof of Lemma 3.2 such a net has the form $V = \langle Uw, h\rangle$, with $U \subset R_1$, and it thus lies in the closure of $\mathfrak{F}_1$. It is easy to see that the most general element of $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2$ is a net isomorphic to $\langle wx, wy, xz\rangle$: for when $V = \langle wx, wy, h\rangle$ has a second linear relation, either w divides h and $V \in \mathfrak{F}_3$, or some $ax + by$ divides h, and after a change in basis for R_1, $V \cong \langle wx, wy, xz\rangle$. A similar discussion for $\langle w^2, wx, h\rangle$ completes the proof that any element of $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2$ is in the closure of the orbit of $V = \langle wx, wy, xz\rangle$, which is also the determinantal ideal of $\begin{pmatrix} x+y & y & 0 \\ z & z & w \end{pmatrix}$. This shows also that $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2 \subset \overline{\mathfrak{F}_{\mathrm{RNC}}}$, and completes the proof that $\mathfrak{F}_2$ contains $\mathfrak{F}_{\mathrm{RNC}}$ as dense open subset.

We recall that $\dim \mathfrak{F}_{\mathrm{RNC}} = 12$. A twisted cubic—a rational normal curve of degree three—is determined by the choice of four degree three forms in the polynomial ring $K[x, y]$, up to common K^*-multiple, mod the action of Pgl(1), yielding dimension $4 \cdot 4 - 4 = 12$ [30].

We have that $\overline{\mathfrak{F}_1}$ and $\overline{\mathfrak{F}_2}$ define two distinct irreducible components of $\mathfrak{F}$, since the subfamily $\mathfrak{F}_2$ parametrizing nets for which there are two linear relations, cannot specialize to any net $V = \langle f, g, h\rangle$ for which f, g, h have a single linear relation; and $\mathfrak{F}_1$, parametrizing nets V each containing a subspace of the form $\ell \cdot U$, cannot specialize to a vector space V for which the ideal (V) is the prime ideal of a twisted cubic. This completes the proof of the initial claims of the lemma.

We now complete the proof of (i), by determining the dimension of $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2$, which is by the above argument equal to the dimension of the Pgl(3)-orbit $\mathscr{B}$ of $\langle wx, wy, xz\rangle$. For $W = \langle w'x', w'y', x'z'\rangle \in \mathscr{B}$, the unordered pair of linear forms (w', x'), each mod K^*-multiple is uniquely determined by W (as each divides a two-dimensional subspace of W): thus there is a morphism $\pi : \mathscr{B} \to \mathrm{Sym}^2(\mathbb{P}^3)$, from $\mathscr{B}$ to the symmetric product, whose image is the nondiagonal pairs. Spaces W in the fibre of π over (w', x') are determined by the choice of the two two-dimensional subspaces, the first $\langle x', y'\rangle$ containing x', the second $\langle w', z'\rangle$ containing w'. Thus, a space W in the fibre is determined by the choice of $y' \in R_1/\langle x'\rangle$ and $z' \in R_1/\langle w'\rangle$, each up to K^*-multiple, and these choices are each made in an open dense subset of $\mathbb{P}^2$ (as z' must not equal x' mod w' for $W \in \mathscr{B}$). Thus, the fibre $\pi^{-1}(w', x') \subset \mathscr{B}$ is isomorphic to an open dense subset of $\mathbb{P}^2 \times \mathbb{P}^2$. It follows that $\mathscr{B}$ and $\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2$ have dimension 10, and codimension 11 in G.

We now show the claim in (ii) that $\overline{\mathfrak{F}_1} - \mathfrak{F}_1 = (\overline{\mathfrak{F}_1} \cap \mathfrak{F}_2) \cup \mathfrak{F}_3$. Suppose that $V \in \overline{\mathfrak{F}_1} - \mathfrak{F}_1$; then evidently there is a two-dimensional subspace $V_1 \subset V$ having a common factor $V_1 = \ell \cdot U$. Letting $V = \langle V_1, h\rangle$ then $V \in \mathfrak{F}_2$ implies h must have a common divisor with an element of V_1. Thus, up to Pgl(3) isomorphism we have $V = \langle wx, wy, xz\rangle$ or $V = \langle w^2, wx, xz\rangle$, both in $\mathfrak{F}_2$ (we may ignore w is a common factor of V since then $V \in \mathfrak{F}_3$). Each of these spaces has basis the minors of a 2×3 matrix of linear forms. This with (3.2) above completes the proof of (ii).

The family $\mathfrak{F}_3$ has as open dense subset the orbit $\mathscr{B}'$ of $V = \langle wx, wy, wz\rangle$. An element $W' = w'V'$, $V' \subset R_1$ of $\mathscr{B}'$ is determined by a choice of $w' \in R_1$ and a codimension one vector space $V' \subset R_1$, thus $\mathscr{B}'$ is an open in $\mathbb{P}^3 \times \mathbb{P}^3$, so has dimension six, codimension 15 in G.

The claim in (iii) that the locus $\mathfrak{F}_{\mathrm{sp}} = \overline{\mathfrak{F}_2} \cap \mathfrak{F}_3$ follows from the well-known classification of the specializations of rational normal curves [24,30]; the dimension count for this locus is five, 3 for the choice of w, and 2 for the choice of $\langle x, y\rangle \subset R_2/\langle w^2\rangle$. This completes the proof of Lemma 3.3. □

Lemma 3.4 (*Minimal resolutions for nets of quadrics in* $\mathfrak{F}$). *There are exactly three possible sets of graded Betti numbers for the ideal generated by a net of quadrics in* $\mathfrak{F}$ (*those having at least one linear relation*):

(i) *Those* V *in the family* $\mathfrak{F}_1$ *have graded Betti numbers that of* (wx, wy, z^2), *with a single linear and two quadratic relations, and Hilbert function* $H = H(R/(V)) = (1, 4, 7, 9, 11, 13, \ldots)$ *where* $H_i = 2i + 3$ *for* $i \geqslant 2$. *Such* V *define a curve of degree* 2, *genus* -2 (*See Lemma* 3.5).
(ii) *For* $V \in \mathfrak{F}_2$, *the ideal* (V) is *Cohen–Macaulay of height two, the Hilbert function* $H = H(R/(V)) = (1, 4, 7, 10, 13, \ldots)$ *where* $H_i = 3i + 1$ *for* $i \geqslant 0$, *and* V *has the standard determinantal minimal resolution with two linear relations*.
(iii) *Those* V *in the family* $\mathfrak{F}_3$ *have graded Betti numbers that of* (wx, wy, wz).

Proof. For (i), Lemma 3.2 implies that the quotient $R/(V)$ determined by an element V of $\mathfrak{F}_1$ is cut out from $R/(wx, wy)$ or $R/(w^2, wx)$ by the nonzero-divisor h, hence the minimal resolution of $R/(V)$ is that of $R/(wx, wy, z^2)$. For (ii) let $V \in \mathfrak{F}_2$. Then by Lemma 3.3(ii), V is has a basis consisting of the minors of a 2×3 matrix of linear forms; an examination of cases shows that V is Cohen–Macaulay of height two, so is determinantal. Thus V has the standard determinantal minimal resolution. The last part (iii) follows immediately from Lemma 3.3(iii), and a computation in Macaulay. □

Lemma 3.5. [24, *Sections* 3.4–3.6] *The Hilbert scheme* $\mathrm{Hilb}^{2,-2}(\mathbb{P}^3)$ *parametrizing curves* $C \subset \mathbb{P}^3$ *of degree* 2, *genus* -2 (*Hilbert polynomial* $2t + 3$) *has two irreducible components*. *A general point of the first parametrizes a scheme consisting of two skew lines union a point off the line*; *this component has dimension* 11. *A general point of the second component parametrizes a planar conic union two points*; *this component has dimension* 14.

Likewise, [24, *Theorem* 3.5.1] $\mathrm{Hilb}^{2,-1}(\mathbb{P}^3)$ (*Hilbert polynomial* $2t+2$) *has the analogous components parametrizing two skew lines, or a planar conic union a point*. *The scheme* $\mathrm{Hilb}^{2,0}(\mathbb{P}^3)$ (*Hilbert polynomial* $2t + 1$) *has a single component, whose generic points parametrize plane conics*.

The following result mostly concerns certain ideals I for which I_3 to I_4 or I_4 to I_5 is of extremal growth in the sense of F.H.S. Macaulay. We thank a referee for the simple argument for (ii). Note that nets V with no linear relation need not define complete intersections, and the ideal (V) need not be saturated: thus (iii) below does not follow from (ii).

Lemma 3.6. *Assume for* (i),(ii) *below that* I *is a saturated ideal of* $R = K[w, x, y, z]$.

(i) *If* $H(R/I) = (1, 4, 7, 10, 13, 16, \dots)$, *then* $I_{\leqslant 3}$ *defines a twisted cubic* (*or specialization not in the closure of the plane cubics*) *or a plane cubic union a point* (*possibly embedded*). *In the former case,* I_2 *lies in* $\overline{\mathfrak{F}_2}$, *and generates* I; *in the latter case* $I_2 \in \mathfrak{F}_3$.

(ii) $H(R/I)$ *cannot be any of* $(1, 4, 7, 8, 10, \dots)$, $(1, 4, 7, b, 9, 11, \dots)$, *or* $(1, 4, 7, 9, 12, \dots)$.

(iii) *If* R/I *is Artinian Gorenstein of socle degree at least* 5, *then* $R/(I_2)$ *cannot have a Hilbert function of the form* $H(R/(I_2)) = (1, 4, 7, 8, 10, \dots)$, $H(R/(I_2)) = (1, 4, 7, b, 9, 11, \dots)$, *or* $H(R/(I_2)) = (1, 4, 7, 9, 12, \dots)$.

Proof. Suppose that a saturated ideal I has the Hilbert function given in case (i). Then 13 to 16 is an extremal growth. So, by the Gotzmann theorem I defines a scheme $\mathfrak{Z} \subset \mathbb{P}^3$, of Hilbert polynomial $3t+1$ so $\mathfrak{Z}$ is a degree three curve of genus zero. The Piene–Schlessinger theorem characterizing the components of $\mathrm{Hilb}^{3,0}(\mathbb{P}^3)$ [30] implies that if $\mathfrak{Z}$ is nondegenerate (not contained in a plane), then $\mathfrak{Z}$ is either a twisted cubic or a specialization, so I_2 is in $\overline{\mathfrak{F}_2}$, or $\mathfrak{Z}$ is the union of a planar cubic and a (possibly embedded) spatial point, and then I_2 is in $\mathfrak{F}_3$. If $\mathfrak{Z}$ is degenerate, then also $I_2 \in \mathfrak{F}_3$. This completes the proof of (i).

The three sequences of (ii) cannot occur for a saturated ideal I: a saturated ideal has depth at least one, so $A = R/I$ has a (linear) nonzero divisor, and the first differences $\Delta H(R/I)$ must be admissible. But $(1, 3, 3, 1, 2, ..)$, $(1, 3, 3, b-7, 9-b, 2, ..)$ and $(1, 3, 3, 2, 3, ..)$ are not O-sequences.

In the first case of (iii) we have that $10 = 8^{(3)}$, so by Theorem 2.3(ii) $\mathfrak{Z} = \mathrm{Proj}\ (R/(I_3))$ is a scheme of Hilbert polynomial $2t + 2$ (degree two and genus -1) and regularity degree no more than 3, the Gotzmann regularity degree of $2t + 2$. By a classical degree inequality, such a scheme is either reducible, or degenerate—contained in a hyperplane [16, p. 173]. Furthermore, by Lemma 3.5 the Hilbert scheme $\mathrm{Hilb}^{2,-1}(\mathbb{P}^3)$ of degree two genus -1 curves has two irreducible components, one whose generic point parametrizes two skew lines, the second, whose generic point parametrizes a planar conic union a point. For either component, the Hilbert function $H(R/I_{\mathfrak{Z}})_2 \leqslant 6$ which by Corollary 2.2 implies $H(R/I)_2 \leqslant 6$, contradicting the assumption. A similar argument handles the second case of (iii): since $9^{(4)} = 11$, $H_{4,5} = (9, 11)$ is maximal growth; by Theorem 2.3(ii) the scheme $\mathfrak{Z} = \mathrm{Proj}\ (R/(I_4))$ has Hilbert polynomial $2t + 1$, of Gotzmann regularity two implying $H(R/I_{\mathfrak{Z}})_2 = 5$, and by Corollary 2.2, $H(R/I)_2 \leqslant 5$, a contradiction. For the last case it suffices by Corollary 2.2 and the Gotzmann Theorem to know that any scheme of Hilbert polynomial $3t$ (degree three and genus one) is a planar cubic or degenerate, a result of the classification of curves [30,24]. □

3.2. Ideals with $I_2 = \langle wx, wy, wz \rangle$

Let $\mathfrak{V}$ denote the vector space $\langle wx, wy, wz \rangle$. In this section we assume $H = (1, 4, 7, \dots, 1)$ and we consider the subfamily $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ parametrizing those algebras $A = R/I$ of Hilbert function H for which I_2 is $\mathrm{Pgl}(3)$ isomorphic to $\mathfrak{V}$. We first determine when $\mathfrak{C}(H)$ is nonempty and give a structure theorem for such A (Theorem 3.7). We then determine the minimal resolution of A (Theorem 3.9). We also determine the tangent space to the family

$\mathfrak{C}(H)$ (Theorem 3.11). To prove our results we connect these Artinian algebras with height three Artinian Gorenstein quotients R'/J_I of $R' = K[x, y, z]$, where $J_I = I \cap R'$, which are well understood [9,12,20,22].

We recall from Lemma 2.1ff. that, given an ideal I of R, we denote by $I^{\perp}$ its inverse system, the perpendicular R-submodule to I in the divided power ring $\mathscr{D} = K_{\mathrm{DP}}[W, X, Y, Z]$, where R acts by contraction.

Theorem 3.7. *Let $H = (1, 4, 7, \ldots)$ of socle degree $j \geqslant 4$ be a Gorenstein sequence, and assume that $I \in \mathfrak{C}(H)$ satisfies $I_2 = \mathfrak{V} = \langle wx, wy, wz\rangle$. Let $F \in \mathscr{D}_j$ satisfy $I = \mathrm{Ann}\ (F)$. Let $R' = K[x, y, z]$. Then,*

(i) *The inverse system $(\mathfrak{V})^{\perp}$ of the ideal $(\mathfrak{V})$, $\mathfrak{V} = \langle wx, wy, wz\rangle \subset R$, satisfies*

$$(\mathfrak{V})_j^{\perp} = \langle K_{\mathrm{DP}}[X, Y, Z]_j, W^{[j]}\rangle. \tag{3.3}$$

(ii) *$F \in K_{\mathrm{DP}}[W, X, Y, Z]_j$ and satisfies*

$$F = G + \alpha \cdot W^{[j]}, \quad G \in K_{\mathrm{DP}}[X, Y, Z]_j,\ \alpha \in K, \tag{3.4}$$

where $G \neq 0, \alpha \neq 0$.
Furthermore, $I = (J_I, \mathfrak{V}, f)$ where $J_I = I \cap R'$ is the height three Gorenstein ideal $Ann_{R'}(G)$ and $f = w^j - g, g \in K[x, y, z]_j, g \neq 0$.
The Hilbert function $H(R/I)_i = H(R'/J_I)_i + 1$ for $1 \leqslant i \leqslant j - 1$, so we have

$$H(R/I) = H(R'/J_I) + (0, 1, 1, \ldots, 1, 0) = (1, 4, \ldots, 4, 1). \tag{3.5}$$

The inverse system $I^{\perp}$ satisfies $I_j^{\perp} = \langle F\rangle$, $I_i^{\perp} = 0$ for $i \geqslant j + 1$, and

$$I_i^{\perp} = (R \circ F)_i = \langle (R' \circ G)_i, W^{[i]}\rangle \quad \textit{for } 1 \leqslant i \leqslant j - 1. \tag{3.6}$$

(iii) *The Gorenstein sequence $H = (1, 4, 7, \ldots)$ satisfies $\mathfrak{C}(H)$ is nonempty if and only if $H' = H - (0, 1, 1, \ldots, 1, 0)$ is a Gorenstein sequence of height three.* (*See Corollary* 4.3).

Proof. We first prove (i). Since $\mathfrak{V} = (wx, wy, wz) = w \cap (x, y, z)$ we have from the properties of the Macaulay duality,

$$(wx, wy, wz)^{\perp} = (w)^{\perp} + (x, y, z)^{\perp} = K_{\mathrm{DP}}[X, Y, Z] + K_{\mathrm{DP}}[W],$$

which is (3.3).

We now show (ii). Since F generates $(I_j)^{\perp}$, $F \in (\mathfrak{V})_j^{\perp}$ can be written $F = G + \alpha W^{[j]}$ as in (3.4). Since $H(R/I) = (1, 4, \ldots)$, we have $G \neq 0$ and $\alpha \neq 0$. The inverse system relation (3.6) is immediate, and gives

$$R \circ F = R'_{\geqslant 1} \circ h + \langle W, W^{[2]}, \ldots, W^{[j-1]}, F\rangle,$$

as well as the Hilbert function equality (3.5). Let $J_I = \mathrm{Ann}\ (G) \cap K[x, y, z]$: evidently, $\mathrm{Ann}\ (G) = (w, J_I)$. Let $h \in I \cap K[x, y, z]$. Then we have $h \circ F = 0$ and $h \circ W^j = 0$, implying $h \circ G = 0$ so $h \in J_I$; conversely, if $h \in J_I = \mathrm{Ann}\ (G) \cap K[x, y, z]$ then $h \circ G = 0$, $h \circ W^j = 0$, implying $h \circ F = 0$, so $h \in I \cap K[x, y, z]$. Thus $J_I = I \cap K[x, y, z]$, as claimed in (ii) is immediate. Now, any form h of degree less than j satisfying $h \cdot F = 0$, and $h \neq (wx, wy, wz)$ must satisfy $h \in K[x, y, z]$ and hence is in J_I. If $f = w^j - g$ with $g \circ G = \alpha$ then we have $f \circ F = 0$ and hence $f \in I$. If $g = 0$ we would have $R_1 \cdot w^{j-1} \in I$, implying that $w^{j-1} \bmod I$ is a socle element of $A = R/I$, contradicting the assumption that A is Artinian Gorenstein of socle degree j. Thus, we have $f = w^j - g$ with $g \neq 0$. Since the lowest-degree third syzygy of I are those in degree four arising from $\mathfrak{V}$, the symmetry of the minimal resolution implies that I has no generators (first syzygies) in degrees greater than j. Thus the ideal $I \in \mathfrak{F}$ is minimally generated as $I = (J_I, \mathfrak{V}, f)$ as claimed, and this completes the proof of (ii).

To show (iii), note that if $I \in \mathfrak{C}(H)$ then H' from (iii) satisfies $H' = H(R/J_I) = H(R'/(I \cap R'))$ with $I \cap R'$ a Gorenstein ideal in R', so H' is a Gorenstein sequence. Conversely if $H' = H - (0, 1, 1, \ldots, 1, 0)$ is a Gorenstein sequence then take J' to be any Gorenstein ideal in R' of Hilbert function H' and let $J' = \mathrm{Ann}_{R'}(G)$. Let $F = G + W^j$. Then $\mathrm{Ann}\,(F) = I = (J', w^j - g, wx, wy, wz)$ where $g \in R'_j$ but $g \notin J'$: the ideal I is a Gorenstein ideal of height four. Then we have $I \in \mathfrak{C}(H)$.

Thus, $\mathfrak{C}(H)$ is nonempty if and only if $H' = H - (0, 1, 1, \ldots, 1, 0) = (1, 3, \ldots, 3, 1)$ is a Gorenstein sequence of height three. This completes the proof. □

The minimal resolution of R/I can be constructed from the minimal resolution of J_I. We construct a putative complex in Definition 3.8; we prove that it is an exact complex in Theorem 3.9. The construction relies on Theorem 3.7(ii).

Suppose that $I \subset R$ defines a Artinian Gorenstein quotient $A = R/I$, that $I_1 = 0$ and $I_2 = \mathfrak{V}$, and that $I = (\mathfrak{V}, J_I, g - w^j)$ with $g \in R'_j$ satisfying $g \neq 0$, and $J' = J_I = I \cap R' = K[x, y, z]$ defining a Artinian Gorenstein quotient $A' = R/J'$ of R'. Let the minimal resolution of $R/J,\ J = J_I R$ be (here $m = 2n + 1$ is odd)

$$\mathbb{J}: \quad 0 \to R \xrightarrow{\alpha^{\mathrm{t}}} R^m \xrightarrow{\phi} R^m \xrightarrow{\alpha} R \to R/J \to 0, \tag{3.7}$$

where ϕ is an $m \times m$ alternating matrix with homogeneous entries, and $\alpha = [J]$ denotes the $1 \times m$ row vector with entries the homogeneous generators of J that are the Pfaffians of ϕ, according to the Buchsbaum–Eisenbud structure theorem for height three Gorenstein ideals (since J is homogeneous, $\mathbb{J}$ may be chosen homogeneous: see [9,12]). Denote by $\mathbb{K}$ the Koszul complex resolving $R/(x, y, z)$ (so $\mathbb{K}_0 = \mathbb{K}_3 = R$):

$$\mathbb{K}: \quad 0 \to R \xrightarrow{\delta_3} R^3 \xrightarrow{\delta_2} R^3 \xrightarrow{\delta_1} R \to R/(x, y, z) \to 0, \tag{3.8}$$

where

$$\delta_1 = [x, y, z], \quad \delta_2 = \begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix},$$

and $\delta_3 = \delta_1^{\mathrm{t}}$. We will let $\mathbb{T} : \mathbb{K} \to \mathbb{J}$ be a map of complexes induced by multiplication by g on R. By degree considerations, we see that $\deg T_3 = 0$, so T_3 is multiplication by $\gamma \in K$.

So we have $T_1 \circ \delta_2 = \phi \circ T_2$, also $T_2 \circ \delta_3 = [J]^{\mathrm{t}}$, and

$$T_2 \circ \begin{bmatrix} z \\ y \\ x \end{bmatrix} = \gamma \circ [J]^{\mathrm{t}}.$$

Definition 3.8. Given I, J, $\mathbb{J}$, $\mathbb{K}$ as above, we define the following complex:

$$\mathbb{F}: \quad 0 \to R \xrightarrow{F_4} R^{m+4} \xrightarrow{F_3} R^{2m+6} \xrightarrow{F_2} R^{m+4} \xrightarrow{F_1} R \to R/I \to 0, \tag{3.9}$$

where $F_1 = (wx, wy, wz, \alpha, w^j - g)$, and F_2 satisfies

$$F_2 = \left(\begin{array}{c|cccc} & 3 & m & m & 3 \\ \hline 3 & \delta_2 & 0 & \frac{-1}{\gamma} E \circ T_2^{\mathrm{t}} & w^{j-1} I_{3\times 3} \\ m & 0 & \phi & w I_{m\times m} & T_1 \\ 1 & 0 & 0 & 0 & -x \ -y \ -z \end{array}\right), \tag{3.10}$$

where $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The map F_3 satisfies

$$F_3 = \left(\begin{array}{c|ccc} & 3 & m & 3 \\ \hline 3 & w^{j-1} I_{3\times 3} & E T_1^{\mathrm{t}} & -z \ -y \ -x \\ m & T_2 & -\gamma w I_{m\times m} & 0 \\ m & 0 & \gamma\phi & 0 \\ 3 & -\delta_2 & 0 & 0 \end{array}\right) \tag{3.11}$$

and $F_4 = (wz, wy, wx, \alpha, w^j - g)^{\mathrm{t}}$.

Theorem 3.9. *Let I be a homogenous height four Gorenstein ideal in $R = K[w, x, y, z]$ with socle degree j and with $I_2 = (wx, wy, wz)$. Then the complex $\mathbb{F}$ of* (3.9) *in Definition* 3.8 *is exact and is the minimal resolution of R/I.*

Proof. We first show that $\mathbb{F}$ is a complex. By (ii) of the structure theorem, we see that I is minimally generated by $J = I \cap K[x, y, z]$, wx, wy, wz, $g - w^j$ where $g \in K[x, y, z]$. So, $g \notin J$. Suppose that $\gamma = 0$. Then $T_2 \circ \delta_3 = 0$, hence we would have $T_2 = T' \circ \delta_2$ for some T'. Then

$$\begin{aligned} &T_1 \circ \delta_2 = \phi \circ T_2 = \phi \circ T' \circ \delta_2; \\ &\text{so} \quad (T_1 - \phi \circ T') \circ \delta_2 = 0, \\ &T_1 - \phi \circ T' = \beta[x, y, z], \quad \beta \in K, \\ &\alpha \circ T_1 = \alpha \circ \beta[x, y, z], \\ &- g[x, y, z] = \alpha\beta[x, y, z]. \end{aligned}$$

This implies $g \in J$ contradicting $g \notin J$. So, we get $\gamma \neq 0$.

We get $F_1 \circ F_2 = 0$ and $F_3 \circ F_4 = 0$ from the following three identities. First, from the exact sequence $\mathbb{J}$ of (3.7) we have

$$\phi\alpha = \alpha^{\mathrm{t}}\phi = 0. \tag{3.12}$$

Second, from

$$\begin{aligned}[x,y,z]\left(\left(\frac{-1}{\gamma}\right)ET_2^{\mathrm{t}}\right) &= \frac{-1}{\gamma}[zyx]T_2^{\mathrm{t}}\\ &= \frac{-1}{\gamma}\left[T_2\begin{bmatrix} z\\ y\\ x\end{bmatrix}\right]^{\mathrm{t}} = \frac{-1}{\gamma}\gamma(\alpha^{\mathrm{t}})^{\mathrm{t}}\\ &= -\alpha\end{aligned}$$

$$\text{we have}\quad [x,y,z]\left[\frac{-1}{\gamma}ET_2^{\mathrm{t}}\right] = -\alpha. \tag{3.13}$$

Third, we have

$$T_1 J = -g[x,y,z]. \tag{3.14}$$

To see that $F_2 \circ F_3 = 0$ we just need to check that

$$\begin{aligned}&\phi T_2 - T_1\delta_2 = 0\\ \text{and}\quad &\delta_2 ET_1^{\mathrm{t}} - \frac{1}{\gamma}ET_2^{\mathrm{t}}(\gamma\phi) = 0.\end{aligned}$$

The first of these follows from the map of complexes $\mathbb{T} : \mathbb{K} \to \mathbb{F}$. For the second we have

$$\begin{aligned}\delta_2 ET_1^{\mathrm{t}} - ET_2^{\mathrm{t}}\phi &= \delta_2 ET_1^{\mathrm{t}} + ET_2^{\mathrm{t}}\phi^{\mathrm{t}}\\ &= \delta_2 ET_1^{\mathrm{t}} + E(\phi T_2)^{\mathrm{t}}\\ &= \delta_2 ET_1^{\mathrm{t}} + E(T_1\delta_2)^{\mathrm{t}}\\ &= \delta_2 ET_1^{\mathrm{t}} + E(\delta_2^{\mathrm{t}})T_1^{\mathrm{t}}\\ &= \left(\delta_2 E + E\delta_2^{\mathrm{t}}\right)T_1^{\mathrm{t}} = 0,\end{aligned}$$

$$\text{since}\quad \delta_2 E + E\delta_2^{\mathrm{t}} = 0.$$

So we get $F_2F_3 = 0$. Thus, $\mathbb{F}$ is a complex.

To see that the complex $\mathbb{F}$ is exact, we use the exactness criterion [8,13, Theorem 20.9]. It suffices to show that $\sqrt{I_{m+3}(F_2)}$ and $\sqrt{I_{m+3}(F_3)}$ have depth at least three, where $I_{m+3}(F_2)$

denotes the Fitting ideal generated by the $(m+3)\times(m+3)$ minors of F_2. We write F_2 as

$$F_2 = \left(\begin{array}{c|cccc} & 3 & m & m & 3 \\ \hline 3 & \begin{smallmatrix} y & z & 0\\ -x & 0 & z\\ 0 & -x & -y\end{smallmatrix} & 0 & \begin{smallmatrix}\tau_{11} & \dots & \tau_{1m}\\ & \dots & \\ \tau_{31} & \dots & \tau_{3m}\end{smallmatrix} & \begin{smallmatrix} w^{j-1} & 0 & 0\\ 0 & w^{j-1} & 0\\ 0 & 0 & w^{j-1}\end{smallmatrix} \\ m & 0 & \phi & \begin{smallmatrix} w & 0 & 0\\ 0 & w & 0\\ 0 & 0 & w\end{smallmatrix} & T_1 \\ 1 & 0 & 0 & 0 & -x \ -y \ -z \end{array}\right), \tag{3.15}$$

where $x\tau_{1i}+y\tau_{2i}+z\tau_{3i}=-\alpha_i$, and $J=(\alpha_1,\dots,\alpha_m)$. Consider the minor M_i of F_2 having all rows except the $(3+i)$th row, and having the columns $1, 2, 4, \dots, 3+i-1, 3+i+1, m+3, m+3+i, 2m+4$. This is the minor

$$M_i = \begin{vmatrix} y & z & & t_{1i} & w^{j-1} \\ -x & 0 & 0 & t_{2i} & 0 \\ 0 & -x & & t_{3i} & 0 \\ & & & & * \\ 0 & & \phi_i & 0 & * \\ & & & & * \\ 0 & & 0 & 0 & -x \end{vmatrix}, \tag{3.16}$$

and it equals

$$\begin{aligned} &\pm x a_i^2 \begin{vmatrix} y & -z & t_{1i} \\ -x & 0 & t_{2i} \\ 0 & -x & t_{3i} \end{vmatrix} \\ &= \pm x a_i^2 x(x\tau_{1i} + y\tau_{2i} + z\tau_{3i}) \\ &= \pm x^3 a_i^2. \end{aligned}$$

Thus $xa_i \in \sqrt{I(F_2)}$. Similarly, $ya_i, za_i \in \sqrt{I(F_2)}$. Thus $mJ \subset \sqrt{I(F_2)}$. Finally, looking at the last $m+3$ rows and the columns $1, 2, m+4, \dots, 2m+4$, we get $\pm x^3 w^m$ in $I(F_2)$. So $wx \in \sqrt{I(F_2)}$, as well as wy, wz, by similar computations. Thus $\sqrt{I(F_2)} \supset (J, wx, wy, wz)$. Similarly $\sqrt{I(F_3)} \supset (J, wx, wy, wz)$. So these Fitting ideals have depth at least three, and the complex $\mathbb{F}$ is exact. This completes the proof. □

Remark 3.10. The above resolution in Theorem 3.9 is similar to but different from the minimal resolution obtained by Kustin and Miller in [23]. They consider ideals of the form (f, g, h, wJ) where (f, g, h) is a regular sequence and J is height three Gorenstein. It turns out that it is not a specialization of their resolution. One reason for the resemblance is that (wx, wy, wz) has three Koszul type relations even though they are not a regular sequence.

If $H(R/I) = (1, 4, 7, h, 7, 4, 1)$, recall that $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ denotes the subfamily parametrizing ideals I such that $I_2 \cong \mathfrak{V} = \langle wx, wy, wz\rangle$, up to a coordinate change. We denote by $v_i(J)$ the number of degree-i generators of J. We will later show that any Gorenstein sequence $H = (1, 4, 7, \dots)$ satisfies $\mathfrak{C}(H)$ nonempty (Theorem 4.2). For $I \in \mathbb{P}\mathrm{Gor}(H)$ we denote by $\mathscr{T}_I$ the tangent space to the affine cone over $\mathbb{P}\mathrm{Gor}(H)$ at the point corresponding to $A = R/I$. Recall that $H' = H - (0, 1, 1, \dots, 1, 0)$. We denote by $\mathscr{T}_{J_I}$ the tangent space to the affine cone over $\mathbb{P}\mathrm{Gor}(H')$, $H' = H(R/J_I)$ from (3.5), at the point corresponding to $A' = R'/J_I$, where $J_I = I \cap K[x, y, z]$.

Theorem 3.11. *Let* $H = (1, 4, 7, \ldots)$ *of socle degree* $j \geqslant 5$. *In* (i), (ii), (iii) *we let* $A = R/I \in \mathfrak{C}(H)$, *and we let* $J_I = I \cap K[x, y, z]$.

(i) *The dimension of* $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ *satisfies*

$$\dim(\mathfrak{C}(H)) = 7 + \dim \mathbb{P}\mathrm{Gor}(H'). \tag{3.17}$$

(ii) *The dimension of the tangent space* $\mathscr{T}_I$ *to the affine cone over* $\mathbb{P}\mathrm{Gor}(H)$ *at the point determined by* $A = R/I \in \mathbb{P}\mathrm{Gor}(H)$ *satisfies,*

$$\dim_K \mathscr{T}_I = 7 + \dim_K \mathscr{T}_{J_I} + v_{j-1}(J_I). \tag{3.18}$$

(iii) *The GA algebra* $A \in \mathfrak{C}(H)$ *is a smooth point of* $\mathbb{P}\mathrm{Gor}(H)$ *if and only if* $v_{j-1}(J_I) = 0$.
(iv) *The subscheme* $\mathfrak{C}(H)$ *of* $\mathbb{P}\mathrm{Gor}(H)$ *is irreducible.*
(v) *When* $j = 6$ *and* $H = H_h = (1, 4, 7, h, 7, 4, 1)$, $7 \leqslant h \leqslant 11$ *we have*

$$\dim(\mathfrak{C}(H)) = 34 - \binom{h^\vee + 1}{2}, \quad h^\vee = 11 - h. \tag{3.19}$$

When also, $8 \leqslant h \leqslant 11$, $\mathfrak{C}(H)$ *is generically smooth.*

Proof. The proof of (i) is immediate from the structure Theorem 3.7(ii): the choice of $\mathfrak{V}$ involves that of w and the vector space $\langle x, y, z\rangle$, so 6 dimensions, and that of the $F = w^j + G$ involves one parameter, given $\langle G\rangle$, which determines J_I.

We now show (ii). Let $A = R/I \in \mathfrak{C}(H)$. We recall from [20, Theorem 3.9] that for a GA quotient $A = R/I$, we have $\dim_K \mathscr{T}_I = \dim_K R_j/(I^2)_j = H(R/I^2)_j$. We have

$$\begin{aligned}(I^2)_j &= I_2 \cdot I_{j-2} \oplus (J^2)_j \\ &= \left(wR'_1 \cdot ((w^{j-3}R'_1 \oplus w^{j-4}R'_2 \oplus \cdots \oplus wR'_{j-3}) \oplus J_{j-2})\right) \oplus (J^2)_j \\ &= \left(w^{j-2}R'_2 \oplus w^{j-3}R'_3 \oplus \cdots \oplus w^2R'_{j-2}\right) \oplus wR'_1J_{j-2} \oplus (J^2)_j.\end{aligned}$$

Hence we have

$$R_j/(I^2)_j \cong w^j \oplus w^{j-1}R'_1 \oplus w(R'_{j-1}/R'_1J_{j-2}) \oplus R'_j/(J_j)^2, \quad \text{and}$$

$$\begin{aligned}\dim_K R_j/(I^2)_j &= 1 + 3 + H'_{j-1} + v_{j-1}(J) + \dim_K R'_j/(J_j)^2 \\ &= 7 + \dim_K \mathscr{T}_{J_I} + v_{j-1}(J_I).\end{aligned}$$

We now show (iii). We use J.-O. Kleppe's result that in codimension 3, $\mathbb{P}\mathrm{Gor}(H')$ is smooth [22]. It follows that for the Gorenstein ideal $J_I \subset R' = K[x, y, z]$, of socle degree j, of Hilbert function $H(R'/J) = H'$ the dimension of the tangent space $\mathscr{T}_{J_I}$ to the affine cone over $\mathbb{P}\mathrm{Gor}(H')$ at J_I satisfies

$$\dim_K T_{J_I} = \dim(\mathbb{P}\mathrm{Gor}(H')) + 1.$$

This, together with (i), (ii) shows that $v_{j-1}(J_I)=0$ implies $\dim_K \mathscr{T}_I = \dim \mathfrak{C}(H)+1$, hence that $\mathfrak{C}(H)$ and $\mathbb{P}\mathrm{Gor}(H)$ are smooth at such points, which is (iii).

We now show (iv). We first show that $\mathfrak{C}(H)$ is irreducible. The scheme $\mathbb{P}\mathrm{Gor}(H')$ is irreducible by Diesel [12] (or by its smoothness [22], discovered later). The scheme $\mathfrak{C}(H)$, is fibred over the family of nets isomorphic to $\mathfrak{V}$ by $\mathbb{P}\mathrm{Gor}(H')$, then by an open in $\mathbb{P}^1$ (to choose F given G), so it is irreducible.

We now show (v). The dimension formula (3.19) results immediately from (i) and the known dimension of $\mathbb{P}\mathrm{Gor}(H')$ (see [20, Theorem 4.1B], [22]). From the latter source, we have that the codimension of $\mathbb{P}\mathrm{Gor}(H')\subset\mathbb{P}^{27}$, $H'=(1,3,6,h-1,6,3,1)$ is $\binom{h^\vee+1}{2}$ where $h^\vee=10-(h-1)$. When also $8\leqslant h\leqslant 11$, we have $\Delta^3(H')_5=0$; it follows simply from [12] (or see [20, Theorem 5.25]) that the generic GA quotient R'/J having Hilbert function H' satisfies $v_5(J)=0$. This completes the proof of (v) and of the Theorem. □

3.3. *Mysterious Gorenstein algebras with* $I_2=\langle w^2,wx,wy\rangle$

Let $\mathfrak{W}$ denote the vector space $\langle w^2,wx,wy\rangle$. In this section we assume $H=(1,4,7,\ldots,1)$ and study graded Artinian Gorenstein algebras $A=R/I$, $R=K[w,x,y,z]$, such that

$$A\in\mathfrak{C}_{\mathrm{sp}}(H):\ \ I_2=\mathfrak{W}. \tag{3.20}$$

We will show that their Hilbert functions are closely related to those of a Gorenstein ideal in three variables (Lemmas 3.17 and 3.19). From these results we can characterize the Hilbert functions H for which $\mathfrak{C}_{\mathrm{sp}}(H)$ is nonempty (Theorem 3.20): these are the same as found in the previous section for Gorenstein algebras $A\in\mathfrak{C}(H)$: those with $I_2\cong\langle wx,wy,wz\rangle$. However, it is an open question whether the Zariski closure $\overline{\mathfrak{C}(H)}$ contains $\mathfrak{C}_{\mathrm{sp}}(H)$, and it is this uncertainty that requires us to consider $\mathfrak{C}_{\mathrm{sp}}$ in detail.

The ideal $(\mathfrak{W})$ generated by $\mathfrak{W}$ satisfies $(\mathfrak{W})=(w^2,x,y)\cap(w)$. The inverse system $\mathfrak{W}^\perp\subset\mathscr{D}$ satisfies

$$\begin{aligned}\mathfrak{W}^\perp &= ((w^2,x,y)\cap(w))^\perp=(w^2,x,y)^\perp+(w)^\perp\\ &= K_{\mathrm{DP}}[Z]+W\cdot K_{\mathrm{DP}}[Z]+K_{\mathrm{DP}}[X,Y,Z],\end{aligned} \tag{3.21}$$

Thus we have for the degree-j component

$$\{\mathfrak{W}^\perp\}_j=K_{\mathrm{DP}}[X,Y,Z]_j+\langle WZ^{[j-1]},Z^{[j]}\rangle.$$

Lemma 3.12. *Let I satisfy* (3.20), *and let* $F\in\mathscr{R}=K_{\mathrm{DP}}[W,X,Y,Z]_j$ *be a generator of its inverse system. Then F may be written uniquely*

$$F=G+WZ^{[j-1]},\quad G\in K_{\mathrm{DP}}[X,Y,Z], \tag{3.22}$$

in the sense that the decomposition depends only on I, and the choice of generators w,x,y,z *of R. Further, after a linear change of basis in R, we may suppose that G in* (3.22) *has no monomial term in* $Z^{[j]}$.

Proof. Since w^2,wx,wy are all in I, by (3.21) the generator F of $I^\perp$ can be written in the form $F=G+\lambda WZ^{[j-1]}$, $G\in K_{\mathrm{DP}}[X,Y,Z]$. Evidently, $\lambda\neq 0$, since otherwise

$H(A) = (1, 3, \ldots)$; so we may choose $\lambda = 1$. The decomposition of (3.22) is certainly unique, given I, and the choice of x, y, z, w. A linear change of basis $w \to w, x \to x, y \to y, z \to z + \beta w$ in R, and the contragradient change of basis $W \to W - \beta Z, X \to X, Y \to Y, Z \to Z$ in $\mathscr{R}$ eliminates any monomial term in $Z^{[j]}$ from G. □

We denote by R' the polynomial ring $R' = K[x, y, z]$.

Lemma 3.13. *Let I be an ideal satisfying* (3.20), *let $F = G + WZ^{[j-1]}$ be a generator of its inverse system as in* (3.22), *and let $J = \text{Ann}\ (G)$, $J' = \text{Ann}\ (G) \cap R'$; then $J = (w, J')$. Let $\alpha(J)$ be the integer*

$$\alpha(J) = \min\{\alpha \geqslant 1 \mid J'_\alpha \nsubseteq (x, y)\} = \min\{\alpha \geqslant 1 \mid J_\alpha \nsubseteq (x, y, w)\}. \tag{3.23}$$

Then $\alpha(J) = \min\{i \mid Z^{[i]} \notin R'_{j-i} \circ G\}$, and we have $2 \leqslant \alpha(J) \leqslant j$.

Proof. The first statement follows from $(x, y)^\perp \cap \mathscr{R}' = K_{\text{DP}}[Z]$. The lower bound on α follows from the assumption of (3.20), which implies that $H(R/J') = (1, 3, \ldots)$, so $2 \leqslant \alpha$. The upper bound on α follows from the fact that $z^j \in J' = \text{Ann}\ (G)$. □

Definition 3.14. Let I satisfy (3.20), let $F = G + WZ^{[j-1]}$ be a generator of its inverse system, as in (3.22), and let $\alpha = \alpha(J)$ as in (3.23). We define a sequence

$$H_\alpha = \begin{cases} (0, 1, 1, \ldots, 1, 2 = h_\alpha, 2, \ldots, 2 = h_{j-\alpha}, 1, \ldots, 1, 0 = h_j) \text{ if } \alpha \leqslant j/2, \\ (0, 1, 1, \ldots, 1 = h_{j-\alpha}, 0, \ldots, 0, 1 = h_\alpha, 1, \ldots 1, 0 = h_j) \text{ if } \alpha > j/2 \text{ and } \\ \qquad j \neq 2\alpha - 1, \\ (0, 1, 1, \ldots, 1, 0 = h_j) \text{ if } j = 2\alpha - 1 \end{cases} \tag{3.24}$$

We let $H_0 = (0, 1, 1, \ldots, 1, 0 = h_j)$.

Note that H_α takes values only 0,1, and 2. When $\alpha \leqslant j/2$, there are $j + 1 - 2\alpha$ 2's in the middle of the sequence H_α; when $\alpha > j/2$ there are $2\alpha + 1 - j$ 0's in the middle of H_α. When $\alpha \leqslant j/2$ the middle run of 2's is bordered on the left by 0 in degree zero, followed by $\alpha - 1$ 1's. When $\alpha > j/2$ the middle run of 0's is bordered on the left by 0 in degree zero followed by $j - \alpha$ 1's.

Definition 3.15. We denote by M the R-submodule of $\mathscr{D}$ generated by $WZ^{[j-1]}$, whose degree-i component satisfies $M_i = \langle Z^{[i]}, W \cdot Z^{[i-1]} \rangle$ for $1 \leqslant i < j$. Given F, G as in (3.22) we define two R-modules

$$\begin{aligned} B &= R \circ \langle F, WZ^{[j-1]} \rangle / R \circ G, \\ C &= R \circ \langle F, WZ^{[j-1]} \rangle / R \circ F. \end{aligned} \tag{3.25}$$

We denote by $H^\vee(B)$ the *dual* sequence $H^\vee(B)_i = H(B)_{j-i}$, and likewise $H^\vee(C)_i = H(C)_{j-i}$.

Evidently we have for F, G as in (3.22)

$$I \cap J = \text{Ann}\ \langle F, G \rangle = \text{Ann}\ \langle F, WZ^{[j-1]} \rangle = \text{Ann}\ \langle G, WZ^{[j-1]} \rangle. \tag{3.26}$$

Our convention will be to specify Hilbert functions of R-submodules of $\mathscr{D}$ (or of $\mathscr{R}$) as subobjects: thus $H(R \circ \{Z^{[2]}, WZ\}) = H(\langle 1; Z, W; Z^{[2]}, WZ\rangle) = (1, 2, 2)$. However, the Hilbert functions $H(B)$, and $H(C)$ are as R-modules: thus, when $F = X^{[2]} \cdot Z^{[2]} + WZ^{[3]}$, the module B from (3.25) satisfies, after taking representatives for the quotient, $B \cong \langle WZ^{[3]}; Z^{[3]}, W \cdot Z^{[2]}; WZ; W\rangle$ so $H(B) = (1, 2, 1, 1)$, and the dual sequence $H'(B) = (0, 1, 1, 2, 1)$.

Lemma 3.16. *We have*

$$\begin{aligned} H(R/(I \cap J)) &= H(R'/J') + H^{\vee}(B) \\ &= H(R/I) + H^{\vee}(C). \end{aligned} \tag{3.27}$$

The R-modules B and C each have a single generator, the class of $WZ^{[j-1]}$.

Proof. Eq. (3.27) is immediate from (3.26), and the definition of $H(B)$, $H(C)$. The last statement is immediate from the definition of B, C. □

Lemma 3.17. *Let I be an ideal satisfying* (3.20), *and let* $F = G + WZ^{[j-1]}$ *be a decomposition as in* (3.22) *of the generator F of the inverse system* $I^{\perp}$. *Let* $J = \text{Ann}\ (G)$ *and* $\alpha = \alpha(J)$ *as in* (3.23). *Then we have*

(i) $I \cap J = \text{Ann}\ \langle G, WZ^{[j-1]}\rangle$, *and* $(I \cap J)^{\perp} = \langle R' \circ G, M\rangle = (J')^{\perp} + M = I^{\perp} + M$.
(ii) $H(B) = (1, 2, \ldots, 2_{j-\alpha}, 1, \ldots, 1, 0)$, $H(C) = (1, 1, \ldots 1_c)$, *with* $c = \alpha$ *or* $c = j - \alpha$. *The case* $c = j - \alpha$ *can occur only if* $\alpha \geqslant j/2$.
(iii) *When* $c = \alpha$, *we have* $H(R/I) - H(R'/J') = H_{\alpha}$; *when* $c = j - \alpha$ *we have* $H(R/I) - H(R'/J') = H_0 = (0, 1, 1, \ldots, 1, 0)$.

Proof. Since $I = \text{Ann}\ (F) = \text{Ann}\ (G + WZ^{[j-1]})$ and $J = \text{Ann}\ (G)$, we have

$$I \cap J = \text{Ann}\ \langle F, G\rangle = \text{Ann}\ \langle G, WZ^{[j-1]}\rangle = \text{Ann}\ \langle F, WZ^{[j-1]}\rangle.$$

This proves (i). To show (ii) we consider the two R-modules B, C defined above. Evidently we have $H(B)_i \leqslant 2$, whence by the Macaulay inequalities $H(B) = (1, 2, \ldots, 2_a, 1, \ldots, 1_b, 0)$, with invariants the length $a - 1$ of the sequence of 2's, and the length $b - a$ of the sequence of 1's. Since $Z^{[i]} \in R \circ F$ for $1 \leqslant i \leqslant j - 1$, we have that the Hilbert function $H(C)$ satisfies $H(C)_i \leqslant 1$, hence, $H(C) = (1, 1, \ldots 1_c, 0)$, with sole invariant the length $c + 1$ of the sequence of 1's. Now

$$\begin{aligned} H(R/(I \cap J))_i - H(R/J)_i = 2 &\Leftrightarrow M_i \oplus (R' \circ G)_i = R_{j-i} \circ \langle G, WZ^{[j-1]}\rangle \\ &\Leftrightarrow Z^{[i]} \notin (R' \circ G)_i \\ &\Leftrightarrow i \geqslant \alpha(J). \end{aligned}$$

Otherwise, for $1 \leqslant i < \alpha(J)$, $H(R/(I \cap J))_i - H(R/J)_i = 1$, since for such i we have

$$WZ^{[i-1]} \in R_{j-i} \circ \langle G, WZ^{[j-1]}\rangle \quad \text{but} \quad WZ^{[j-1]} \notin R'_{j-i} \circ G,$$

and for $i = 0$ the difference is 0. Hence, taking into account that $H^{\vee}(B) = H(R/(I \cap J)) - H(R'/J')$, we have $a = j - \alpha(J)$ and $b = j - 1$. Since both $H(R/I)$ and $H(R'/J')$ are symmetric about $j/2$, so is their difference

$$H(R/I) - H(R'/J') = H^{\vee}(B) - H^{\vee}(C). \tag{3.28}$$

This difference can be symmetric only if $c = \alpha$ or $c = j - \alpha$.

Suppose now that $c = j - \alpha$, and $\alpha < j/2$. We will show that $H(R/I)_\alpha = H(R/J)_\alpha + 2$. By definition of α, J'_α has a generator of the form $z^\alpha - g$, $g \in (x, y)R'$; it follows that $z^{j-\alpha} - g' \in J'$, $g' = z^{j-2\alpha}g \in (x, y)R'$. Consider the subset

$$((x, y) \cdot R') \circ G = ((x, y) \cdot R') \circ F.$$

Note that $z^{j-\alpha} \circ G \in (x, y)R' \circ G$. However, $z^{j-\alpha} \circ F$ has a term $WZ^{[\alpha-1]}$, and $wz^{j-\alpha-1} \circ F = Z^{[\alpha]}$. By Lemma 3.13 $Z^{[\alpha]} \notin R' \circ G$, it follows that dim $R_{j-\alpha} \circ F =$ dim $R'_{j-\alpha} \circ G + 2$, as claimed. This implies that $H(R/I) = H(R/J) + H_\alpha$ is the only possibility when $\alpha < j/2$.

The statement (iii) is immediate from (ii) and (3.28). □

Remark 3.18. Note that, *given the Hilbert function* $H' = H(R/J)$ the condition $\alpha(J) \geqslant \alpha_0$ is a closed condition on the family $\mathbb{P}\mathrm{Gor}(H')$. That is, it is rarer to have higher values of $\alpha(J)$. However, the situation is quite different if the Hilbert function is allowed to change, for example if a term $\lambda Z^{[j]}$ is added to the dual generator G of J: see Lemma 3.24, where the effect of such a change is described.

Lemma 3.19. *Let I be an ideal satisfying* (3.20), *and suppose that* $F = G + WZ^{[j-1]}$ *be a decomposition as in* (3.22) *of a generator F of the inverse system* $I^\perp$. *Let* $\alpha = \alpha(J)$, $J =$ Ann (G) *be the integer of* (3.23). *Then we have*

(i) $H(R/I)$ *satisfies either* $H(R/I) = H(R'/J') + H_\alpha$ *or* $H(R/I) = H(R'/J) + H_0$; *the second possibility may occur only if* $\alpha \geqslant j/2$.

(ii) *If* $H(R/I) = H(R/J') + H_\alpha$, *then*

$$H(R/(I \cap J)) = H(R/I) + (0, 0, \ldots, 0, 1_{j+1-\alpha}, 1, \ldots, 1_j) \textit{ and}$$
$$(I \cap J)^\perp = I^\perp \oplus \langle WZ^{[j-\alpha]}, \ldots, WZ^{[j-1]}\rangle$$
$$\textit{also } H(R/(I \cap J)) = H(R'/J') + (0, 1, \ldots, 1, 2_\alpha, 2, \ldots 2_{j-1}, 1_j), \textit{ and}$$
$$(I \cap J)^\perp = (J')^\perp \oplus \langle W, WZ, \ldots, WZ^{[j-1]}; Z^{[\alpha]}, Z^{[\alpha+1]}, \ldots, Z^{[j-1]}\rangle.$$

(iii) *If* $H(R/I) = H(R/J') + H_0$, *then* $H(R/(I \cap J))$ *and* $H(R'/J')$ *are related as above, but*

$$H(R/(I \cap J)) = H(R/I) + (0, 0, \ldots, 0, 1_\alpha, 1, \ldots, 1_j).$$

Proof. The lemma is an immediate consequence of Lemma 3.17 and (3.28). □

Recall that a Gorenstein sequence H of height 3 is a nonnegative sequence of integers $H = (1, 3, \ldots, 1 = h_j, 0, \ldots)$, symmetric about $j/2$, that occurs as the Hilbert function of a graded Artinian Gorenstein algebra $A \cong K[x_1, \ldots, x_r]/I$. Recall that then $(\Delta H)_i = H_i - H_{i-1}$.

Theorem 3.20. *Let I be an ideal satisfying* (3.20). *Then* $H = H(R/I)$ *satisfies*

(i) $\Delta H_{\leqslant j/2}$ *is an O-sequence.*
(ii) $H = H' + H_0 = H' + (0, 1, 1, \ldots, 1, 0)$ *for some Gorenstein sequence* H' *of height three.*

Warning: the H' of (ii) above is *not* in general equal to $H(R'/J')$, except when $c = j - \alpha$.

Proof. By Lemma 3.17(ii) we have $c = \alpha$ or $c = j - \alpha$. The result of the theorem is obvious in the case $c = j - \alpha$, since then by Lemma 3.17(iii) $H(R/I) = H(R'/J') + H_0$. So we assume $c = \alpha$. By Lemma 3.19 we have $H(R/I) = H(R'/J') + H_\alpha$. Here $J' = (\text{Ann } G) \cap K[x, y, z]$ from Lemma 3.13 has a generator in degree α, since by its definition (3.23) α is the lowest degree for which $J'_i \not\subseteq \langle x, y\rangle \cdot R'_{i-1}$.

First, assume $\alpha < j/2$, when $H_\alpha = (0, 1, \ldots, 1, 2_\alpha, \ldots, 2_{j-\alpha}, 1, \ldots, 1, 0)$, from Definition 3.14. We let $H' = H(R/I) - H_0$, and we have

$$H' = H(R'/J') + (0, \ldots, 0, 1_\alpha, 1, \ldots, 1_{j-\alpha}, 0, \ldots). \tag{3.29}$$

Thus, to show (ii) here it would suffice to show that H' of (3.29) is a height three Gorenstein sequence. Assuming that the order of J' is v, we have

$$\Delta H' = \Delta H(R'/J') + (0, 0, \ldots, 1_\alpha, 0, \ldots, 0, -1_{j+1-\alpha}, 0, \ldots), \text{ and}$$
$$\Delta H(R'/J') = (1, 2, 3, \ldots, v, t_v, \ldots, -2, -1), \tag{3.30}$$

with $v \geqslant t_v \geqslant \ldots \geqslant t_{\lfloor j/2 \rfloor}$. Furthermore, a result of A. Conca and G. Valla is that the maximum number of degree-i generators possible for any Gorenstein ideal J' of Hilbert function $H(R/J')$ is

$$\max\{v_i\} = \begin{cases} -(\Delta^2 H(R'/J'))_i = t_{i-1} - t_i & \text{when } i \leqslant j/2 \text{ and } i \neq v \\ 1 - (\Delta^2 H(R'/J'))_v = 1 + v - t_v & \text{when } i = v. \end{cases}$$

(see [11] or [20, Theorem B.13]). Since J' has a generator in degree α it follows when $\alpha > v$ that $t_{\alpha-1} \geqslant t_\alpha + 1$. Thus, for $\alpha \geqslant v$ adding one in degree α to the first difference $(\Delta H(R'/J'))_{\leqslant j/2}$ yields a sequence $\Delta H'$ as in (3.30) that is still an *O*-sequence: for height two this condition is simply that the sequence $\Delta H'$ must rise to a maximum value v', then be nonincreasing. This implies that H' is indeed a height three Gorenstein sequence, and completes the proof when $\alpha \leqslant j/2$.

Now assume that $c = \alpha$ and $\alpha > j/2$. Let

$$H'' = H(R'/J') + (0, \ldots, 0, -1_{j+1-\alpha}, -1, \ldots, -1_{\alpha-1}, 0, \ldots).$$

Then we have in this case $H(R/I) = H'' + H_0$. Thus, to show (ii) here it would suffice to show that H'' also is a height three Gorenstein sequence. We have

$$\Delta H'' = \Delta H(R'/J') + (0, 0, \ldots, -1_{j+1-\alpha}, 0, \ldots, 0, 1_\alpha, 0, \ldots). \tag{3.31}$$

that J' has a generator in degree $\alpha > j/2$, implies that $\Delta^2(H(R'/J))_\alpha \leqslant -1$, which is equivalent by the symmetry of $\Delta^2(H(R'/J'))$ to $\Delta^2(H(R'/J'))_{j+2-\alpha} \leqslant -1$. This in turn implies $(\Delta H(R'/J'))_{j+2-\alpha} < (\Delta H(R'/J'))_{j+1-\alpha}$. Thus, lowering $(\Delta H(R'/J'))_{j+1-\alpha}$ by 1 in degree $j+1-\alpha$ to obtain $\Delta H''_{\leqslant j/2}$ as in (3.31) preserves the condition that $(\Delta H'')_{\leqslant j/2}$ is the Hilbert function of some height two Artinian algebra. This completes the proof of the theorem. □

The following examples illustrate Lemma 3.19. In particular we explore how the Hilbert functions $H(R/I)$, $H(R/J)$ change (recall that $I = \text{Ann }(F)$, $J = \text{Ann }(G)$) as we alter the coefficient of $Z^{[j]}$ in F, G. Here there is a marked difference for the cases $\alpha(J) \leqslant j/2$, and $\alpha(J) > j/2$. The subsequent Lemma 3.24 explains some of the observations.

Example 3.21. Letting $G = X^{[4]}Z^{[2]} - X^{[4[}YZ$, $F = G + WZ^{[5]}$, we have $J = \text{Ann }(G) = (w, yz+z^2, y^2, x^5)$, so $\alpha(J)=2$, and $I = \text{Ann }(F) = (w^2, wx, wy, y^2, yz^2, xyz+xz^2, x^4y+wz^4, x^5, z^6)$. Also $H(R/J) = (1, 3, 4, 4, 4, 3, 1)$, and

$$H(R/I) = (1, 4, 6, 6, 6, 4, 1) = H(R/J) + H_2.$$

Changing G by adding a $Z^{[6]}$ term, we have $G_1 = X^{[4]}Z^{[2]} - X^{[4]}YZ + Z^{[6]}$, $F_1 = G_1 + WZ^{[5]}$, $J(1) = \text{Ann }(G_1) = (w, y^2, yz^2, xyz + xz^2, x^4y + z^5, x^5)$, so $\alpha(J(1)) = 5$, and $I(1) = \text{Ann }(F_1) = (w^2, wx, wy, y^2, yz^2, xyz + xz^2, x^4y + wz^4, x^5, wz^5 - z^6)$. Also $H(R/J(1)) = (1, 3, 5, 5, 5, 3, 1)$, and

$$H(R/I(1)) = (1, 4, 6, 6, 6, 4, 1) = H(R/J(1)) + H_0.$$

Example 3.22. In this example, we chose $G = (Z + X)^{[6]} + (Z + 2X)^{[6]} + (Z + Y)^{[6]} + (Z + 2Y)^{[6]} + (Z + X + Y)^{[6]} + (Z + 2X + 2Y)^{[6]}$, the sum of 6 divided powers, and let $J = \text{Ann }(G)$. Then $H(R/J)$ has the expected value $H(R/J) = (1, 3, 6, 6, 6, 3, 1)$ (see [20]), and $\alpha(J) = 3$. From Lemma 3.19, letting $I = \text{Ann }(F)$, $F = G + WZ^{[5]}$ we have

$$H(R/I) = H(R/J) + H_3 = (1, 4, 7, 8, 7, 4, 1).$$

Here

$$\begin{aligned} I = (&w^2, wx, wy, y^3 - 3y^2z + 2yz^2, x^2y - xy^2, x^3 - 3x^2z + 2xz^2,\\ &51xy^2z - 18x^2z^2 - 99xyz^2 - 18y^2z^2 - 12wz^3 + 34xz^3 + 34yz^3,\\ &5y^2z^3 + 4wz^4 - 9yz^4, yz^5 - z^6). \end{aligned}$$

Omitting the pure $Z^{[6]}$ term from G and F, to obtain G_1, F_1 we have $H(R/\text{Ann }(G_1)) = (1, 3, 6, 7, 6, 3, 1)$, $\alpha(\text{Ann }(G_1)) = 4$ and

$$H(R/\text{Ann }F_1) = H(R/I) = H(R/\text{Ann }(G_1)) + H_0.$$

This example shows that it is not the inclusion of a $Z^{[6]}$ term in G that keys the simpler case $H(R/I) = H(R/J) + H_0$. The Hilbert function $H(R/I)$ is always invariant under a change in the $Z^{[j]}$ term of F: this follows from $z^i \circ F = WZ^{[j-1-i]} + z^i \circ G$, linearly disjoint from $\langle R_i \bmod z^i\rangle \circ F$.

Example 3.23. When $j=8$, $G=X^{[3]}Y^{[5]}+X^{[2]}Y^{[4]}Z^{[2]}+Y^{[5]}Z^{[3]}$, then

$$J=\text{Ann }\ G=(w, x^3-z^3, z^4, xz^3, x^2z^2-yz^3, y^6, xy^5+x^2y^3z-y^4z^2, x^2y^4-y^5z-xy^3z^2).$$

We have $\alpha(G)=3$, $H(R/\text{Ann }(G))=(1,3,6,9,9,9,6,3,1)$, and $I=\text{Ann }(F)$, $F=G+WZ^{[7]}$, satisfies

$$H(R/I)=(1,4,7,11,11,11,7,4,1)=H(R/\text{Ann }(G))+H_3.$$

Here

$$I=(w^2, wx, wy, xz^3, x^2z^2-yz^3, x^3z, x^3y-yz^3, x^4, y^5z-wz^5, y^6, xy^5+x^2y^3z-y^4z^2, x^2y^4-xy^3z^2-wz^5, z^8).$$

Adding a $Z^{[8]}$ term to G to form G_1 leads to $J(1)=\text{Ann }(G_1)$ with $\alpha(J(1))=6$ and F_1, $I(1)=\text{Ann }(F_1)$ satisfying

$$H(R/I(1))=H(R/I)=H(R/J(1))+H_0.$$

It might be thought from the previous examples, that adding $\lambda Z^{[j]}$ with λ generically chosen, will "improve" G to a G_λ such that $J(\lambda)=\text{Ann }\ G_\lambda$ and $I_\lambda=\text{Ann }\ F_\lambda$, $F_\lambda=G_\lambda+WZ^{[j-1]}$ will satisfy $H(R/I_\lambda)=H(R/J_\lambda)+H_0$. This change would indeed be an improvement, since when $H(R/I)=H(R/J)+H_0$ the minimal resolutions of the ideals I, J appear to be closer than they are when $H(R/I)=H(R/J)+H_\alpha$. In the next lemma we show that this "improvement" must occur when $\alpha(J)\leqslant j/2$, but can occur either never, or for a single value of λ when $\alpha(J)>j/2$. We suppose that $\lambda\in K$.

Lemma 3.24. *Let* $J=\text{Ann }(G)$, $I=\text{Ann }(F)$, $F=G+WZ^{[j-1]}$ *be such that I satisfies* (3.20), *and define* $G_\lambda=G+\lambda Z^{[j]}$, $F_\lambda=F+\lambda Z^{[j]}$, $J(\lambda)=\text{Ann }(G_\lambda)$, $I(\lambda)=\text{Ann }(F_\lambda)$. *Then we have*

(i) $(I\cap J)+m^j=(I(\lambda)\cap J(\lambda))+m^j$ *and* $(I\cap J)_j$ *differs from* $(I(\lambda)\cap J(\lambda))_j$ *by replacing* $z^j-u, u\in J\cap((x,y)\cap K[x,y,z])$ *by* $z^j-u', u'\in J(\lambda)\cap((x,y)\cap K[x,y,z])$.

(ii) $H(R/I)=H(R/I(\lambda))$, *and* $H(R/(I\cap J))=H(R/(I(\lambda)\cap J(\lambda)))$;

(iii) *If* $\alpha(J)\leqslant j/2$ *and* $\lambda\neq 0$ *then* $\alpha(J(\lambda))=j+1-\alpha(J)$, *and*

$$H(R/J(\lambda))=H(R/J)+(0,\dots,0_{\alpha-1},1_\alpha,1,\dots,1_{j-\alpha},0_{j+1-\alpha},\dots,0_j).$$

In this case $H(R/I(\lambda))=H(R/J(\lambda))+H_0$.

(iv) *Let* $\alpha(J)>j/2$ *then* $\alpha(J(\lambda))=\alpha(J)$ *or* $\alpha(J(\lambda))=j+1-\alpha(J)$. *In the former case* $H(R/J(\lambda))=H(R/J)$. *The latter case may occur for at most a single value* λ_0; *if it occurs, then for* $\lambda=\lambda_0$, $\alpha=\alpha(J)$,

$$H(R/J(\lambda_0))=H(R/J)-(0,\dots,0_{j-\alpha},1_{j+1-\alpha},1,\dots,1_{\alpha-1},0_\alpha,\dots,0_j).$$

(a) *If* $H(R/I)=H(R/J)+H_\alpha$ *then* $\alpha(J(\lambda))=\alpha(J)$ *and* $H(R/J(\lambda))=H(R/J)$.

(b) *If* $H(R/I) = H(R/J) + H_0$, *then for all values of* λ *except possibly a single value* $\lambda_0 \neq 0$ *we have* $\alpha(J(\lambda)) = \alpha(J)$ *and* $H(R/J(\lambda)) = H(R/J)$.

Proof. Since for $i \leqslant j-1$, $Z^{[i]} = wz^{j-1-i} \circ WZ^{[j-1]}$, $(I \cap J)_i = (I(\lambda) \cap J(\lambda))_i$ for $i \leqslant j-1$. The second statement in (i) is evident. The first claim in (ii) follows since the two ideals I, $I(\lambda)$ are isomorphic, under a change of variables. The second claim in (ii) follows from (i).

Suppose that $\alpha \leqslant j/2$ and $\lambda \neq 0$, and that $h = z^{\alpha} - g$, $g \in (x, y) \cdot K[x, y, z] \in J$. Then for $0 \leqslant u \leqslant j - 2\alpha$ we have

$$(z^u h) \circ (G + \lambda Z^{[j]}) = z^u h \circ (\lambda Z^{[j]}) = \lambda Z^{[j-\alpha-u]}.$$

It follows that for $i \leqslant j - \alpha$, $Z^{[i]} \in R \circ G(\lambda)$. This implies that for $\alpha \leqslant i \leqslant j - \alpha$, we have $H(R/J(\lambda))_i = H(R/J)_i + 1$, since by Lemma 3.19 $Z^{[i]} \neq R \circ G$ for $i \geqslant \alpha(J)$. The claims in (iii) now follow from the symmetry of $H(R/J(\lambda))$, $H(R/J)$ and hence of $H(R/J(\lambda)) - H(R/J)$.

Suppose that $\alpha(J) > j/2$. The symmetry of $H(R/J(\lambda)) - H(R/J)$ and Lemma 3.17(ii) show the first claim concerning $\alpha(J(\lambda))$ in (iv). This and (ii) show (iva). The same symmetry, and (iii) also prove (ivb), and completes the proof of (iv) that the exceptional case may occur for at most a single value λ_0. □

Example 3.25. Letting $G = X^{[3]}Z^{[3]} - Y^{[4]}X^{[2]} + Y^{[2]}Z^{[4]} + XY^{[2]}Z^{[3]} + Z^{[6]}$, $F = G + WZ^{[5]}$ we have $J = \text{Ann}\,(G) = (x^3 + x^2z - y^2z, y^3z, y^4 - x^2z^2 + y^2z^2 + xz^3 - z^4, xy^2z + x^2z^2 - y^2z^2, xy^3 - xyz^2 + yz^3, x^2yz, x^2y^2 + x^2z^2 - y^2z^2 + z^4)$, $\alpha(J) = 4$, and $H(R/J) = (1, 3, 6, 9, 6, 3, 1)$. Then

$$I = (wy, wx, w^2, x^2z - y^2z + wz^2, x^3 - wz^2, y^3z, xy^3 - xyz^2 + yz^3, x^2y^2 + y^4 + xz^3, wz^5 - z^6), \quad \text{and}$$

$$H(R/I) = (1, 4, 7, 9, 7, 4, 1) = H(R/J) + (0, 1, 1, 0, 1, 1, 0) = H(R/J) + H_4.$$

This is an example of Lemma 3.24(iva) where $H(R/J(\lambda)) = H(R/J)$ for every λ.

4. Hilbert functions $H = (1, 4, 7, h, \ldots, 4, 1)$

We now consider Gorenstein sequences—Hilbert functions of Artinian Gorenstein algebras, so symmetric about $j/2$—having the form

$$H = (1, 4, 7, h, b, \ldots, 4, 1), \tag{4.1}$$

of any socle degree $j \geqslant 6$ for any possible b. We show in Theorem 4.2 that each such Gorenstein sequence must satisfy the *SI condition* that $\Delta H_{\leqslant j/2}$ is an O-sequence. This condition was shown by Stanley and Buchsbaum and Eisenbud to characterize Gorenstein sequences of height three (see [9,31,17]). When a Gorenstein sequence H satisfies this condition we can construct Artinian Gorenstein algebras, elements of $\mathbb{P}\text{Gor}(H)$, as quotients of the coordinate ring of suitable punctual schemes, and we have good control over their

Betti numbers (Lemma 4.7 and Corollary 4.8). In particular, when $H = (1, 4, 7, h, \ldots)$ satisfies the SI condition and $7 \leqslant h \leqslant 10$ we may choose $A \in \mathbb{P}\mathrm{Gor}(H)$ such that I_2 has only two linear relations: thus $A \notin \overline{\mathfrak{C}(H)}$, the locus where $I_2 \cong \langle wx, wy, wz\rangle$, implying for most such Hilbert functions H that $\mathbb{P}\mathrm{Gor}(H)$ has at least two irreducible components (Theorems 4.6 and 4.9).

Our first result is relevant also to the open question of whether all height four Gorenstein sequences satisfy the SI condition. Despite our positive result we doubt that this is true in general (see Remark 4.5). We now set some notation. When H is clear we usually write h_i for H_i below. We set $\Delta H_i = h_i - h_{i-1}$. By $H_{i,i+1}$ we mean (h_i, h_{i+1}). Given a Hilbert function H_3, we define $\mathrm{Sym}(H_3, j)$ as the symmetrization of $(H_3)_{\leqslant j/2}$ about $j/2$:

$$\mathrm{Sym}(H_3, j)_i = \begin{cases} (H_3)_i & \text{if } i \leqslant j/2, \\ (H_3)_{j-i} & \text{if } i > j/2. \end{cases} \tag{4.2}$$

Lemma 4.1. *Let $j \geqslant 6$ and suppose that the Gorenstein sequence H of socle degree j satisfies* (4.1). *Then $7 \leqslant h \leqslant 11$. If $j \geqslant 7$, then the minimum value of $b = H_4$ that can occur is $b = h$, and the maximum values of b that can occur in* (4.1) *are*

h	7	8	9	10	11
$b_{\max}$	7	9	11	13	16

(4.3)

Equivalently, a Gorenstein sequence H satisfying (4.1) *must satisfy $\Delta H_{\leqslant 4}$ is an O-sequence. Also, each initial sequence $(1, 4, 7, h, b)$ satisfying $7 \leqslant h \leqslant 11$ and $h \leqslant b \leqslant b_{\max}$ occurs for $j = 8$.*

Finally, if H satisfies (4.1) *and $j \geqslant 6, h \leqslant 10$ then $\Delta H_{\leqslant j/2}$ is an O-sequence if and only if its subsequence $\Delta H_{1 \leqslant i \leqslant j/2} = (3, 3, h-7, b-h, \ldots)$ is both nonnegative and nonincreasing.*

Proof. We showed $7 \leqslant h \leqslant 11$ in Corollary 2.8. We now show the upper bounds $b \leqslant b_{\max}$ of (4.3). When $h = 11$, the upper bound of (4.3) is just the Macaulay upper bound. When $h = 10$, the impossibility of $(h, b) = (10, 15)$ follows from Corollary 2.6. The impossibility of $(h, b) = (10, 14)$ follows from two considerations. First, by Theorem 3.7(iii) and Theorem 3.20(ii) I_2 cannot be Pgl(3)-isomorphic to $\langle wx, wy, wz\rangle$ or $\langle w^2, wx, wz\rangle$, as $H' = H - (0, 1, 1, \ldots, 1, 0) = (1, 3, 6, 9, 13, \ldots)$ is not a height three Gorenstein sequence, since $\Delta H'_{\leqslant j/2} = (1, 2, 3, 3, 4, \ldots)$ is not an O-sequence in two variables [9,12]. Thus I_2 cannot have a common factor, so has two linear relations. By Lemma 3.4(ii) I_2 has a basis given by the 2×2 minors of a 2×3 matrix; since I_2 has no common factor, the quotient $R/(I_2)$ has height two, I_2 is determinantal and has the usual determinantal minimal resolution. In particular we have $H(R/(I_2))_i = 3i + 1$, for all $i \geqslant 0$, so as before $H(R/I)_4 \leqslant H(R/(I_2))_4 = 13$.

When $h = 8$ or 9 the upper bound of (4.3) is one less than the Macaulay upper bound. The impossibility of the Macaulay upper bound for $H(R/I)_{3,4}$ in the cases $h = 8, 9$ follow from Lemma 3.6(iii). When $h = 7$, the upper bound $b \leqslant 7$ is shown in the $h = 7$ case of the proof of Theorem 4.2 below. This completes the proof of the upper bounds $b \leqslant b_{\max}$ of (4.3).

We next show the lower bound on b: when $j \geqslant 7$, then $b \geqslant h$. Evidently, when $j = 7$, the symmetry of H implies $b = h$, so we may assume $j \geqslant 8$. The symmetry of H implies $(H_{j-4,j-3}) = (b, h)$. The Macaulay Theorem 2.3(i) applied to $(H_{j-4,j-3})$ eliminates all triples (j, h, b) where $b \leqslant h - 2$ except the triple $(j, h, b) = (8, 5, 4)$. For this triple $H_{4,5} = (b, h) = (9, 11)$ is extremal growth as $9^{(4)} = 11$; then we have a contradiction by Corollary 2.6.

We now assume $j \geqslant 8$ and $b = h - 1$. We have $h \neq 11$ by Theorem 3.7(iii) and Theorem 3.20. Since in Macaulay's inequality of Theorem 2.3(i) $b^{(d)} = b$ when $b \leqslant d$, and $h_{j-4} = b, h_{j-3} = b + 1$ we must have $b > j - 4$, so $h \geqslant j - 2$. Except for the triples $(j, h, b) = (8, 10, 9)$ or $(8, 11, 10)$, then $H_{j-4,j-3}$ has extremal Macaulay growth, a contradiction by Corollary 2.6. The second triple has $h = 11$, already ruled out. The first triple occurs only for $H = (1, 4, 7, 10, 9, 10, 7, 4, 1)$ where $\Delta^4 H_6 = -12$; by symmetry of the minimal resolution of R/I, the number of degree six generators of I satisfies $v_6(I) \geqslant 6$, implying that $H(R/(I_5))_{5,6} = (10, 13)$, contradicting the Macaulay bound which requires $H(R/(I_5))_6 \leqslant 10^{(6)} = 11$. This completes the proof of the lower bound on b, that $h \leqslant b$ in (4.1).

It is easy to see that these bounds are just the condition that $\Delta H_{\leqslant 4}$ be an O-sequence, as claimed.

That each extremal pair (h, b) satisfying $h \leqslant b \leqslant b_{\max}$ from (4.3) occurs in socle degree 8 can be shown by choosing the ring A to be a general enough socle-degree 8 Artinian Gorenstein quotient of the coordinate ring of any smooth punctual scheme of degree b, having Hilbert function $H_{\mathfrak{Z}} = (1, 4, 7, h, b, b, \ldots)$. Since $b \geqslant h$, $\Delta H_{\mathfrak{Z}}$ is an O-sequence and there are Artinian algebras of Hilbert function $\Delta H_{\mathfrak{Z}}$; then there is a smooth punctual schemes of Hilbert function $H_{\mathfrak{Z}}$, by the result of Maroscia [27,14,28]. That the general socle-degree j GA quotient of $\Gamma(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}})$ has the expected symmetrized Hilbert function $H = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ satisfying $(\mathrm{Sym}(H_{\mathfrak{Z}}, j))_i = (H_{\mathfrak{Z}})_i$ for $i \leqslant j/2$, is well known: see [3,28,20, Lemma 6.1].

The last statement of Lemma 4.1 that $j \geqslant 6$, $h \leqslant 10$ and $\Delta H_{\leqslant j/2}$ an O-sequence is equivalent to $\Delta H_{2 \leqslant i \leqslant j/2}$ being nonnegative and non-increasing, follows from $\Delta H = (1, 3, 3, h - 7, \ldots)$, with $h - 7 \leqslant 3$: by Macaulay's inequality Theorem 2.3(i), we have for any O-sequence T that $t_i \leqslant i$ implies $t_{i+1} \leqslant i$. □

Theorem 4.2. *Every Gorenstein sequence H beginning* $H = (1, 4, 7, \ldots)$ *satisfies the condition,* $\Delta H_{\leqslant j/2}$ *is an O-sequence.*

Proof. We assume $H = H(R/I)$ for an Artinian Gorenstein quotient R/I satisfies (4.1) that $H = (1, 4, 7, h, b, \ldots)$ and consider each value of h in turn. We show that each occurring sequence H satisfies the criterion from Lemma 4.1 for ΔH to be an O-sequence.

Case $h = 7$: We have $H(R/I)_{j-3,j-2} = H(R/I)_{2,3} = (7, 7)$; if $j \geqslant 10$ then H is extremal in degrees $j - 3$ to $j - 2$, and we have that $\mathfrak{Z} = \mathrm{Proj}\ (I_{j-3})$ is a degree-7 punctual scheme satisfying by Lemma 2.9 $H(\mathfrak{Z})_i = 7$ for all $i \geqslant 3$: by Corollary 2.2, we have $H(R/I)_i = 7$ for $3 \leqslant i \leqslant j - 2$. So we may assume that $j = 8$ or 9. We have $b \leqslant 7^{(3)} = 9$. Should $b = 9$ then Proj $(R/(I_3))$ would define a degree-2 curve of genus zero and regularity two, so its Hilbert function would satisfy $H(R/(I_{\mathfrak{Z}}))_2 \leqslant 5$, by Corollary 2.2 contradicting $H(R/I)_2 = 7$. We now suppose that $h = 7$, $b = 8$, and suppose the socle degree $j = 8$ or 9. When $j = 8$, $H = (1, 4, 7, 7, 8, 7, 7, 4, 1)$, since $\Delta^4 H_5 = -7$, the ideal I has v_5 generators (first syzygies) and μ_5 third syzygies in degree 5, with $7 \leqslant v_5 + \mu_5$; by symmetry of the minimal resolution $v_7 = \mu_5$ and $\mu_7 = v_5$; thus we have either $v_5 \geqslant 3$ or $v_7 \geqslant 4$; but $v_5 \leqslant 2$ and $v_7 \leqslant 4$ by Macaulay's Theorem 2.3. If $v_7 = 4$ then the ideal $(I_{\leqslant 6})$ would satisfy $H(R/(I_6))_{6,7} = 7, 8$ of extremal growth, a contradiction with $\Delta H_3 = 0$, by Corollary 2.6 and Lemma 4.1. For $j = 9$ we would have similarly $\Delta^4 H_5 = -6$, so $v_5 + v_8 \geqslant 6$, but $v_5 \leqslant 2$, and when $v_8 = 4$ we'd have $H(R/(I_6))_{7,8} = 7, 8$, and a similar contradiction. We have shown that a Gorenstein sequence beginning (1, 4, 7, 7) continues with a subsequence of 7's followed by (4, 1).

Case $h = 8$: Macaulay extremality shows $h_i \leqslant i + 5$ and $\Delta H_{i+1} \leqslant 1$ for $i \geqslant 3$. Suppose by way of contradiction that $\Delta H_i < 0$, for some $i \leqslant j/2$ (this is equivalent to H being nonunimodal). Letting $i' = j - i$, we have by the symmetry of H that $h_{i'+1} = h_{i'} + 1 = h_{i-1} \leqslant i - 1 + 5 \leqslant i' + 4$; it follows from Theorem 2.3 that either this is impossible (when $h_{i'} \leqslant i'$) or H is extremal in degrees i' to $i' + 1$, a contradiction by Corollary 2.6 and Lemma 4.1. Now suppose $4 \leqslant i < k$, $\Delta H_i = 0$ but $\Delta H_k = 1$. Then $H_k = (H_{k-1})^{(k-1)}$, and we have a contradiction by Corollary 2.6 and Lemma 4.1. It follows that H satisfies, ΔH_i, $2 \leqslant i \leqslant j/2$ is nonnegative and nonincreasing, thus $\Delta H_{\leqslant j/2}$ is an O-sequence.

Case $h = 9$: Lemma 3.6(iii) implies that $h_4 \leqslant 11$; applying Macaulay extremality inductively we have for $i \geqslant 4$ that $h_i \leqslant 2i + 3$ and $\Delta H_i \leqslant 2$. Suppose by way of contradiction that $\Delta H_i < 0$, for some $i \leqslant j/2$; then $h_i = i + a$ with $a \leqslant i$. We now use the symmetry of H about $j/2$. Letting $i' = j - i$, we have $h_{i'} = i + a = i' + a'$, $a' = a - (i' - i)$; since $a' < a < i'$ we must have $h_{i'} \leqslant 2i'$ whence $h_{i'+1} \leqslant h_{i'} + 1$ by the Macaulay Theorem 2.3(i), so $\Delta H_{i'+1} = -\Delta H_i = 1$, and $h_{i'+1} = h_{i'} + 1$ is extremal, a contradiction by Corollary 2.6 and Lemma 4.1.

Now suppose that for some $i \leqslant j/2$ we have $\Delta H_{i-1} = 1$, but $\Delta H_i = 2$: then by Theorem 2.3 we would have Proj $(R/(I_i))$ defines a degree 2 curve union some points, of Hilbert polynomial $2t + a$, $a \leqslant 2$, of regularity degree at most 3 by Corollary 2.5, hence by Lemma 2.9 and Corollary 2.2 we would have $h_3 \leqslant 8$, a contradiction.

Finally, suppose that for some $i \leqslant j/2$ we have $\Delta H_{i-1} = 0$ but $\Delta H_i > 0$. By Corollary 2.6 we have $\Delta H_i \neq 2$, so $\Delta H_i = 1$. If also there is a previous u, $4 \leqslant u \leqslant i - 2$ with $\Delta H_u < 2$ then $h_i \leqslant 2i$, implying that $H_i = (H_{i-1})^{(i-1)}$, a contradiction by Corollary 2.6. Thus to complete the case $h = 9$, we need only consider sequences

$$H = (1, 4, 7, 9, \ldots, h_u = 2u + 3, \ldots, h_{i-2} = h_{i-1} = 2i - 1, h_i = 2i, \ldots, 7, 4, 1) \tag{4.4}$$

with possible consecutive repetition of the maximum value $2i$. We have $\Delta^4 H_{i+1} = -5$ if $h_{i+1} = h_i$, and -6 if $j = 2i$ so $h_{i+1} = h_i - 1$. In either case, we obtain $v_{i+1} + v_{j+3-i} \geqslant 5$. This is impossible since on the one hand $v_{j+3-i} \geqslant 3$ would imply that $H(R/(I_{j+2-i}))_i = h_{i-2} = 2i - 1$, $H(R/(I_i))_{j+3-i} = h_{i-3} + 3 = 2i - 3 + 3 = 2i$, which is extremal growth of H, a contradiction by Corollary 2.6. On the other hand if $v_{i+1} \geqslant 1$ when $h_{i+1} = h_i$, or if $v_{i+1} \geqslant 2$ when $h_{i+1} = h_i - 1$ we would have $H(R/I)_i = 2i$, $H(R/(I_i))_{i+1} = 2i + 1$ implying extremal growth, a contradiction with (4.4) by Corollary 2.6. This completes the proof that ΔH is an O-sequence when $h = 9$.

Case $h = 10$: By Lemma 4.1 $h_4 \leqslant 13$; also when I_2 has a common factor Theorems 3.7(iii) and Theorem 3.20 show that $\Delta H_{\leqslant j/2}$ is an O-sequence. We suppose henceforth in our analysis of $h = 10$ that I_2 does not have a common factor. Then by Lemma 3.4(ii) I_2 defines a rational normal curve, satisfying $H(R/(I_2))_t = 3t + 1$ for all $t \geqslant 0$. Notice also that if $H(R/I)_t \leqslant 3t-1$, and $t \geqslant 4$, then the Macaulay inequality Theorem 2.3(i) implies $\Delta H(R/I)_{i+1} \leqslant 2$. We next rule out various perturbations in the Hilbert function sequence.

First, $\Delta H_{i+1} \leqslant -2$ for some $i < j/2$ is impossible from the Macaulay bound and the symmetry of H. We would have $\Delta H_{i'+1} \geqslant 2$ for $i' = j-i-1 \geqslant i+1$; then letting $h_i = 3i+1-e$, $e \geqslant 0$ we have $h_{i'} = h_{i+1} \leqslant h_i - 2 = 3i-(e+1) = 2i+(i-e-1) = 2i'+b$, $b \leqslant i-e-3$; thus, the Macaulay bound here implies $\Delta H_{i'+1} \leqslant 2$, so there is equality $\Delta H_{i'+1} = 2$, a contradiction by Corollary 2.6. Also $\Delta H_{i+1} = -1$ for some $i < j/2$, and $j > 5i+e$, is impossible by a similar calculation that $\Delta H_{i'+1} = 1$ the maximum possible, again a contradiction by Corollary 2.6.

Suppose $\Delta H_{i+1} = -1$ with $i \leqslant j/2 - 1$ and no restriction on j; suppose that i is the maximum such integer. Letting $c = h_{i+1}$ we write the consecutive subsequence $(h_{i-1}, \ldots, h_{i+3})$ as

$$(a+c, 1+c, c, 1-\alpha+c, b+c). \tag{4.5}$$

Then $v_{i+3}(I) + v_{j+5-i}(I) \geqslant -\Delta^4 H_{i+3} = -\Delta^4 H_{j+5-i} = 8-a-b-4\alpha$. We have

$$H(R/(I_{i+2}))_{i+2,i+3} = (1-\alpha+c, b+c+v_{i+3}) \quad \text{and}$$
$$H(R/(I_{j+4-i}))_{j+4-i,j+5-i} = (1+c, a+c+v_{j+5-i}).$$

Thus the sum $\delta+\delta'$, $\delta = \Delta H(R/(I_{i+2}))_{i+3}$, $\delta' = H(R/(I_{j+4-i}))_{j+5-i}$ satisfies

$$\delta+\delta' = (b+v_{i+3}+\alpha-1) + (a+v_{j+5-i}-1) \geqslant 6-3\alpha.$$

So if $\alpha \leqslant 1$ at least one of δ, δ' is two, and the corresponding Hilbert function has extremal growth of two, a contradiction by Corollary 2.6. If $\alpha = 2$, then $i+1 \leqslant j/2-1$ (by the symmetry of H), and $\Delta H_{i+2} = -1$, contradicting the assumption on i; and $\alpha \geqslant 3$ has already been ruled out. We have shown $\Delta H_{i+1} = -1$ for $i \leqslant j/2-1$ is impossible.

We cannot have both $\Delta H_u \leqslant 2$ and $\Delta H_{i+1} = 3$ for a pair u, i satisfying $u < i < j/2$, since then $h_i \leqslant 3i$. This is possible only if $h_i = 3i$ and $h_{i+1} = h_i^{(i)}$, a contradiction by Corollary 2.6. We cannot have both $\Delta H_u \leqslant 1$ and $\Delta H_{i+1} = 2$ for $u < i < j/2$, since then $h_i = 3i-1-e$, $e \geqslant 0$, and $H_{i,i+1}$ is extremal, again a contradiction by Corollary 2.6.

Suppose that for some i, $2 \leqslant i \leqslant j/2-1$, we have $\Delta H_i = 0$, but $\Delta H_{i+1} = 1$. Then, letting $c = h_i$ the consecutive subsequence $(h_{i-2}, \ldots, h_{i+2})$ is

$$(a+c, c, c, 1+c, b+c). \tag{4.6}$$

Then $v_{i+2}(I) + v_{j+6-i}(I) \geqslant -\Delta^4 h_{i+2} = -\Delta^4 h_{j+6-i} = 4-(b+a)$. It follows that the sum $\Delta H(R/(I_{i+2}))_{i+3} + \Delta H(R/(I_{j+5-i}))_{j+6-i} = a+b-1+4-(a+b) = 3$, hence one of the two differences is at least two, which is here extremal growth, since $H_{i+2} \leqslant 3(i+2)$ and similarly $H_{j+5-i} \leqslant 3(j+5-i)$. Then Corollary 2.6 implies a contradiction with (4.6).

This completes the proof in the case $h = 10$.

Case $h = 11$: In this case I_2 must have a common linear factor. Theorem 3.7(iii) for $I_2 \cong \langle wx, wy, wz \rangle$ and Theorem 3.20 for $I_2 \cong \langle w^2, wy, wz \rangle$ show that $H = H' + (0, 1, 1, \ldots, 1, 0)$, which implies that $\Delta H_{\leqslant j/2}$ is an O-sequence.

This completes the proof of the theorem. □

For H satisfying (4.1), recall that we denote by $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ the subfamily parametrizing ideals I such that $I_2 \cong \mathfrak{V} = \langle wx, wy, wz\rangle$, up to a coordinate change. By Theorem 3.7(ii) we have that $\mathfrak{C}(H)$ is nonempty if and only if $\mathbb{P}\mathrm{Gor}(H')$ is nonempty, where $H' = (1, 3, 6, h-1, i-1, \ldots, 3, 1)$.

Corollary 4.3. *Let $H = (1, 4, 7, \ldots)$. The following are equivalent.*

(i) *The sequence H is a Gorenstein sequence.*
(ii) *The sequence $\Delta H_{\leqslant j/2}$ is an O-sequence.*
(iii) *The sequence $H' = H - (0, 1, 1, \ldots, 1, 0)$ is a height three Gorenstein sequence.*
(iv) *$\Delta H'_{\leqslant j/2}$ is an O-sequence.*
(v) *$\Delta H'_{\leqslant j/2} = (1, 2, 3, \ldots, i+1, h_v, h_{v+1}, \ldots)$ with $i+1 \geqslant h_v \geqslant h_{v+1} \geqslant \cdots$.*

Under this assumption, the subfamily $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ is always nonempty.

Proof. That (i) is equivalent to (ii) is Theorem 4.2. That (ii) is equivalent to (iv) is immediate from the last statement of Lemma 4.1, and an easy verification when $H = (1, 4, 7, 11, \ldots)$. That (iii) is equivalent to (iv) follows from the Buchsbaum–Eisenbud structure theorem [9,31]. That specific criterion (iv) is equivalent to (v) is well known—see for example [20, Theorem 5.25] and [21, Corollary C6]. That $\mathfrak{C}(H)$ is always nonempty when H satisfies these conditions follows from Theorem 3.7 and (iii). □

The following result handles height four Gorenstein sequences below those considered in Theorem 4.2.

Proposition 4.4. *A symmetric sequence $H = (1, 4, a, \ldots, 4, 1)$, $a \leqslant 6$ of socle degree j is a Gorenstein sequence if and only if $\Delta H_{\leqslant j/2}$ is an O-sequence, or, equivalently, if $\Delta H_{\leqslant j/2}$ is nonincreasing once it does not increase. The values $a = 2, 3$ cannot occur.*

Proof. When $a = 6$, then $h_3 = 10$, the maximum under Macaulay's theorem, would imply $h_1 = 3$, by Corollary 2.6. Assume $H = H(A)$ for an Artinian Gorenstein $A = R/I$ and let α_i denote the number of relations (first syzygies) in degree i. When $a = 6$ and $h_3 = 9$, $h_4 = b$, then the fourth differences of H satisfy $\Delta^4(H)_4 = \Delta^4(H)_j = b - 15$, so by the symmetry of the minimal resolution of A we have $\alpha_4 + \alpha_j \geqslant 15 - b$. Since $\Delta H(R/(I_{j-1}))_j = \alpha - 3$ and $j - 1 \geqslant 5$, the Macaulay bound implies that growth from $h_{j-1} = 4$ to $H(R/(I_{j-1}))_j = 1 + \alpha$ would be maximal when $\alpha_j = 3$. But $\alpha_j = 3$, is impossible by Corollary 2.6. However, $\alpha_j \leqslant 2$, implies $\alpha_4 \geqslant 13 - b$; thus $H(R/(I_3))_4 \geqslant b + 13 - b = 13$,contradicting the Macaulay bound of $9^{(3)} = 12$. We have shown $H = (1, 4, 6, 9, \ldots)$ to be impossible. Establishing the result for $H = (1, 4, 6, b, \ldots)$ with $b \leqslant 8$ is relatively simple, requiring only Theorem 2.3 and Corollary 2.2 without using the symmetry of the minimal resolution: we leave this to the reader.

When $a = 5$, then the Macaulay bound gives $h_3 \leqslant 7$; and $H = (1, 4, 5, 7, b, \ldots)$ is not possible by Corollary 2.6.

The remaining cases are simpler, and we leave them as an exercise. Note that $a = 2, 3$ are impossible, since by the symmetry of H, we would have $h_{j-2} = a$ and $h_{j-1} = 4$: however, the Macaulay bound gives $a^{(j-2)} \leqslant a$ when $a \leqslant j - 2$, and here $j - 2 \geqslant 4$. □

Remark 4.5 (*Do height four Gorenstein sequences satisfy $\Delta H_{\leqslant j/2}$ is an O-sequence?*). The height four Gorenstein sequences of the form $H=(1, 4, 7, \dots)$ are probably close to an upper bound of those which may be shown to satisfy the condition $\Delta H_{\leqslant j/2}$ is an O-sequence, by the kind of arguments we have used for Theorem 4.2. Notice that we were not able to rule out the nonoccurring, sequence $H = (1, 4, 7, 10, 14, 10, 7, 4, 1)$ by a simple application of Macaulay bounds and the Gotzmann method of Lemma 2.3, together with calculation of $\Delta^4 H$. Rather, we needed to use Lemma 3.4, which involves the twisted cubic. Likewise, in proving other parts of Theorem 4.2, we use at times detailed information about low degree curves in $\mathbb{P}^3$.

Thus we are inclined to conjecture that there are height four Gorenstein sequences that do not satisfy the condition that $\Delta H_{\leqslant j/2}$ is an O-sequence.

Recall that we denote by $v_i(J)$ the number of degree-i generators of the ideal J. The next result follows from Theorems 3.11 and 4.2. Recall that the socle degree of H is the highest j such that $h_j \neq 0$.

Theorem 4.6. *Assume that the Gorenstein sequence H satisfies* $H=(1, 4, 7, h, b, \dots, 4, 1)$, *of socle degree* $j \geqslant 6$, *where* h, b *are arbitrary integers satisfying the necessary restrictions of Lemma* 4.1.

(i) *the dimension of the tangent space* $\mathcal{T}_I$ *on* $\mathbb{P}\mathrm{Gor}(H)$ *to a general element I of* $\mathfrak{C}(H) \subset \mathbb{P}\mathrm{Gor}(H)$ *satisfies*,

$$\dim_K \mathcal{T}_I = \dim \mathfrak{C}(H) + 1 + v_{j-1}(J) \tag{4.7}$$

where J is a generic element of $\mathbb{P}\mathrm{Gor}(H')$, $H'=(1, 3, 6, h-1, b-1, \dots, h-1, 6, 3, 1)$.

(ii) *When* $j \geqslant 6$, *the Zariski closure* $\overline{\mathfrak{C}(H)}$ *is a generically smooth irreducible component of* $\mathbb{P}\mathrm{Gor}(H)$ *when, equivalently*

(a) $v_{j-1}(J) = 0$ *for J generic in* $\mathbb{P}\mathrm{Gor}(H')$;
(b) *a generic* $J \in \mathbb{P}\mathrm{Gor}(H')$ *has no degree*-4 *relations*;
(c) $3h - b - 17 \geqslant 0$.

Proof. Here (i) follows immediately from Theorem 3.11(i), (ii). This shows (iia); by the symmetry of the minimal resolution of J, (iia) is equivalent to (iib). The third difference satisfies $(\Delta^3 H')_4 = 17 + b - 3h$, and under the assumption $j \geqslant 6$, it gives, when positive, the number of degree-4 relations—the linear relations among those generators of J having degree 3; when 0 or negative there are no such relations. This completes the proof of the equivalence of (iib) and (iic). □

We now show that there are monomial ideals in $R' = K[x, y, z]$, having certain Hilbert functions T' and having a small number of generators. This prepares a key step for Theorem 4.9. We consider Hilbert functions of the form $T' = (1, 3, 3, \dots, 2_a, \dots, 1_c, \dots, 0, \dots)$ where degree a is the first degree in which $T'_a < 3$, and c is the first degree $c \geqslant 3$ in which $T'_i \leqslant 1$, and d is the first positive degree in which $T'_i = 0$: we allow equalities among a, c, d, so if $a = c = 4$, $d = 5$, $T' = (1, 3, 3, 3, 1, 0, \dots)$. The following result is easy to verify.

Lemma 4.7. (i) *The Artinian algebra* $A = R'/J_{a,c,d}$, $J_{a,c,d} = (xy, xz, yz, x^a, y^c, z^d)$, $3 \leqslant a \leqslant c \leqslant d$ *has Hilbert function* $T'(a, c, d) = (1, 3, 3, \ldots, 2_a, \ldots, 1_c, \ldots, 0_d, \ldots)$ *in the sense above.*

(ii) *The Artinian algebra* $A = R'/K_{a,c}$, $K_{a,c} = (x^2, xy, z^2, x^{a-1}z, y^c)$, $3 \leqslant a \leqslant c$ *has Hilbert function* $T'(a, c) = (1, 3, 3, 2, \ldots, 1_a, \ldots, 0_c, \ldots)$.

Corollary 4.8 (*Artinian Gorenstein algebras with related minimal resolution*). (i) Maroscia [27,14,20,28]. *Let* $s = \sum_{i \geqslant 0} T'(a, c, d)_i$, *or* $\sum_{i \geqslant 0} T'_{a,c}$, *respectively. Then there are smooth degree-s punctual schemes* $\mathfrak{Z} = \mathfrak{Z}(a, c, d) \subset \mathbb{P}^3$ *or* $\mathfrak{Z} = \mathfrak{Z}(a, c) \subset \mathbb{P}^3$, *respectively, whose coordinate rings have the same minimal resolutions as the Artinian algebras defined by* $J_{a,c,d}$ *or* $K_{a,c}$, *respectively.*

(ii) (Boij [3]). *Furthermore, let* $j \geqslant 2c$, *or* $j \geqslant 2b$, *respectively, and let* $A = A(a, c, d, j, F)$ *or* $A = A(a, c, j, F)$, *respectively, denote a general enough GA quotient of* $\mathcal{O}_{\mathfrak{Z}}$, $\mathfrak{Z} = \mathfrak{Z}(a, c, d)$ *or* $\mathfrak{Z} = \mathfrak{Z}(a, c)$ *having socle degree j, defined by* $A = R/\mathrm{Ann}\ (F)$, $F \in (I_{\mathfrak{Z}})_j^{\perp}$. *The minimal resolution of A agrees with that of the corresponding coordinate ring* $\mathcal{O}_{\mathfrak{Z}}$ *in degrees up to* $j/2$.

Proof. P. Maroscia's well-known result deforms a given monomial ideal defining an Artinian algebra to a graded ideal defining a smooth punctual scheme $\mathfrak{Z}$, and having the same minimal resolution. M. Boij showed that a general enough GA quotient of $\mathfrak{Z}$ has a related minimal resolution. □

Theorem 4.9 (*Families* $\mathbb{P}\mathrm{Gor}(H)$ *with several components*). (i) *Assume that H is a Gorenstein sequence of socle degree* $j \geqslant 6$ *satisfying* (4.1), *namely* $H = (1, 4, 7, h, b, \ldots)$ *and that* $h \leqslant 10$. *Then there is a GA quotient of the coordinate ring of a smooth punctual scheme* $\mathfrak{Z}$ *having Hilbert function H, and* $H = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$.

(ii) *Assume further that* $3h - b - 17 \geqslant 0$ *and* $8 \leqslant h \leqslant 10$. *Then* $\mathbb{P}\mathrm{Gor}(H)$ *has at least two irreducible components, the component* $\overline{\mathfrak{C}}$, *and a component containing suitable GA algebras* $A = A(a, c, d, j, F)$ *or* $A = A(a, c, j, F)$, *respectively, that are quotients of the coordinate ring of smooth punctual schemes.*

Proof. Assume that $H = (1, 4, 7, h, b, \ldots, 1)$ has socle degree $j \geqslant 6$ and let $T' = \Delta H_{\leqslant j/2}$. By Theorem 4.2, T' is an *O*-sequence; since $h \leqslant 10$ Lemma 4.1 implies T' satisfies $T' = (1, 3, 3, h - 7, b - h, \ldots)$, with $h - 7 \leqslant 3$, with $T'_{\leqslant j/2}$ nonnegative, and nonincreasing after degree 0 to 1. Thus $T' = T'(a, c, d)$ or $T' = T'(a', c')$ for suitable (a, c, d) or (a', c'). Lemma 4.7 and Corollary 4.8(ii) imply that there is a Artinian Gorenstein algebra $A = R/I$ of Hilbert function *H*, such that the beginning of its minimal resolution is that of $R'/J(a, b, c)$ or $R'/K(a, b)$. In particular I_2 has at most two linear relations. Since one cannotspecialize from a GA algebra $A = R/I \in \overline{\mathfrak{C}(H)}$ where I_2 has three linear relations, to a GA algebra $A = A(a, c, d, j, F)$ or $A(a, c, j, F)$ where I_2 has at most two linear relations, the claim of the theorem follows. □

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