ANNALS OF MATHEMATICAL LOGIC – Volume 4, No. 1 (1972) pp. 31–73

DEGREES OF UNSOLVABILITY COMPLEMENTARY BETWEEN RECURSIVELY ENUMERABLE DEGREES, PART I.

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Received 11 January 1971

Given a set S of mutually incomparable degrees and a pair of degrees **a** and **b** we say that S is *complementary between* **a** and **b** whenever **a** is the greatest lower bound of the members of S and **b** is the least upper bound. A degree **a** is *minimal* if **O** is the least upper bound of the degrees strictly less than **a**. We obtain an indication of the variety of decision problems to be found amongst degrees of a particular type of looking at the pairs of degrees between which sets of degrees of that type are complementary. If S complementary between **O** and **a** we say that S is *complementary below* **a** and we prove below that there is a pair of minimal degrees complementary below **O**′.

Spector [8] showed that minimal degrees exist and Sacks [6] construated one below **O**′, the largest recursively enumerable degree. Shoenfield [7] proved that given any degree strictly between **O** and **O**′ we may find a minimal degree below **O**′ which is incomparable with it. Lachlan [3] proved that no pair of recursively enumerable (r.e.) degrees is complementary below **O**′ even though there is a pair of r.e. degrees complementary below some r.e. degree (see Yates [10] and Lachlan [3]). We construct below a pair of minimal degrees with join **O**′. Shoenfield's theorem is an immediate corollary of this. Since the theorem yields a pair complementary below **O**′ we have that no dramatic generalisation of Lachlan's theorem is possible. Related results proved elsewhere are: (1) there is a pair of degrees complementary below any given r.e. degree other than **O**, (2) there is a r.e. degree other than **O** below which no set of minimal degrees is complementary (although Yates [11] has shown there to be countably many minimal predecessors for each non-zero r.e. degree), (3) there are three r.e. degrees complementary below $\mathbf{O}'$

We take $\{\Phi_e | e \geqslant 0\}$ to be a standard enumeration of the partial recursive functionals. $\{\Phi_{e,s} | e, s \geqslant 0\}$ is a double sequence of finite approximations to these functionals satisfying the following: (i) $\{\Phi_{e,s}\}$ is a recursive set, (ii) $\Phi_{e,s} \subseteq \Phi_{e,s+1}$ for each e and each $s \geqslant 0$, (iii) $\Phi_e = \bigcup_{s \geqslant 0} \Phi_{e,s}$ for each $e \geqslant 0$, (iv) for each s $\Phi_{e,s}$ is empty for all but a finite set of numbers. The last condition is included in order to avoid an infinite search occurring at a stage of the construction. $\{R_e\}$ will be a standard list of the recursively enumerable sets with double sequence $\{R_{e,s}\}$ of approximations with properties similar to (i)–(iv) above for $\{\Phi_{e,s}\}$. And $\{F_e\}$ is an enumeration of the partial recursive functions, each F_e having its recursive tower $\{F_{e,s} | s \geqslant 0\}$ of finite approximations.

σ is said to be a *string* of *length* $n+1$ if it is an initial segment (or *beginning*) $C[n]$ of a characteristic function C defined on exactly $n+1$ numbers. If σ is a string of length $n+1$ and $m \leqslant n$ we write $\sigma[m]$ for the beginning of σ of length $m+1$. If we write $\mathrm{lh}(\sigma)$ for the length of σ and $y(\sigma_1, \sigma_2)$ for the least number y for which $\sigma_1(y) \neq \sigma_2(y)$, there is a natural ordering $\leqslant$ of the strings defined by:

$$\sigma_1 \leqslant \sigma_2 \leftrightarrow$$
$$\sigma_1 = \sigma_2 \text{ or } \mathrm{lh}(\sigma_1) < \mathrm{lh}(\sigma_2) \text{ or } \mathrm{lh}(\sigma_1) = \mathrm{lh}(\sigma_2) \text{ and } \sigma_1(y(\sigma_1, \sigma_2)) < \sigma_2(y(\sigma_1, \sigma_2)).$$

Define an ordering $\leqslant$ on the ordered pairs of strings by:

$$(\sigma_1, \sigma_2) \leqslant (\pi_1, \pi_2) \leftrightarrow$$
$$\sigma_1[y(\sigma_1, \sigma_2) - 1] < \pi_1[y(\pi_1, \pi_2) - 1] \text{ or}$$
$$\sigma_1[y(\sigma_1, \sigma_2) - 1] = \pi_1[y(\pi_1, \pi_2) - 1] \text{ and } \sigma_1 < \pi_1 \text{ or } \sigma_1 = \pi_1 \text{ and } \sigma_2 \leqslant \pi_2.$$

This will enable us to talk of the least pair of strings with a given property.

$\emptyset$ is the string defined nowhere and 0 and 1 are the strings with domain $\{0\}$ and respective ranges $\{0\}$ and $\{1\}$.

$\sigma * \tau$ is the string defined by:

$$\sigma * \tau(x) = \begin{cases} \sigma(x) \text{ if } x < \mathrm{lh}\sigma, \\ \tau(x - \mathrm{lh}\sigma) \text{ if } \mathrm{lh}\sigma \leqslant x < \mathrm{lh}\sigma + \mathrm{lh}\tau, \\ \text{undefined otherwise.} \end{cases}$$

If σ and τ are beginnings of some characteristic function C then we say that σ and τ are *compatible*, and write $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$ according to $\text{lh}\sigma \leqslant \text{lh}\tau$ or $\text{lh}\tau \leqslant \text{lh}\sigma$. Otherwise σ and τ are *incompatible*.

A *tree* T is a mapping from the strings into the strings such that if $T(\tau * i)$ is defined where i is 0 or 1 then so are $T(\tau * 1 - i)$ and $T(\tau)$, and such that the partial ordering induced on the domain of T coincides with the ordering $\subseteq$ on the range of T. The terms 'recursive tree' and 'partial recursive tree' will be used a natural informal way.

If $T(\tau * 0)$, $(\tau * 1)$ $(=T(\tau * 0)$, $T(\tau * 1))$ are defined then they comprise the *syzygy on T based on* $T(\tau)$. Otherwise if $T(\tau)$ is defined then $T(\tau)$ is an *end string for T*. A string σ is *compatible with* a tree T if σ lies on T (i.e., is in the range of T) or in an extension of an end string for T. T' is *compatible with* T if every string on T' is compatible with T.

We say that two strings σ_1, σ_2 *split* τ *for* e *through* x if $\sigma_1, \sigma_2 \supset \tau$ and $\Phi_e(\sigma_1, x)$ (σ_2, x) and $\Phi_e(\sigma_1, x)$, (σ_2, x) are defined and unequal. σ_1, σ_2 *split* τ *for* e *through* x *at stage* s if $\sigma_1, \sigma_2 \supset \tau$ and $\Phi_{e,s}(\sigma_1, x)$, (σ_2, x) are defined and unequal. Then σ_1, σ_2 split τ for e through x if and only if σ_1, σ_2 split τ for e through x at some stage $s \geqslant 0$ since $\Phi_e = \bigcup_{s \geqslant 0} \Phi_{e,s}$, and if σ_1, σ_2 split τ for e through x at stage s then σ_1, σ_2 split τ for e through x at every stage $s' > s$ because $\Phi_{e,s'} \supseteq \Phi_{e,s}$.

Before proving the main theorem we give a short proof of a weaker result.

Theorem 1. *There is a pair of degrees complementary below* **O**$'$.

Proof. Let D be a set of degree **O**$'$ such that D is recursive in every infinite subset of D (i.e., D is intro-reducible in the sense of [2]). We construct at stages $n \geqslant 0$ beginnings α_n, β_n of characteristic functions A and B respectively and take the required pair to be the degrees of A and B. For each n we will have $\text{lh}(\alpha_n) = \text{lh}(\beta_n)$. Strings α and β with $\alpha \supset \alpha_n$ and $\beta \supset \beta_n$ are said to be *admissible at stage* $n+1$ if for no $x \geqslant \text{lh}(\alpha_n)$ do we have $\alpha(x)$ and $\beta(x)$ defined and each equal to 0.

Stage 4e *of the construction.*
Define

$$x_0 = \text{the least number in } D,$$
$$x_{n+1} = \text{the least element of } D \text{ greater than } \text{lh}(\alpha_{4n+3}).$$

Let $\alpha \supseteq \alpha_{4e-1}$ and $\beta \supseteq \beta_{4e-1}$ comrpise the least pair of strings admissible at stage $4e$ with $\text{lh}\alpha = \text{lh}\beta = x_e$.
Define

$$\alpha_{4e}, \beta_{4e} = \alpha * 0, \beta * 0 \qquad \text{respectively.}$$

Stage $4e+1$

Look for the least triple (β, x, s) (under some recursive ordering) for which $\beta \supset \beta_{4e}$ and $\Phi_{e,s}(\beta, x)$ is defined and such that if $\Phi_e(\beta, x) = 1$ then $\beta(x) \neq 0$.
If no such (β, x, s) exists set

$$\alpha_{4e+1}, \beta_{4e+1} = \alpha_{4e} * 1, \beta_{4e} * 1 \qquad \text{respectively.}$$

Otherwise let α, β' be the least pair with $\alpha \supset \alpha_{4e}, \beta' \supset \beta_{4e}, \alpha, \beta$ admissable at stage $4e+1$ with $\beta' \supseteq \beta$, $\text{lh}(\alpha) = \text{lh}(\beta')$ and such that $\alpha(x)$ is defined and is not equal to $\Phi_e(\beta, x)$.
Define

$$\alpha_{4e+1}, \beta_{4e+1} = \alpha, \beta' \qquad \text{respectively.}$$

Stage $4e+2$.

The same as stage $4e+1$ but with α and β interchanged and $4e+2$, $4e+1$ written for $4e+1$, $4e$ respectively.

Stage $4e+3$.

Let (m, n) be the e^{th} apri of numbers (in some recursive ordering).

We look for the least quadruple (β^1, β^2, x, s) for which β^1, β^2 split $4e+2$ for n through x at stage s.

If (β^1, β^2, x, s) does not exist set

$$\alpha_{4e+3}, \beta_{4e+3} = \alpha_{4e+2} * 1, \beta_{4e+2} * 1 \qquad \text{respectively, and}$$

otherwise look for the least pair (α, s) with $\alpha \supset \alpha_{4e+2}$ such that α, β^1 and α, β^2 are admissable pairs and $\Phi_{m,s}(\alpha, x)$ is defined.

If α exists let β^i be the least of the strings β^1, β^2 such that

$$\Phi_m(\alpha, x) \neq \Phi_n(\beta^i, x)$$

and take α^*, β^* to be the least admissable pair of strings of equal length with $\alpha^* \supseteq \alpha$ and $\beta^* \supseteq \beta^i$.
Define

$$\alpha_{4e+3}, \beta_{4e+3} = \alpha^*, \beta^* \qquad \text{respectively.}$$

Otherwise take $\alpha_{4e+3}, \beta_{4e+3}$ to be the least pair of admissable strings of equal length with $\alpha_{4e+3} \supset \alpha_{4e+2}$ and $\beta_{4e+3} \supset \beta_{4e+2}$ and with $\mathrm{lh}(\alpha_{4e+3}) \geqslant \mathrm{lh}(\beta^1) + \mathrm{lh}(\beta^2)$.

Lemma 1. *A and B are recursive in* $\mathbf{O}'$.

Proof. We examine the questions asked during the construction. The result will follows from the fact that they are uniformly recursive in $\mathbf{O}'$ and in what we have defined at previous stages of the construction so that we could define α_n, β_n by a recursion schema using $\mathbf{O}'$ recursive functions.
(1)–(4) below correspond to the stages $4e$ to $4e+3$ of the construction.

(1) We require the number x_e, which depends only on $D \in \mathbf{O}'$ and on the strings α_{4e-1} and β_{4e-1} already defined (the admissable pairs form a recursive set).

(2) The set of triples (β, x, s) that we are interested in is a r.e. set qualified by a predicate recursive in α_{4e} and β_{4e}.

(3) Similarly for the triples (α, x, s).

(4) The quadruples (β^1, β^2, x, s) and the pairs (α, s) each form the intersection of an α_n, β_n recursive set and a fixed r.e. set.

It follows that if we write $\mathbf{a} = \deg A$ and $\mathbf{b} = \deg B$ then $\mathbf{a} \cup \mathbf{b} \leqslant \mathbf{O}'$.

Lemma 2. $\mathbf{O}' \leqslant \mathbf{a} \cup \mathbf{b}$.

Proof. If we inspect the construction we find that the only stages at which we fail to choose an admissable pair α, β as extensions of α_n, β_n respectively are the stages $4e \geqslant 0$ when α_{4e}, β_{4e} are chosen to be admissable apart from the fact that

$$\alpha_{4e}(x_e) = \beta_{4e}(x_e) = 0.$$

This means that $A \cap B$ is a subset of D and is infinite since infinitely many numbers x_e are chosen. Since D is intro-reducible we have $D \leqslant_T A \cap B$ where $\deg A \cup B \leqslant \mathbf{a} \cup \mathbf{b}$.

It follows from lemmas 1 and 2 that $\mathbf{O}' = \mathbf{a} \cup \mathbf{b}$.

Lemma 3. a *and* b *are incomparable*

Proof. Assume that

$$A = \Phi_e(B)$$

for some number e.

If a triple (β, x, s) exists satisfying stage $4e+1$ of the construction then we have that $\Phi_e(\beta_{4e+1}, x)$ is defined and is not equal to α_{4e+1}, which would mean that $\Phi_e(B, x) \neq A(x)$.

So for every pair (β, x) such that $\beta \supset \beta_{4e}$ and $\Phi_e(\beta, x)$ is defined we have that $\Phi_e(\beta, x) = 1$ which implies that A is empty, contradicting the fact that $A \cap B$ is an infinite subset of D.

Lemma 4. *If* $\Phi_m(A)$, $\Phi_n(B)$ *are total and* $\Phi_m(A) = \Phi_n(B)$ *then* $\Phi_m(A)$ *is recursive.*

Proof. Let (m, n) be the e^{th} pair of numbers. Then at stage $4e+3$ we look for a pair β^1, β^2 which split β_{4e+2} for n through some number x at a stage $s \geqslant 0$. If β^1, β^2 do not exist then $\Phi_n(B)$ will be recursive. In order to compute $\Phi_n(B, x)$ for a given number x we need only generate recursively the functionals $\Phi_{n,s}$ and also the extensions σ of β_{4e+2}, and if for some such σ and some $s \geqslant 0$ we have

$$\Phi_{n,s}(\sigma, s) = \delta$$

then we have that

$$\Phi_n(B, x) = \delta .$$

Otherwise there is a beginning β of B, which we can choose to properly extend β_{4e+2}, for which

$$\Phi_n(\beta, x) = \delta' \neq \delta,$$

so that for some $s^* > s$ we have

$$\Phi_{n,s^*}(\beta, x) = \delta' \neq \delta = \Phi_{n,s^*}(\sigma, x)$$

(since $\Phi_n = \bigcup_{s \geqslant 0} \Phi_{n,s}$ and $\Phi_{n,s} \subseteq \Phi_{n,s+1}$ for each s) and so β, σ split β_{4e+2} through x for n at stage s^*.

Say (β^1, β^2, x, s) exists.

If (α, s) does not exist then since $\beta^1 . \alpha_{4e+3}$ and $\beta^2 . \alpha_{4e+3}$ are admissable pairs at stage $4e+3$ and $\mathrm{lh}\alpha_{4e+3} \geqslant \max \ \mathrm{lh}\beta^i | i = 1$ or 2 there can be no extension α' of α_{4e+3} for which $\Phi_m(\alpha', x)$ is defined, so that $\Phi_m(A, x)$ is not defined.

If α exists then by choice of α_{4e+3} and β_{4e+3} we have that

$$\Phi_n(\beta_{4e+3}, x),\ \ \Phi_m(\alpha_{4e+3}, x)$$

are defined and unequal so that

$$\Phi_n(B) \neq \Phi_m(A).$$

It follows from the lemma that $\mathbf{a} \cap \mathbf{b}$ exists and is equal to $\mathbf{O}$.

We can adapt the proof so as to replace $\mathbf{O}$, $\mathbf{O}'$ by $\mathbf{c}$, $\mathbf{c}'$ for any given $\mathbf{c} \geqslant \mathbf{O}$. This has the corollary that every degree is a non-trivial meet of a pair of degrees. Lachlan [3] has shows that if $\mathbf{c}$ is r.e. and strictly below $\mathbf{O}'$ then we cannot in general choose the pair of degrees to be r.e. But we can ask:

(1) Is every degree below $\mathbf{O}'$ a non-trivial meet of two degrees below $\mathbf{O}'$?, or

(2) Is there some general class of r.e. degrees with non-trivial r.e. meets (e.g., Robert Robinson's low degrees [5]) ?

Sacks [6] examines lattice embeddings for the degrees as a whole and Lachlan [4] and Thomason [9] obtain results about lattice embeddings in the r.e. degrees, but little is known about embeddings which preserve greatest and least elements in the degrees below $\mathbf{O}'$ or in the r.e. degrees between two compatable r.e. degrees.

Theorem 2. *There exists a pair of minimal degrees with least upper bound* $\mathbf{O}'$.

Proof. Let f be a recursive function which enumerates without repetitions a r.e. set D of degree $\mathbf{O}'$. At stages $s \geqslant 0$ we construct strings α_s^0 and α_s^1 and take the pair of degrees to be the degrees of A^0 and A^1 where

$$A^i(x) = \lim_s \alpha_s^i(x)$$

for each $i \leqslant 1$ and each x. The strings α_s^0 and α_s^1 will be chosen to lie on certain finite trees $T_{e,s}^i$ with $i \leqslant 1$ where at any given stage $s \geqslant 0$ there will only be a finite number of these trees different from $\emptyset$.

If $\sigma \subseteq \alpha_s^p$ for some $p \leqslant 1$ then σ is said to have *rank e of the p^{th} kind at stage* $s+1$ where e is the least number for which

$$\sigma \subseteq T_{e,s}^p(\delta)$$

for some $\delta \leqslant 1$. We order the pairs (e, p) lexicographically upwards.

The method by which we make A^0, A^1 to be of minimal degree is a constructivisation of that of Spector's in [8] but different from that of [11] in that not every *syzygy* defined on a tree $T_{e,s}^p$ at a stage $2s+p-1 \geqslant 0$ will be a splitting pair for e, and also in that we will not expect the limit trees

$$T_e^p = \lim_s T_{e,s}^p$$

to be partial recursive, although if A^p lies on an infinite splitting portion of T_e^p then we will be able to select a partial recursive splitting subtree of T_e^p on which A^p also lies.

If $T_{e,s}^p(\tau)$, say, is defined and has been chosen as a member of *syzygy* which splits for e then if there is no *syzygy* for $T_{e,s}^p$ based on $T_{e,s}^p(\tau)$ which splits for e at stage s we say that $T_{e,s}^p(\tau)$ is a *boundary string for $T_{e,s}^p$ at stage s.*

The method by which we make D recursive in the join of the degrees of A^0 and A^1 is to ensure that if there is a stage s such that $T_{e+1,s}^0$ and $T_{e+1,s}^1(0)$ are beginnings of A^0 and A^1 respectively then

$$D_s(e) = D(e)$$

where $D_s = \{f(k) | k \leqslant s\}$.

Stage 0 *of the construction.*

Define

$$T_{-1,0}^p = I \qquad \text{(the identity tree)}$$

for each $p = 0$ or 1.

$$T_{e,0}^p = \emptyset \qquad \text{otherwise}.$$

Define

$$\alpha_0^p = \emptyset \qquad \text{for each } p = 0 \text{ or } 1.$$

Stage $2s + p + 1$.

Define

$$T^p{}_{1,s+1} = I\,.$$

Assume that $T^p_{i,s+1}$ has been defined for each $i < e$ and that $T^p_{e,s+1}(\tau)$ has been defined where τ is a string other than $\emptyset$ and that

$$T^p_{e,s+1}(\tau) = T^p_{e,s}(\tau)\,.$$

We may now base a *syzygy* on $T^p_{e,s}(\tau)$ at stage $s+1$ through one of the following cases:

Case I.

Let $T^p_{e,s}(\tau)$ have rank k of the p^{th} kind at stage $s+1$.

Assume that $T^p_{e,s}(\tau * 0), (\tau * 1)$ are defined and are compatible with each tree $T^p_{i,s+1}$ with $i < e$.

Also assume that one of the following hold:

(1) $T^p_{e,s}(\tau * 0), (\tau * 1)$ split for e at stage $s+1$, or

(2) there is no pair of strings $\sigma_1, \sigma_2 \supset T^p_{e,s}(\tau)$ which split for e at stage $s+1$ and which satisfy the following conditions:

(i) σ_1, σ_2 are compatible with every tree $T^p_{i,s+1}$ with $i < e$ and neither of σ_1, σ_2 properly extend a boundary string $T^p_{i,s+1}(\pi)$ with $i < e$ and

$$T^p_{e,s}(\tau) \subset T^p_{i,s+1}(\pi)\,,$$

(ii) if σ_1 or σ_2 extends some *prohibited string* π (a term to be defined later) where

$$T^p_{e,s}(\tau) \subset \pi$$

then we may *free* π by *stretching* a string of rank k^* of the $(1-p)^{\text{th}}$ kind where

$$(k, p) < (k^*, 1-p)\,,$$

(iii) by defining

$$\sigma_1, \sigma_2 = T^p_{e,s+1}(\tau * 0), (\tau * 1)$$

respectively we do not make some string π of rank k^* of the q^{th} kind at stage $s+1$ *liable to require attention at a stage greater than* $2s + p + 1$ (again a term to be defined later) through a number $e' > k^*$ where

$$(k, p) \geqslant (k^*, q) \text{ and } q \leqslant 1, \text{ or}$$

(3) $$T^p_{e,s}(\tau) \nsubseteq \alpha^p_s .$$

We define

$$T^p_{e,s+1}(\tau * 0), (\tau * 1) = T^p_{e,s}(\tau * 0), (\tau * 1) \text{ respectively.}$$

Case II.

Assume that case I does not hold and that none of (1)–(3) of case I holds.

So there does exist a pair σ_1, σ_2 as described in (2). We define

$$T^p_{e,s+1}(\tau * 0), (\tau * 1) = \sigma_1, \sigma_2$$

respectively, and we require a string of rank k^* of the $(1-p)^{\text{th}}$ kind at stage $s+1$ to free all the prohibited strings π such that

$$T^p_{e,s}(\tau) \subset \pi \subseteq \sigma_1$$

or $$T^p_{e,s}(\tau) \subset \pi \subseteq \sigma_2 ,$$

where the choose k^* to be the largest possible such number.

Case III.

If cases I and II do not hold but

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

define

$$T^p_{e,s+1}(\tau * 0), (\tau * 1) = \sigma'_1, \sigma'_2$$

respectively where σ'_1, σ'_2 is the least pair of incompatible strings which extend $T^p_{e,s}(\tau)$ and which are compatible with every tree $T^p_{i,s+1}$ with $i < e$. This concludes case III.

We say that e^* *is liable to require attention through* $x-1$ *for* q *at stage* $2s+p+1$ if

$$D_s(x-1) = 1$$

and e^* is the largest number for which there is a string $\sigma \supset T^q_{e^*,s+p \dot{-} q}(0)$ which is incompatible with each $T^q_{x,w}(0)$, $w \leqslant s$, such that

$$T^{1-q}_{x,w}(0) \subseteq \alpha^{1-q}_{s+p \dot{-} (1-q)}\ ,$$

and which is compatible with each tree $T^q_{i,s+p \dot{-} q}$ such that $i < e^*$.

At stage $2s+p+1$ we make a string π of rank k^* of the q^{th} kind *liable to require attention at a stage greater than* $2s+p+1$ if at end of stage $2s+p+1$ we have that k^*, k^{**} are liable to require attention through some $x-1$ for q, $1-q$ respectively at stage $2s+p+2$, and

$$(k^*, q) > (k^{**}, 1-q).$$

Assume now that the extensions σ_1, σ_2 of $T^p_{e,s+1}(\tau)$ as described in I(2) do exist except that (iii) fails to gold. Then σ_1 or σ_2 extends a string $T^p_{x,t}(0)$ where $t \leqslant s$ and $x-1$ is greater than the rank of $T^p_{e,s}(\tau)$. If e^* is liable to require attention through $x-1$ for p at stage $2s+p+1$ we *require* $T^p_{e^*,s}(0)$ *to be stretched at stage* $2s+p+1$ unless this has already been done at some earlier stage for the potential *syzygy* σ_1, σ_2.

The new number enumerated in D at stage $s+1$

Let

$$f(s+1) = x-1\ .$$

If $T^p_{e^*,s}(0)$ is liable to require attention through $x-1$ at stage $2s+p+1$ for some $e^* \geqslant 0$ then $T^p_{e^*,s}(0)$ requires attention at stage $2s+p+1$ through $x-1$. We will try to ensure at every subsequent stage $w > s$ that we either have

$$T^p_{x,t}(0) \not\subseteq \alpha^p_\omega \quad \text{or} \quad T^{1-p}_{x,t}(0) \not\subseteq \alpha^{1-p}_w$$

for each $t \leqslant s$, and so as to achieve this certain strings $T^p_{x,t}(0)$ with $t \leqslant s$ may become strings *prohibited through* x.

At stage $2s+p$ we may have required some string to free a prohibited string π where we defined extensions of some string through case II at stage $2s+p$ one of which extended π. Assume that π was prohibited at stage $2s+p$ by virtue of being a string $T^{1-p}_{y,t}(0)$ for some y, t where $t \leqslant t'$ and $f(t'+1) = y-1$. Then we choose $T^p_{e^*,s}(0)$ in a similar way to that above to be a string for which there is a proper extension σ compatible with all the trees $T^p_{i,s}$ with $i < e^*$ and incompatible with each string $T^p_{y,t}(0)$ such that $t \leqslant t'$ and

$$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p-1} .$$

And $T^p_{e^*,s}(0)$ is the string which is required to free π at stage $2s+p+1$ if (and only if) $T^p_{e^*,s+1}(\emptyset)$ is defined and $T^p_{e^*,s}(0)$, (1) and σ are compatible with each tree $T^p_{i,s+1}$ with $i < e$. Also we have that each string $T^p_{y,t}(0)$ with $t \leqslant t'$ and

$$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p-1}$$

is prohibited through y at each stage $t^* > 2s+p$ such that we have not required $T^p_{y,t}(0)$ to be freed at a stage t^{**} such that

$$t^* > t^{**} > 2s+p .$$

We define $T^p_{e,s+1}(0)$, (1) *at stage* $2s+p+1$ *if* $T^p_{e,s+1}(\emptyset)$ *is defined and is not equal to* α^p_{s+1}.

Case (a).

Assume that $T^p_{e,s}(0)$, (1) are defined and are compatible with every tree $T^p_{i,s+1}$ with $i < e$.
If e requires attention because $f(s+1) = x-1$ or if $T^p_{e,s}(0)$ is required to free a string π where π is prohibited by virtue of being a string $T^{1-p}_{x,t}(0)$ for some $t \leqslant s$, or if $T_{e,s}(0)$ is required to be stretched because we would have defined strings $T^p_{i,s+1}(\tau * 0)$, $(\tau * 1)$ through case II apart from the fact that $T^p_{i,s+1}(\tau * 0)$ or $(\tau * 1)$ extends a string $T^p_{x,t}(0)$ for some $t \leqslant s$ then let $\sigma \supset T^p_{e,s}(0)$ be the least string incompatible with each $T^p_{y,t}(0)$ for

$$t \leqslant \mu\ t'(f(t'+1) = y-1 \text{ or } t' = s)$$

and

$$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p},$$

where $z < y \leqslant x$ where z is chosen to be the least number for which there exists such a string and such that σ is compatible with each tree $T^p_{i,s+1}$ with $i < e$.
Define

$$T^p_{e,s+1}(0),\ (1) = \sigma,\ T^p_{e,s}(1)$$

respectively and in the former case every string $T^q_{x,t}(0)$ with $t \leqslant s$ and

$$T^{1-q}_{x,t}(0) \subseteq \alpha^{1-q}_{sg(p+q)+s}$$

becomes prohibited through x.
We now inductively make changes in the definitions of some of the strings $T^p_{i,s+1}(\tau)$, $i < e$. Assume that the necessary changes have been made on all trees $T^p_{j,s+1}$, $j < i$. Let $T^p_{i,s+1}(\tau)$ be the least string such that either $T^p_{i,s+1}(\tau)$ is not compatible with some tree $T^p_{j,s+1}$ with $j < i$, or

$$T^p_{e,s}(0) \subseteq T^p_{i,s+1}(\tau) \subset \sigma$$

and $T^p_{i,s+1}(\tau)$ is a boundary string for i at stage $s+1$. If no such string exists we make no changes. Otherwise we re-define

$$T^p_{i,s+1}(\tau) = \sigma,$$

and $T^p_{i,s+1}(\tau * \pi)$ is undefined for each $\pi \supset \emptyset$. We say that $T^p_{e,s}(0)$ *is stretched to* σ $(= T^p_{e,s+1}(0))$.
Otherwise we define

$$T^p_{e,s+1}(0),\ (1) = T^p_{e,s}(0),\ (1) \qquad \text{respectively}$$

Case (b).

If $T^p_{e,s}(0)$, (1) are not defined and compatible witj every tree $T^p_{i,s+1}$, $i < e$, let σ_1, σ_2 be the least pair of incompatible extensions of $T^p_{e,s+1}(\emptyset)$ compatible with every tree $T^p_{i,s+1}$ with $i < e$ where if one of these strings extends no string $T^p_{e,t}(0)$ with $t \leqslant s$ we take it to be σ_2.

Define

$$T^p_{e,s+1}(0), (1) = \sigma_1, \sigma_2 \qquad \text{respectively.}$$

In either of cases (a) or (b) if

$$T^p_{e,s+1}(0), (1) \neq T^p_{e,s}(0), (1),$$

define

$$T^p_{e,s+1}(0) = \alpha^p_{s+1} = T^p_{e+1,s+1}(\emptyset)\,.$$

otherwise merely defining

$$T^p_{e,s+1}(0) = T^p_{e+1,s+1}(\emptyset)\,.$$

Lemma 5. *For each number $e \geqslant 0$ and each $p \leqslant 1$ $T^p_e(0) = \lim_s T^p_{e,s}(0)$ is defined.*

Proof. First of all we show that there is a stage after which $T^p_{e,s}(0), (1)$ do not change other than by being stretched. As inductive hypothesis we take:

(i) for all $s >$ some t $T^q_{i,s}(0), (1)$ change value only through being stretched if $(i, q) < (e, p)$.

(ii) $D_{t*}[e] = D[e]$ where

$$t = 2t^* + q^* + 1\,,$$

(iii) for each $i <$ some $e' < e$, if $s > t$ then $T^p_{e,s}(0), (1)$ do not change value because of the definition of a new *syzygy* for $T^p_{i,w}$ at a stage $2w+p-1 > 2s+p-1$.

We inductively verify the validity of the hypothesis for every $e' < e$ and from this obtain the first part of the step in the main induction.

We may assume that at no stage $s > t^*$ is $T^p_{e-1,s}(0)$ stretched. To see this we look at the three ways in which $T^p_{e-1,s}(0)$ might be stretched:

1. $T^p_{e-1,s}(0)$ may be required to free some prohibited string π through the definition of strings on a tree through case II.

But in order that this should happen the string for which new extensions are defined must have rank k where

$$(k, 1-p) < (e-1, p)\,.$$

And this means that some string $T^{1-p}_{e^*,s}(0)$ where

$$(e^*, 1-p) < (e, p)$$

changes at a stage $s > t^*$ and not through being stretched which contradicts (i) of the inductive hypothesis.

2. $e-1$ may require attention at some stage greater than t for p.

We show that this can happen at most a finite number of times. At stage t $e-1$ can only be liable to require attention through a finite number of numbers $x-1$ since $T^p_{x,t'}(0)$ is only defined for a finite number of numbers x with $t' \leq t$, and $e-1$ can only require attention at most once through each of these numbers. Also it is easy to see that if $T^p_{x,t'}(0)$ is defined for no $t' \leq t$ then $e-1$ cannot require attention through $x-1$ at a stage $2s+p+1 > t$. Since $e-1$ is not liable to require attention through $x-1$ at stage t and since

$$D_{t^*}[e] = D[e].$$

we must define extensions at some stage $>t$ of some string $T^q_{i,s}(\tau)$ of rank e' which renders $e-1$ liable to require attention. This is because if $e-1$ becomes liable to require attention through $x-1$ through some e' requiring attention at a stage $t' > t$ through a number $x'-1$ then we have $x < x'$, since if a string of rank e' is stretched to be incompatible with each string onto which the x'^{th} tree of the r^{th} kind maps 0 at stage $2u+r+1 < t' = 2s'+r+1$ where

$$\alpha^{1-r}_{s'+r} \supseteq T^{1-r}_{x',u+1}(0)$$

then it will be stretched to be incompatible with all such strings of greater rank. And $x > x'$ since otherwise by the construction there can have been no string of rank $\geq e'$ of the r^{th} kind incompatible with each $T^r_{x,u}(0)$ defined before stage t' with

$$\alpha^{1-r}_{s'+r} \supseteq T^{1-r}_{x,u}(0)$$

and compatible with all the i^{th} trees at stage t' with $i < e'$.

So at some stage $t' > t$ we base a *syzygy* on a string $T^q_{i,s+1}(\tau)$ of rank e' of the q^{th} kind at stage $s+1$ which renders $e-1$ liable to require attention at some stage greater than t. There are two possibilities:

(a) $(e', q) \geqslant (e-1, p)$.

But this cannot happen since

$$D_{t^*}[e] = D[e]$$

and because $T^q_{i,s+1}(\tau * 0), (\tau * 1)$ are defined through case II and must satisfy condition (2) (iii) of the construction at stage t'.

(b) As for case 1. above we cannot have

$$(e', q) < (e-1, p)$$

because of (i) of the inductive hypothesis.

3. We may require $T^p_{e-1,s}(0)$ to be stretched at a stage $2s+p+1 > t$.

This means that there are potential extensions σ_1, σ_2 of a string $T^p_{e',s}(\tau)$ which satisfy all the requirements of case II at stage $2s+p+1$ except for (iii) where $T^p_{e',s}(\tau)$ is of rank $\leqslant e-1$ of the p^{th} kind at stage $s+1$. Then the assumption implies that if e^* is liable to require attention through $x-1$ for $1-p$ at stage $2s+p+1$ where $T^p_{e-1,s}(0)$ is required to be stretched because one of the potential extensions σ_1 or σ_2 extends a string $T^p_{x,w}(0)$ with $w \leqslant s$ then $x > e$ and

$$(e^*, 1-p) < (e-1, p)$$

and $e-1$ is liable to require attention through $x-1$ for p at stage $2s+p+1$

$$(e^*, 1-p) < (e-1, p)$$

since no alterations are made to trees of the $(1-p)^{\text{th}}$ kind at stage $2s+p+1$ and so if by taking

$$\sigma_1, \sigma_2 = T^p_{e',s+1}(\tau * 0), (\tau * 1)$$

respectively we would have made a string π of rank k^* of the q^{th} kind liable to require attention at some stage greater than $2s+p+1$ then we have

$$(e-1, p) > (k^*, q) \geqslant (e^*, 1-p) .$$

We show that there can only be finitely many such numbers x, or more specifically, if e^*, $e-1$ are liable to require attention for $1-p$, p respectively through $x-1$ at stage $2s+p+1 > t$ where

$$(e^*, 1-p) < (e-1, p)$$

then e^*, $e-1$ are liable to require attention for $1-p$, p respectively through $x-1$ at stage t. This is because if the former holds then $e-1$ is liable to require attention through $x-1$ at stage $2s+p+1$ and from part 2. we know that in this case $e-1$ must have been liable to require attention through $x-1$ at stage t.

Lastly we notice that $T^p_{e-1,s}(0)$ can only be stretched by being required to be stretched at a finite number of stages through a given number $x-1$, for if $T^p_{e-1,s}(0)$ is stretched through being required to be stretched at a stage $2s+p+1 > t$ through $x-1$ then $e-1$ is not liable to require attention for p through $x-1$ at stage $2s+p+3$ since $x > e$, and in fact is not liable to require attention for p through $x-1$ at a stage $> 2s+p+3$ by a similar argument to that in which we limited the relevant numbers $x-1$ to a finite set.

So $T^p_{e-1,s}(0)$ is stretched at no stage $2s+p+1 > t$ and hence by the inductive hypothesis $T^p_e(\emptyset)$ exists where

$$T^p_e(\emptyset) = \lim_s T^p_{e,s}(\emptyset) = \lim_s T^p_{e-1,s}(0)$$

and

$$T^p_{e,t}(\emptyset) = T^p_e(\emptyset) .$$

We may assume that for all $i < e$ either there is a string τ^i for which

$$T^p_{i,s}(\tau^i) = T^p_e(\emptyset)$$

for all $s > t^*$ or else $T^p_{e,s}(\emptyset)$ lies on $T^p_{i,s}$ for no $s > t^*$. If $T^p_{e,s}(0)$, (1) are to change at a stage $2s+p+1 > t$ other then through being stretched we must at stage $2s+p+1$ have $T^p_{i,s+1}(\tau^i * 0)$, $(\tau^i * 1) \neq T^p_{i,s}(\tau^i * 0)$, $(\tau^i * 1)$ respectively for some $i < e$.

We take as the hypothesis for a sub-induction:

There is a stage $2t(i)+p+1 > t$ such that for each $j < i$ wither for each $s > t(i)$ $T^p_{j,s}(\tau^j * 0)$, $(\tau^j * 1)$ split $T^p_{j,s}(\tau^j)$ for j at stage $s+1$ or $T^p_{j,s}(\tau^j * 0)$, $(\tau^j * 1)$ split for j at no stage $s+1 > t(i)$; and also for each $j < i$, each $\pi \supset \emptyset$, if for some $s > t(i)$ and every $\pi' * q$ with $q \leqslant 1$ and $\pi' * q \subseteq \pi$ we have that $T^p_{j,s}(\tau^j * \pi' * q)$, $(\tau^j * \pi' * 1-q)$ split $T^p_{j,s}(\tau^j)$ for j at stage $s+1$ and are not boundary strings for a tree $T^p_{k,s}$ with $k < j$ then $T^p_{j,w}(\tau^j * \pi)$ changes at no stage $2w+p+1 > 2s+p+1$ except as a result of being stretched.

There are two possibilities for the number i:

(a) at no stage $2s+p+1 > 2t(i)+p+1$ do we define strings $T^p_{i,s+1}(\tau^i * 0)$, $(\tau^i * 1)$ which split $T^p_{i,s}(\tau^i)$ for i at stage $s+1$. In this case the next stage of the induction follows immediately.

(b) at a stage $2s+p+1 > 2t(i)+p+1$ the strings $T^p_{i,s+1}(\tau^i * 0)$, $(\tau * 1)$ are defined and split for i at stage $s+1$.

If σ_1, σ_2 are respective extensions of $T^p_{i,s+1}(\tau^i * 0)$, $(\tau^i * 1)$ then σ_1, σ_2 split for i at each stage $w+1 \geqslant s+1$. This means that if there is a stage $w+1 > s+1$ such that $T^p_{i,w+1}(\tau^i * 0)$, $(\tau^i * 1)$ do not split for i at stage $w+1$ then at some stage $2u+p+1 > 2s+p+1$ we must have

$$(T^p_{i,u+1}(\tau^i * 0), (\tau^i * 1)) \neq (T^p_{i,u}(\tau^i * 0), (\tau^i * 1))$$

other than as a result of a member of the latter syzygy being stretched at a stage $2u+p+1$. That is we must define strings $T^p_{j,u+1}(\pi * 0)$, $(\pi * 1)$ through case II at stage $2u+p+1$ where $j < i$ and $T^p_{j,u}(\pi)$ is a boundary string for some tree $T^p_{k,u}$ with $k \leqslant j$ and where

$$T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)$$

for some $q \leqslant 1$ (If $T^p_{j,u}(\pi)$ is not such a boundary string then we would define strings $T^p_{j,u}(\pi)$ is a boundary string for some tree $T^p_{k,u}$ with $k \leqslant j$ and where

$$T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)$$

for some $q \leqslant 1$ (If $T^p_{j,u}(\pi)$ is not such a boundary string then we would define strings $T^p_{j,u+1}(\pi' * 0)$, $(\pi' * 1)$ through case II where $\pi' \subset \pi$ and by the construction this would preclude such a definition for $T^p_{j,u+1}(\pi * 0)$, $(\pi * 1)$ at stage $2u+p+1$).
Since $u > t^*$ we have

$$T^p_{i,u}(\tau^i) = T^p_i(\tau^i)$$

and so

$$T^p_{i,u}(\tau^i) \subseteq T^p_{j,u}(\pi),$$

and since $u > t(i)$ we cannot have $\pi = \tau^i$ by the inductive hypothesis which means that

$$T^p_{i,u}(\tau^i) \subset T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)$$

for some $q \leqslant 1$.

Choose $v \geqslant s$ to be the least number for which we have that $T^p_{j,v+1}(\pi)$ is a boundary string for a tree $T^p_{k,v+1}$ with $k \leqslant j$ and for which we have that

$$T^p_{i,v+1}(\tau^i) \subset T^p_{j,v+1}(\pi) \subset T^p_{i,v+1}(\tau^i * q)\ .$$

Let

$$T^p_{j,v+1}(\pi) = T^p_{k,v+1}(\pi^*)\ .$$

There are now three possible ways in which the first part of the next step of the sub-induction can fail with $t(i+1) = s$:

( ) either

$$T^p_{k,v}(\pi^*) \subseteq T^p_{i,v}(\tau^i)$$

and $T^p_{k,v}(\pi^*)$ alters through stretching at stage $2v+p+1$, or

$$T^p_{i,v}(\tau^i * q) \subseteq T^p_{k,v}(\pi^*)$$

and $T^p_{i,v}(\tau^i * q)$ alters through stretching at stage $2v+p+1$,

(ii) $T^p_{k,\nu+1}(\pi^*)$ is defined at stage $2\nu+p+1$ through case II of the construction,

(iii) $T^p_{k,\nu}(\pi^* * 0)$, $(\pi^* * 1)$ split for k at stage ν but $T^p_{k,\nu+1}(\pi^* * 0)$, $(\pi^* * 1)$ do not split for k at stage $\nu+1$.

If the first part of (i) occurs then

$$T^p_{k,\nu+1}(\pi^*) = T^p_{i,\nu+1}(\tau^i)$$

if the latter is to be defined.

For the second part we notice that if

$$T^p_{i,\nu+1}(\tau^i * q) \supset T^p_{k,\nu+1}(\pi^*)$$

then by the nature of the stretching operation $T^p_{k,\nu+1}(\pi^*)$ cannot be a boundary string for $T^p_{k,\nu+1}$.

If (ii) holds then there is a $\pi' \subset \pi^*$ such that

$$T^p_{k,\nu}(\pi') \subset T^p_{k,\nu+1}(\pi^*) \subset T^p_{i,\nu+1}(\tau^i * q)$$

and such that $T^p_{k,\nu}(\pi')$ is a boundary string for a tree $T^p_{k',\nu+1}$ with

$$k' \leqslant k \leqslant i\,.$$

Arguing as above we must also have

$$T^p_{i,\nu}(\tau^i) \subset T_{k,\nu}(\pi')$$

which contradicts the choice of ν.

Finally (iii) cannot occur since by the second part of the hypothesis of the sub-induction it would mean that there is a $\pi' * q'$ where $q' \leqslant 1$ such that

$$\tau^k \subset \pi' * q' \subset \pi^* * r$$

for some $r \leqslant 1$ and such that $T^p_{k,\nu}(\pi' * q')$, $(\pi' * 1 - q')$ do not split for k at stage ν. And this would imply by definition of case II of the construction that we have a string

$$T^p_{k,v}(\tau^k * \sigma) \subseteq T^p_{k,v}(\pi^*)$$

with $\sigma \supset \emptyset$ which is a boundary string for some tree $T^p_{k,v}$ with

$$k' < k \leqslant j .$$

Since

$$T^p_{k,v}(\tau^k * \sigma) \supset T^p_{k,v}(\tau^k) = T^p_{i,v}(\tau^i)$$

and

$$T^p_{k,v}(\pi^*) \subset T^p_{i,v}(\tau^i * q)$$

this contradicts the definition of v again.

The second half of the $(t+1)^{\text{th}}$ step of the sub-induction proceeds exactly as does the proof of the first hald when case (b) applies. The only difficulty is that we must deal with the relevant splitting pairs $T^p_{i,s+1}(\pi * 0), (\pi * 1)$ on $T^p_{i,s+1}$ above $T^p_{i,s+1}(\tau^i)$ by induction on the length of π where the base of the induction is given by the first part of the sub-induction.

It follows that $t(e)$ exists.

Let $i < e$ be the greatest number for which $T^p_{i,t(e)+1}(\tau^i * 0), (\tau^i * 1)$ are defined and split for i at stage $t(e)+1$. Then from the proof of the sub-induction for each $w > t(e)$ we have

$$T^p_{i,w}(\tau^i * 0), (\tau^i * 1) \subseteq T^p_{i,w+1}(\tau^i * 0), (\tau^i * 1)$$

respectively and if $i < j < e$ and $T^p_{j,w+1}(\tau^j * 0), (\tau^j * 1)$ are defined then

$$T^p_{j,w+1}(\tau^j * 0), (\tau^j * 1) = T^p_{i,w+1}(\tau^i * 0), (\tau^i * 1)$$

respectively.
So at each stage $2w+p+1 > 2t(e)+p+1$ we have

$$T^p_{e,w+1}(0), (1) \supseteq T^p_{i,t(e)+1}(0), (1)$$

respectively where we only fail to have equality when $T^p_{e,w+1}(0), (1)$ have been stretched for some reason.

As in the proof of the first part of the sub-induction we never have a boundary string π for a tree $T^p_{j,w+1}$ with $j < e$ where

$$T^p_{e,w+1}(\emptyset) \subset \pi \subset T^p_{e,w+1}(0)$$

or

$$T^p_{e,w+1}(\emptyset) \subset \pi \subset T^p_{e,w+1}(1)$$

and hence

$$T^p_{e,w+1}(0), (1) \supseteq T^p_{e,w}(0), (1)$$

respectively for each $w > t(e)$ and $T^p_{e,w}(0), (1)$ only change value at a stage $2w+p+1$ through being stretched.

It follows easily from the lemma that $\lim_s T^p_{e,s}$ exists for all e and for each $p \leqslant 1$.

From the proof of lemma 5 we have that $\lim_s T^p_{e,s}(0)$ exists for each e, p. If there is a stage t such that

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

for no $s > t$ then by construction if

$$(T^p_{e,s+1}(\tau * 0), (\tau * 1)) \neq (T^p_{e,s}(\tau * 0), (\tau * 1))$$

for some $s > t$ other than through a member of the syzygy being stretched we have that $T^p_{e,w}(\tau * 0), (\tau * 1)$ are defined for no $w > s$. And since we only stretch strings $T^p_{i,s}(0)$ such that

$$T^p_{i,s}(0) \subseteq \alpha^p_s$$

at stage $2s+p+1$, we cannot stretch $T^p_{e,s}(\tau * 0), (\tau * 1)$ at a stage $s > t$. If

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

for each $s > a$ stage t we notice that if τ has length K then neither of $T^p_{e,s}(\tau * 0)$ or $(\tau * 1)$ have rank greater than $e+K+1$ at any stage $s \geqslant 0$. Hence $\lim_s T^p_{e,s}(\tau * 0), (\tau * 1)$ exist since $\lim_s T^p_{e+K+1,s}(0)$ exists.

Lemma 6. *D is recursive in the recursive join of A^0 and A^1.*

Proof. Since $\lim_s T^p_{e,s}(0)$ exists for each $e \geqslant 0$ and each $p = 0$ or 1 we have that if

$$A^0, A^1 = \lim_s \alpha^0_s, \lim_s \alpha^p_s$$

respectively then A^0, A^1 are well defined sets of degree less than or equal to $\mathbf{O}'$.

We show that whenever $s(e)$ is a number for which $T^0_{e+1,s(e)}(0)$ and $T^1_{e+1,s(e)}(0)$ are respective beginnings of A^0 and A^1 it happens that

$$D_{s(e)}(e) = D(e)\,.$$

The lemma follows from the fact that the whole construction proceeds uniformly recursively and from the fact that there always exists such a number $s(e)$.

Assume that there are numbers s and e such that

$$e \in D$$

but for which

$$D_s(e) = 1$$

and $T^p_{e+1,s}(0)$ is a beginning of A^p for each number $p \leqslant 1$.
Let

$$s^* = \mu s(e \in D_{s+1})$$

so that $s \leqslant s^*$ and either some number $e(0)$ requires attention through e at step $2s^*+1$ or some number $e(1)$ requires attention through e at step $2s^*+2$. We need only verify that some number $e^* \geqslant 0$ is liable to require attention through e for 0 or 1 at stage $2s^*+1$ or stage $2s^*+2$ respectively, which is easy since at worst we can take

$$e(p) = 0$$

for each $p = 0$ or 1.
To prove this for each $p \leqslant 1$ take as inductive hypothesis:

$$T^p_{0,w}(0) \text{ is defined and if } T^p_{0,w}(0) = \pi$$

then for some string σ we have that $\pi * \sigma$ is incompatible with each $T^p_{e+1,u}(0)$ with $u \leqslant w$.

The base of the induction is given by $w = 1$ since $T^p_{0,1}(0)$, (1) are defined for each $p \leqslant 1$ but $T^p_{y,u}(0)$ is defined for no numbers y, u, p where

$$y > 0,\ \ 0 \leqslant p \leqslant 1 \ \text{ and } \ 0 \leqslant u \leqslant 1 .$$

Assuming that the induction fails let the hypothesis hold for $w = W$ but not for $w = W + 1$, and let

$$T^p_{0,W}(0) = \Pi$$

and let $\Pi * \Sigma$ be incompatible with each $T^p_{e+1,u}(0)$ with $u \leqslant W$. So

$$\Pi * \Sigma \subseteq T^p_{e+1,W+1}(0)$$

or

$$T^p_{0,W+1}(0) \neq \Pi.$$

If

$$\Pi * \Sigma \subset T^p_{e+1,W+1}(0)$$

then they hypothesis holds for $w = W + 1$ for

$$\pi * \sigma = T^p_{e+1,W+1}(1).$$

We cannot have

$$\Pi * \Sigma = T^p_{e+1,W+1}(0)$$

unless the hypothesis hold for $w = W + 1$ with more than one string σ (say Σ and Σ^*) since by the construction of $T^p_{e+1,W+1}(0)$, (1) we would not have a $u < W + 1$ for which

$$T^p_{e+1,u}(0) \subseteq T^p_{e+1,W+1}(1)$$

unless

$$T^p_{e+1,u}(0) \subseteq T^p_{e+1,W+1}(0) .$$

So if

$$\Pi * \Sigma = T^p_{c+1,W+1}(0)$$

the hypothesis would follow for $w = W + 1$ with

$$\sigma = \Sigma^* .$$

If

$$T^p_{0,W+1}(0) \neq \Pi$$

then since

$$T^p_{-1,W} = T^p_{-1,W+1} = I$$

for each $p \leqslant 1$ it must happen that $T^p_{0,W}(0)$ is stretched to $T^p_{0,W+1}(0)$ at stage $2W+p+1$. If

$$T^p_{0,W+1}(0) \subset \Pi * \Sigma$$

then the inductive step follows using

$$\pi * \sigma = \Pi * \Sigma$$

again. If

$$T^p_{0,W+1}(0) \supseteq \Pi * \Sigma$$

then we may take for $w = W + 1$

$$\pi = T^p_{0,W+1}(0),\ \sigma = \pi * q$$

for some $q \leqslant 1$ such that α^p_{W+1} is incompatible with $\pi * q$.
By the construction if $T^p_{0,W+1}(0)$ is incompatible with $\Pi * \Sigma$ then since $\Pi * \Sigma$ satisfies the hypothesis for $w = W$ we must have that $T^p_{0,W+1}(0) * q$ satisfies the hypothesis for $w = W + 1$ for some $q \leqslant 1$.

So $e(n)$ requires attention at step $2s^*+p+1$ for some $p \leqslant 1$ which means that

$$T^q_{c+1,s}(0) \not\subseteq \alpha^q_{s^*+1}$$

for some $q \leqslant 1$.

Let $t^* > s^*$ be the least number such that

$$T^q_{e+1,s}(0) \subseteq \alpha^q_{w+1}$$

for each $q \leqslant 1$ and each $w \geqslant t^*$.
Inspection of the construction gives us that at each stage greater than $2s^*+p+1$ for each $u < s^*+1$ if

$$T^q_{e+1,u}(0) \subseteq \alpha^q_w$$

for some $q \leqslant 1$ then $T^{q'}_{e+1,u}(0)$ is a string prohibited through $e+1$ for some $q' \leqslant 1$, and so at each stage $2w+p+1 > 2t^*+p+1$ there is a string σ prohibited through $e+1$ such that

$$\sigma \subseteq \alpha^0_{w+1} \qquad \text{or} \qquad \sigma \subseteq \alpha^1_{w+1} \ .$$

By the construction if there is a string σ prohibited through $e+1$ for q at the end of stage $2t^*+q+1$ where

$$\sigma \subseteq \alpha^q_{t^*+1}$$

but

$$\sigma \nsubseteq \alpha^q_{t^*}$$

then this cannot occur through a string $T^q_{e^*,t^*}(\tau)$ being stretched where

$$T^q_{e^*,t^*}(\tau) \subseteq \alpha^q_{t^*}$$

and

$$\sigma \subseteq T^q_{e^*,t^*+1}(\tau) \subseteq \alpha^q_{t^*+1} \ .$$

This is because as in the proof of the above of the above induction we can show that there is an extension of $T^q_{e^*,t^*}(\tau)$ compatible with each tree T^q_{i,t^*+1} with $i < e^*$ but incompatible with each string $T^q_{e+1,u}(0)$ such that

$$T^q_{e+1,u}(0) \nsubseteq T^q_{e^*,t^*}(\tau)$$

and $u < s^*+1$. By the choice of t^* there is no string $T^q_{e+1,u}(0)$ with

$v < s^* + 1$ and

$$T^q_{e+1,v}(0) \subseteq T^q_{e^*,t^*}$$

and so by the definition of the stretching operation

$$\sigma \nsubseteq T^q_{e^*,t^*+1} .$$

This means that we require a string to free σ at stage $2t^*+q+2$. And each string $T^{1-q}_{e+1,u}(0)$ with $u \leqslant s^*$ and

$$T^q_{e+1,u}(0) \subseteq \alpha^q_{t^*+1}$$

becomes prohibited for $1-q$ at stage $2t^*+q+2$.

We construct a function $E(2w+r+1)$ where $r \leqslant 1$ which we take to be undefined for

$$2w + r + 1 \leqslant 2t^* + q + 1,$$

and take as inductive hypothesis:
At stage $2w+r+1 > 2t^* + q + 1$ we define strings $T^r_{e',w+1}(\tau * 0), (\tau * 1)$ through case II of the construction resulting in a requirement for a string to free a string prohibited through $e+1$ at stage $2w+r+2$ where $T^r_{e',w+1}$ has rank $E(2w+r+1)$ and

$$(E(2w + r + 1), r) < (E(2w + r), 1 - r)$$

if $E(2w + r)$ is defined.

We examine stage $2W+R+1$ assuming the result for each stage $2w+r+1$ with

$$2W + R + 1 > 2w + r + 1 > 2t^* + q + 1 .$$

At stage $2W+R+1$ a string of rank k is required to free a string σ prohibited through $e+1$ and all strings $T^R_{e+1,u}(0)$ with $u \leqslant s^*$ and

$$T^{1-R}_{e+1,u}(0) \subseteq \alpha^{1-R}_{W+R}$$

are prohibited for R at stage $2W+R+1$ by virtue of the fact that extensions of some string of rank k' were defined at stage $2W+R$ through case II where

$$(k', 1-R) < (k, R) .$$

We can only fail to free σ if we define strings $T^R_{e',W+1}(\tau * 0), (\tau * 1)$ through case II for some $e' \geqslant 0$ one of which extends a string σ' progibited through $e+1$. But in this case we require a string to free σ' at stage $2W+R+2$, and since such a string cannot have rank greater than k', and by the conditions laid down for case II of the construction we must have that

$$(k', 1-R) > (\text{rank } T^R_{e',W+1}(\tau), R) .$$

If $E(2W+R)$ is defined so that

$$k' = E(2W+R)$$

we obtain the result by defining

$$E(2W+R+1) = \text{rank } T^R_{e',W+1}(\tau) .$$

But from this we see that we have obtained an infinite descending sequence of numbers and so there is no such t^* and the lemma follows.

Lemma 7. *A^0 and A^1 are of minimal degree.*

Proof. We show for each $p \leqslant 1$ and each $e \geqslant 0$ that if $\Phi_e(A^p)$ is total then either $\Phi_e(A^p)$ is recursive or A^p is recursive in $\Phi_e(A)$. It will follow that the degrees of A^0 and A^1 are minimal by lemma 6 and from the fact that $\mathbf{O}'$ is neither recursive nor minimal.

We say that trees *T and T' are mutually compatible* if $T(\emptyset)$ and $T'(\emptyset)$ are compatible and (considering a tree as an array of strings) we have that

$$\{\sigma | \sigma \in T \quad \text{and} \quad \sigma \supseteq T'(\emptyset)\}$$

is compatible with

$$\{\sigma | \sigma \in T' \quad \text{and} \quad \sigma \supseteq T(\emptyset)\}$$

and vice-versa. We write $T \simeq T'$.

We describe a uniformly recursive set of trees

$$\{\Psi^p_{e,s} | e, s \geqslant 0, 1 \geqslant p \geqslant 0\}$$

whose members have the following properties:

(1) $\sigma \in \Psi^p_{e,s} - \Psi^p_{e,s+1} \rightarrow \sigma$ is an end string for $\Psi^p_{e,s}$ and there is a string σ' such that $\sigma \subsetneq \sigma'$ and

$$\sigma' \in \Psi^p_{e,s+1} .$$

(2) $\Psi^p_{e+1,s} \subseteq \Psi^p_{e,s}$

for each e, s, p,

(3) $\Psi^p_{e,s} \sim T^p_{e,s}$

for each e, s, p and no string σ on $\Psi^p_{e,s}$ is a boundary string for a tree $T^p_{i,s}$ with $i \leqslant e$ unless σ is an end string for $\Psi^p_{e,s}$.

(4) either $\Psi^p_{e,s}$ is a splitting tree for e at stages $s \geqslant 0$ or there are only finitely many pairs of strings σ_1, σ_2 such that for some $s \geqslant 0$

$$\sigma_1, \sigma_2 \in \Psi^p_{e,s}$$

and σ_1, σ_2 split for e at stage s.

(5) for each e, p we have that

$$\Psi^p_e = \lim_s \Psi^p_{e,s}$$

exists and contains infinitely many beginnings of A^p.

Assume that $\Psi^p_{e,s}$ has been defined for each $e < e^*+1$ and each $s \geqslant 0$ for some given $p \leqslant 1$ (We take $\Psi^p_{-1,s} = I$ for each $s \geqslant 0$ and each $p \leqslant 1$).

If for every

$$\pi \in \Psi^p_{e^*} \cap \{A^p[n] | n \geqslant 0\}$$

there is a pair

$$T^p_{e^*+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*}$$

which split π for e^*+1 define $s(e^*+1)$ to be the least number for which there is a string

$$T^p_{e^*+1,s(e^*+1)}(\tau) = T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} \cap \Psi^p_{e^*,s(e^*+1)}$$

and take $\pi(e^*+1)$ to be the least such string $T^p_{e^*+1,s(e^*+1)}(\tau)$ which is a beginning of A^p. There must be such a string as long as we can prove (5) for $\Psi^p_{e^*}$ and since by the construction every beginning of A^p is compatible with $T^p_{e^*+1}$ and since by assumption there is a string

$$T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} .$$

Then $\Psi^p_{e^*+1,s}$ is defined to be empty if $s < s(e^*+1)$ and otherwise is the set of strings

$$\{T^p_{e^*+1,s}(\tau) \in \Psi^p_{e^*,s} \mid \text{for each } T^p_{e^*+1,s}(\tau' * q)$$

with $q \leqslant 1$ and $\pi \subset T^p_{e^*+1,s}(\tau' * q) \subseteq T^p_{e^*+1,s}(\tau)$ we have that $T^p_{e^*+1,s}(\tau' * q)$, $(\tau' * 1-q)$ split for $e^*+1\}$ arranged in a tree-like array.

Otherwise choose a

$$\pi \in \Psi^p_{e^*} \cap \{A^p[n] \mid n \geqslant 0\}$$

such that no pair

$$T^p_{e^*+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*}$$

split π for e^*+1.

Define $s(e^*+1)$ to be the least number for which there is a

$$T^p_{e^*+1,s(e^*+1)}(\tau) = T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} \cap \Psi^p_{e^*s(e^*+1)}$$

with

$$T^p_{e^*+1}(\tau) \supset \pi$$

if such a number exists and take $\pi(e^*+1)$ to be the least such string

$$T^p_{e^*+1}(\tau) \subset A^p .$$

And if $s(e^*+1)$ is still not determined take it to be $s(e^*)$ and take $\pi(e^*+1) = \pi$.

In both of the latter cases $\Psi^p_{e^*+1,s}$ is nowhere defined for $s < s(e^*+1)$ and is

$$\{\Psi^p_{e^*,s}(\tau) \supseteq \pi(e^*+1)\}$$

otherwise with the tree ordering induced by $\Psi^p_{e^*,s}$.

We now verify the facts (1)–(5) for

$$\{\Psi^p_{e^*+1,s} \mid s \geqslant 0\}$$

using these facts for each set

$$\{\Psi^p_{e,s} \mid s \geqslant 0\}$$

with $e \leqslant e^*$ and also using any relevant details arising from the inductive definitions.

From the uniform recursiveness of the approximating trees and from (1) it will follow that each Ψ^p_e is 'almost' partial recursive so that by a modified Spector-type argument the lemma will follows from (4) and (5).

We distinguish three cases in the definition of $\{\Psi^p_{e^*+1,s}\}_{s \geqslant 0}$ and treat each in turn.

Case 1. Say

$$\Psi^p_{e^*+1,s}(\tau) = T^p_{e^*+1,s}(\tau') \notin \Psi^p_{e^*+1,s+1}.$$

From the definition of $\Psi^p_{e,s}$ for $e \leqslant e^*+1$ we see that if $T^p_{e,s}(\sigma)$ is a boundary string for $T^p_{e,s}$ and

$$T^p_{e,s}(\sigma) \subseteq \Sigma$$

for some string $\Sigma \in \Psi^p_{e,s}$ then

$$T^p_{e,s}(\sigma) \in \Psi^p_{e,s}$$

or

$$T^p_{e,s}(\sigma) \subset \pi(e).$$

In the former case $T^p_{e,s}(\sigma)$ is an end string for $\Psi^p_{e,s}$ by (3) and so

$$T^p_{e,s}(\sigma) \not\subset \Psi^p_{e^*+1,s}(\tau)$$

and in the latter case $T^p_{e,s}(\sigma)$ is a boundary string for $T^p_{e,s+1}$ by the choice of $\pi(e)$. This means that $T^p_{e^*+1,s+1}(\tau')$ is defined and

$$T^p_{e^*+1,s+1}(\tau') \supset T^p_{e^*+1,s}(\tau')$$

since $T^p_{e^*+1,s}(\tau')$ can only change through being stretched. And since only boundary strings are stretched we have that $T^p_{e^*+1,s}(\tau')$ is a boundary string for some tree $T^p_{e,s}$ with $e \leqslant e^* + 1$ and so by (3) and the definition of $\Psi^p_{e^*+1,s}$ we have that $T^p_{e^*+1,s}(\tau')$ is an end string for $\Psi^p_{e^*+1,s}$. Since $T^p_{e^*+1,s}(\tau')$ is a member of a splitting syzygy for e^*+1 at stage s, $T^p_{e^*+1,s+1}(\tau')$ is a member of syzygy splitting for e^*+1 at stage $s+1$. Finally

$$T^p_{e^*+1,s+1}(\tau') \in \Psi^p_{e^*,s+1}$$

since otherwise let $e < e^*+1$ be the least number for which

$$T^p_{e^*+1,s+1}(\tau') \notin \Psi^p_{e,s+1} \ .$$

Say there is a string Π which is a boundary string for $T^p_{e,s+1}$ where

$$\Pi \subset T^p_{e^*+1,s+1}(\tau') \ .$$

Then by definition of the stretching operation we must have

$$\Pi \subset T^p_{e^*+1,s}(\tau') = \Psi^p_{e^*+1,s}(\tau)$$

which contradicts (3) by definition of $\Psi^p_{e^*+1,s}$. $\Psi^p_{e,s+1}$ will be defined through case (1) since otherwise every end string for $\Psi^p_{e,s+1}$ is an end string for $\Psi^p_{e-1,s+1}$. So $T^p_{e^*+1,s+1}(\tau')$ lies on $T^p_{e,s+1}$ and there is an end string Π for a tree $\Psi^p_{e',s+1}$ with $e' < e$ such that

$$\Pi \subset T^p_{e^*+1,s+1}(\tau')$$

which contradicts the way in which we choose e.

This proves (1) for e^*+1.

We obtain $\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s}$ directly from the construction.

To see that

$$\Psi^p_{e^*+1,s} \simeq T^p_{e^*+1,s}$$

for each s we first note that every string on $\Psi^p_{e^*+1,s}$ also lies on $T^p_{e^*+1,s}$ and so $\Psi^p_{e^*+1,s}$ is compatible with $T^p_{e^*+1,s}$ and $T^p_{e^*+1,s}(\emptyset)$ and $\Psi^p_{e^*+1,s}(\emptyset)$ are compatible by the construction.

Assume that

$$\{\sigma \mid \sigma \in T^p_{e^*+1,s} \quad \text{and} \quad \sigma \supseteq \Psi^p_{e^*+1,s}(\emptyset)\}$$

is not compatible with $\Psi^p_{e^*+1,s}$.
Then for some $T^p_{e^*+1,s}$ with

$$T^p_{e^*+1,s}(\tau) \supset \Psi^p_{e^*+1,s}(\emptyset)$$

we have that $T^p_{e^*+1,s}(\tau)$ neither lies on $\Psi^p_{e^*+1,s}$ nor extends an end string for $\Psi^p_{e^*+1,s}$.
So for some

$$\Psi^p_{e^*+1}(\pi) = T^p_{e^*+1,s}(\tau')$$

we have that

$$\Psi^p_{e^*+1,s}(\emptyset) \subset T^p_{e^*+1,s}(\tau) \subset \Psi^p_{e^*+1,s}(\pi)$$

which by the definition of $\Psi^p_{e^*+1,s}$ implies that

$$T^p_{e^*+1,s}(\tau) \notin \Psi^p_{e^*,s}\ .$$

Let e be the least number such that

$$T^p_{e^*+1,s}(\tau) \notin \Psi^p_{e,s}\ .$$

Since

$$\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e,s}$$

so that

$$T^p_{e^*+1,s}(\tau) \supseteq \Psi^p_{e,s}(\emptyset)$$

we must have $\Psi^p_{e,s}$ defined by means of case 1 and so by the definition of $\Psi^p_{e,s}$ and the fact that

$$T^p_{e^*+1,s}(\tau) \subset \Psi^p_{e^*+1,s}(\pi)$$

we have that $T^p_{e^*+1,s}(\tau)$ lies on $T^p_{e,s}$. Say

$$T^p_{e^*+1,s}(\tau) = T^p_{e,s}(\pi')$$

where

$$\Psi^p_{e,s}(\emptyset) \subset T^p_{e,s}(\pi') \subset \Psi^p_{e,s}(\pi')$$

some π'. Then by the definition of $\Psi^p_{e,s}$

$$T^p_{e,s}(\pi') \in \Psi^p_{e,s}$$

since

$$T^p_{e^*+1,s}(\tau) \in \Psi^p_{e',s}$$

for each $e' < e$, which is a contradiction.

Now let

$$\Psi^p_{e^*+1,s}(\tau) \subset \Psi^p_{e^*+1,s}(\tau') ,$$

some τ, be a boundary string for a tree $T^p_{e,s}$ with $e \leqslant e^*+1$, and choose e to be the least such number. Since

$$\Psi^p_{e',s} \supseteq \Psi^p_{e^*+1,s}$$

for each $e' \leqslant e^*+1$, $\Psi^p_{e^*+1,s}(\tau)$ is an end string for no tree $\Psi^p_{e,s}$ with $e' \leqslant e^*+1$. By the definition of a case 1 construction $\Psi^p_{e,s}$ cannot be defined as a splitting tree for e. But neither of the other cases can hold since $\Psi^p_{e^*+1,s}(\tau)$ being a boundary string for $T^p_{e,s}$ would contradict the choice of $s(e)$ and $\pi(e)$.

By the definition of $\Psi^p_{e^*+1,s}$ we have that $\Psi^p_{e^*+1,s}$ is a splitting tree for e^*+1 at each stage $s \geqslant 0$.

From the proof of (1) we see that if $\Psi^p_{e^*+1,s}(\tau)$ is defined and is not an end string for $\Psi^p_{e^*+1,s}$ then

$$\Psi^p_{e^*+1,s}(\tau) = \Psi^p_{e^*+1,w}(\tau)$$

for each $w \geqslant s$, and if $\Psi^p_{e^*+1,s}(\tau)$ is an end string for $\Psi^p_{e^*+1,w}$ then for some σ we have that for each $w \geqslant s$

$$\Psi^p_{e^*+1,w}(\tau) = T^p_{e^*+1,w}(\sigma)$$

where $T^p_{e^*+1,w}(\sigma)$ is defined and changes only by virtue of being stretched. Since $\lim_s T^p_{e^*+1,s}(\sigma)$ exists so does $\lim_s \Psi^p_{e^*+1,s}(\tau)$.

By definition

$$\pi(e^* + 1) = \Psi^p_{e^*+1}(\emptyset)$$

is a beginning of A^p. Let $\Psi^p_{e^*+1}(\tau)$ be some beginning of A^p where

$$\Psi^p_{e^*+1}(\tau) = T^p_{e^*+1}(\sigma)\,.$$

Since case 1 applies there is a pair

$$T^p_{e^*+1}(\sigma * \rho * 0), (\sigma * \rho * 1) \in \Psi^p_{e^*}$$

which split $T^p_{e^*+1}(\sigma)$ for e^*+1. By the second part of (3) we deduce that $T^p_{e^*+1}(\sigma * 0), (\sigma * 1)$ split $T^p_{e^*+1}(\sigma)$ for e^*+1, and since

$$T^p_{e^*+1}(\sigma) \subset A^p$$

$T^p_{e^*+1}(\sigma * q)$ is a beginning of A^p for some $q \leqslant 1$. So as in the proof of the first part of (3) and by (5) for each tree T^p_e with $e < e^*+1$ we have that

$$T^p_{e^*+1}(\sigma * q) \in \Psi^p_e$$

for each $e < e^*+1$. This means that $T^p_{e^*+1}(\sigma)$ is a boundary string for no tree T^p_e with $e \leqslant e^*+1$.

We show that $T^p_{e^*+1}(\sigma * 1-q)$ lies on each tree Ψ^p_e with $e < e^*+1$.

Assume that e is the least number for which

$$T^p_{e^*+1}(\sigma * 1-q) \notin \Psi^p_e ,$$

so that Ψ^p_e is defined by case 1 and

$$T^p_{e^*+1}(\sigma) = T^p_e(\rho)$$

for some ρ.

Since $T^p_{e^*+1}(\sigma * 0), (\sigma * 1)$ split for e^*+1 and since $T^p_{e^*+1}(\sigma)$ is not a boundary string for T^p_e but $T^p_e(\rho)$ is a member of a pair which splits for e by definition of Ψ^p_e we have that

$$T^p_{e^*+1}(\sigma * 1-q) \in T^p_e .$$

Otherwise we would have that for some string π $T^p_e(\rho * \pi)$ is a boundary string for T^p_e
and

$$T^p_{e^*+1}(\sigma) \subset T^p_e(\rho * \pi) \subset T^p_{e^*+1}(\sigma * 1-q)$$

which would contradict condition (i) of case II of the main construction. From this we get

$$T^p_{e^*+1}(\sigma * 1-q) \in \Psi^p_e ,$$

a contradiction. So the definition of $\Psi^p_{e^*+1}$ implies that

$$T^p_{e^*+}(\sigma * 0), (\sigma * 1) \in \Psi^p_{e^*+1}$$

and so there are beginnings of A^p of arbitrarily long length on $\Psi^p_{e^*+1}$.

Cases 2 and 3.

The only real difference between these cases lies in the definition of $\pi(e^*+1)$, which will appear in the proof of (5).

If

$$\sigma \in \Psi^p_{e^*+1,s} - \Psi^p_{e^*+1,s+1}$$

then by the definition of $\Psi^p_{e^*+1,s+1}$ we have that

$$\sigma \in \Psi^p_{e^*,s} - \Psi^p_{e^*,s+1}$$

and so by the inductive hypothesis σ is an end string for $\Psi^p_{e^*,s}$ and for some ρ we have that

$$\Psi^p_{e^*,s+1}(\rho) \supset \sigma .$$

By the definition of $\Psi^p_{e^*+1,s+1}$

$$\Psi^p_{e^*,s+1}(\rho) \in \Psi^p_{e^*+1,s+1}$$

since

$$\Psi^p_{e^*,s+1}(\rho) \supset \pi(e^* + 1) .$$

By definition we have

$$\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s} .$$

By the choice of $\pi(e^*+1)$ there is no pair

$$T^p_{e^*+1,s+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*,s+1}$$

above $\pi(e^* + 1)$ which is defined through case II. So for each string τ and each number s such that $T^p_{e^*+1,s+1}(\tau * 0)$, $(\tau * 1)$ are defined and compatible with $\pi(e^*+1)$ and are beginnings of strings on $\Psi^p_{e^*,s+1}$ there is a string π and a number $e < e^*+1$ for which $T^p_{e,s+1}(\pi)$, $(\pi * 0)$, $(\pi * 1)$ are defined and equal to $T^p_{e^*+1,s+1}(\tau)$, $(\tau * 0)$, $(\tau * 1)$ respectively. So the tree T consisting of those strings σ such that

$$\sigma \in T^p_{e^*+1,s}$$

and σ is compatible with $\pi(e^* + 1)$ and σ is a beginning of a string on $\Psi^p_{e^*+1,s}$ is mutually compatible with $T^p_{e^*,s}$. Also

$$T^p_{e^*,s} \simeq \Psi^p_{e^*,s}$$

by the inductive hypothesis and

$$\Psi^p_{e^*,s} \simeq \Psi^p_{e^*+1,s}$$

by definition of $\Psi^p_{e^*+1,s}$. Hence

$$\Psi^p_{e^*+1,s} \simeq T$$

which implies that

$$T^p_{e^*+1,s} \simeq \Psi^p_{e^*+1,s}$$

Since

$$T^p_{e^*+1,s} \neq \emptyset$$

implies that

$$T^p_{e^*+1,s}(\emptyset) = \pi(e^*+1) \in \Psi^p_{e^*+1,s}$$

the first part of (3) follows for e^*+1.

Since

$$\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s}$$

and there are no boundary strings for trees $T^p_{e,s}$ with $e \leqslant e^*$ on $\Psi^p_{e^*,s}$ other than end strings, and since there are no boundary strings for $T^p_{e^*+1,s}$ on $\Psi^p_{e^*,s}$ since case 2 or 3 applies, the second part of (3) follows.

We show that the second part of (4) holds for $\Psi^p_{e^*+1}$ and treat cases 2 and 3 separately.

Assume that $\Psi^p_{e^*+1,s}$ is defined through case 2 at each stage $s \geqslant 0$ but that there are infinitely many pairs

$$\sigma_1, \sigma_2 \in \Psi^p_{e^*+1}$$

which split for e^*+1.

We know that $\Psi^p_{e^*+1}(\emptyset)$ is a beginning of A^p and lies on $T^p_{e^*+1}$ and that no string on $\Psi^p_{e^*+1}$ which is not an end string for $\Psi^p_{e^*+1}$ can be a boundary string for a tree T^p_e with $e \leqslant e^*+1$. Also we know that there is no pair

$$T^p_{e^*+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*+1}$$

which split for e^*+1.
So there are infinitely many pairs σ_1, σ_2 such that at some stage $s \geqslant 0$ we have:

(a) $\sigma_1, \sigma_2 \in \Psi^p_{e^*+1,s}$,

(b) σ_1, σ_2 split $\Psi^p_{e^*+1,s}(\emptyset)$ for e^*+1 at stage s where

$$\Psi^p_{e^*+1,s}(\emptyset) \subseteq \alpha^p_s ,$$

(c) $T^p_{e^*+1,s}(\tau)$ is defined and

$$T^p_{e^*+1,s}(\tau) = \Psi^p_{e^*+1,s}(\emptyset) = T^p_{e^*+1}(\tau) ,$$

(d) if $\pi \subseteq \sigma_1$ or σ_2 and π is a boundary string for a tree $T^p_{e,s}$ for some $e \leqslant e^*+1$ then

$$\pi \subseteq T^p_{e^*+1,s} .$$

Since we have (3) for each $e \leqslant e^*+1$, (d) gives that σ_1 and σ_2 are compatible with each tree $T^p_{e,s}$ with $e \leqslant e^*+1$.

Looking at case II of the main construction we see that either:

[1] there are infinitely many beginnings of A^p which are beginnings of string π prohibited at a stage $s \geqslant 0$ where we are unable to free π at stage $s+1$ other than by stretching a string of rank k^* of the $(1-p)^{\text{th}}$ kind where

$$(k^*, 1-p) < (\text{rank } T^p_{e^*+1,s}, p)$$

(since by lemma 6 no beginning of A^p is prohibited at infinitely many stages), or

[2] at stage $2s+p+1$ we have

$$T^p_{e^*+1,s}(\tau) = T^p_{e^*+1}(\tau)$$

and there are strings σ_1 and σ_2 on $\Psi^p_{e^*+1,s}$ which we would define to be

$T^p_{e^*+1,s+1}(\tau * 0)$, $(\tau * 1)$ respectively if it were not for the fact that condition (iii) for case II does not hold for σ_1, σ_2, where we can choose (σ_1, σ_2) and s to be as large as we like.

To see that [1] does not apply we notice that for each x there can only be finitely many prohibited strings $T^p_{x,t}(0)$ and that since

$$T^p_e = \lim_s T^p_{e,s}$$

exists for each e there are only finitely many strings

$$T^{1-p}_{x,t}(0) \subseteq T^{1-p}_{e,s}(0)$$

at some stage $s \geqslant 0$ with

$$(e-1, 1-p) < (\text{rank } T^p_{e^*+1,s}(\tau), p)\,.$$

So eventually we must be able to choose our splitting pair σ_1, σ_2 such that if

$$T^p_{x,t}(0) \subseteq \sigma_1 \qquad \text{or} \qquad T^p_{x,t}(0) \subseteq \sigma_2$$

where $T^p_{x,t}(0)$ is prohibited then $T^p_{x,t}(0)$ can be freed by stretching a string $T^{1-p}_{e',s}(0)$ where

$$(e-1, 1-p) < (e', 1-p)\,.$$

Again the fact that there are only finitely many strings

$$T^{1-p}_{x,t}(0) \subseteq T^{1-p}_{e,s}(0)$$

at some stage $s \geqslant 0$ with

$$(e-1, 1-p) < (\text{rank } T^p_{e^*+1,s}(\tau), p)$$

implies that we can only make strings of rank e with

$$(e, 1-p) < (\text{rank } T^p_{e^*+1,s}(\tau), p)$$

liable to require attention through a finite set of numbers. Let $X-1$ be the largest such number. If we take t^* to be a stage such that

$$T^q_{x,s}(0) = T^q_x(0)$$

for each $q \leqslant 1$ each $s > t^*$ then [2] cannot occur at a stage $2s+p+1 > 2t^*+p+1$ since in this case a string of rank less than X of the p^{th} kind would be required to be strecthed at stage $2s+p+1$.

If the second part of (4) does not hold for $\Psi^p_{e^*+1}$, then $\Psi^p_{e^*+1}$ is not defined through case 3. If there is no string $T^p_{e^*+1}(\tau)$ such that

$$T^p_{e^*+1}(\tau) \in \Psi^p_{e^*}$$

then since A lies on $\Psi^p_{e^*}$ and by the construction either A lies on $T^p_{e^*+1}$ or some beginning of A is an end string for $T^p_{e^*+1}$ we have that for some $t^* > 0$, some τ, each $s > t^*$, $T^p_{e^*+1,s}(\tau)$ is defined and

$$T^p_{e^*+1,s}(\tau) = T^p_{e^*+1,s-1}(\tau)$$

and there is no *syzygy* for $T^p_{e^*+1,s}$ based on $T^p_{e^*+1,s}(\tau)$ which contradicts case III of the construction of $T^p_{e^*+1,s}$

Since (5) holds for $e = e^*$ (5) holds for $e = e^*+1$.

The end of the proof is a straight-forward modification of the arguments of [8].

Assume that Ψ^p_{e+1} is defined through case 2 or case 3. Choose a $\pi \supseteq \Psi^p_{e+1}(\emptyset)$ above which no pair of strings on Ψ^p_{e+1} split for e.
Define

$s(x) = \mu s[\Phi_{e,s}(\sigma, x)$ is defined with $\sigma \in \Psi^p_{e+1,s}$ and $\sigma \supset \pi]$ and $\sigma_x = \mu\sigma[\Phi_{e,s(x)}(\sigma, x)$ is defined with $\sigma \in \Psi^p_{e+1,s(x)}$, $\sigma \supset \pi]$ and

$$f(x) = \Phi_{e,s(x)}(\sigma_x, x)\,.$$

f is partial recursive and since A^p is on Ψ^p_{e+1} if $\Phi_e(A)$ is total then f is recursive. Say $f \neq \Phi_e(A)$. Then for some beginning $A[n]$ of A and some $x \geqslant 0$ we have $A[n] \in \Psi^p_{e+1}$ and $\Phi_e(A[n], x)$ is defined and

$$\Phi_{e,s(x)}(\sigma_x, x) \neq \Phi_e(A[n], x)\,.$$

So by (1) and (5) there is a $\sigma \supseteq \sigma_x$ such that $\sigma \in \Psi_e^p$ and $\sigma, A[n]$ split π for e, a contradiction.

Assume that Ψ_{e+1}^p is defined through case 1. We show how to compute arbitrarily large beginnings of A whenever $\Phi_e(A)$ is total by asking questions uniformly recursive in $\Phi_e(A)$. Assume that $A[n]$ is given where

$$A[n] = \Psi_{e+1,s}^p(\tau)$$

for some $s \geqslant 0$, some τ.
Wait until $\Psi_{e+1,t}^p(\tau * 0), (\tau * 1)$ are defined for some $t \geqslant s$, so that

$$\Psi_{e+1,t}^p(\tau) \supseteq A[n]$$

by (1) and is a beginning of A by (5), which implies that

$$\Psi_{e+1,t}^p(\tau * q) \subset A$$

for some $q \leqslant 1$. By the construction $\Psi_{e+1,s}^p(\tau * 0), (\tau * 1)$ split for e through some $x \geqslant 0$ at stage t and so $\Psi_{e+1,t}^p(\tau * q)$ is a beginning of A where

$$\Psi_{e+1,t}^p(\tau * q) \supset A[n]$$

and

$$\Phi_{e,t}(\Psi_{e+1,t}^p, x) = \Phi_e(A, x)\,.$$

Hence

$$A \leqslant_T \Phi_e(A)\,.$$

Corollary (Shoenfield). *There is a minimal degree below* $\mathbf{O}'$ *incomparable with any given degree strictly between* $\mathbf{O}$ *and* $\mathbf{O}'$.

Another problem concerning joins is that of characterising the joins of degrees of sets satisfying particular separation properties. Also does theorem 2 remain true when we include the degrees of partial functions? Case [1] has shown that the degrees constructed in the proof of theorem 2 will not be minimal partial degrees.

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