Some new bounds on the spectral radius of graphs

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Received 1 October 2001; received in revised form 27 July 2003; accepted 6 August 2003

Abstract

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. This paper presents some upper and lower bounds on the greatest eigenvalue and a lower bound on the smallest eigenvalue.

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Keywords: Graph; Adjacency matrix; Spectral radius

1. Introduction

Let $G = (V, E)$ be a simple graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. For $v_i \in V$, the degree of $v_i$, the set of neighbors of $v_i$ and the average of the degrees of the vertices adjacent to $v_i$ are denoted by $d_i$, $N_G(v_i)$ and $m_i$ respectively. Let $D(G)$ be the diagonal matrix of vertex degrees of a graph $G$. Also let $A(G)$ be the adjacency matrix of $G$ and $A(G) = (a_{ij})$ be defined as the $n \times n$ matrix $(a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E, \\ 0 & \text{otherwise}. \end{cases}$$

It follows immediately that if $G$ is a simple graph, then $A(G)$ is a symmetric $(0, 1)$ matrix where all diagonal elements are zero. We shall denote the characteristic polynomial of $G$ by

$$P(G) = \det(xI - A(G)) = \sum_{i=0}^{n} a_i x^{n-i}.$$
Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as
\[ \lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \cdots \geq \lambda_n(A(G)). \]

Denote $\lambda_i(A(G))$ simply by $\lambda_i(G)$. The sequence of $n$ eigenvalues is called the spectrum of $G$.

2. Upper bound for spectral radius

The largest eigenvalue $\lambda_1(G)$ is often called the spectral radius of $G$. We now give some known upper bounds for the spectral radius $\lambda_1(G)$.

Let $G$ be a simple graph with $n$ vertices and $e$ edges.

(1) (Collatz and Sinogowitz [4]). If $G$ is a connected graph of order $n$, then
\[ \lambda_1(G) \leq \lambda_1(K_n) = n - 1. \]

The upper bound occurs only when $G$ is the complete graph $K_n$.

(2) (Collatz and Sinogowitz [4]). If $G$ is a tree of order $n$, then
\[ \lambda_1(G) \leq \lambda_1(K_{1,n-1}) = \sqrt{n-1}. \]

The upper bound occurs only when $G$ is the star $K_{1,n-1}$.

(3) (Hong [8]). If $G$ is a connected unicyclic graph, then
\[ \lambda_1(G) \leq \lambda_1(S^3_n), \]
where $S^3_n$ denotes the graph obtained by joining any two vertices of degree one of the star $K_{1,n-1}$ by an edge. The upper bound occurs only when $G$ is the graph $S^3_n$.

(4) (Brualdi and Hoffman [3]). If $e = \binom{k}{2}$, then
\[ \lambda_1(G) \leq k - 1, \]
where the equality holds if $G$ is a disjoint union of the complete graph $K_k$ and some isolated vertices.

(5) (Stanley [17]).
\[ \lambda_1(G) \leq (-1 + \sqrt{1 + 8e})/2, \]
where the equality occurs if $e = \binom{k}{2}$ and $G$ is a disjoint union of the complete graph $K_k$ and some isolated vertices.

(6) (Hong [10]). If $G$ is a connected graph, then
\[ \lambda_1(G) \leq \sqrt{2e - n + 1}, \]
where the equality holds if $G$ is one of the following graphs:

(a) the star $K_{1,n-1}$;
(b) the complete graph $K_n$. 
(7) (Hong et al. [12]). Let $G$ be a connected graph with $n$ vertices and $e$ edges. Let $d_n$ be the minimum degree of vertices of $G$. Then
$$\lambda_1(G) \leq \frac{d_n - 1 + \sqrt{(d_n + 1)^2 + 4(2e - d_n)}}{2},$$
where the equality holds iff $G$ is either a regular graph or a bidegreed graph in which each vertex is of degree either $d_n$ or $n - 1$.

(8) (Berman and Zhang [1]). If $G$ is a connected graph, then
$$\lambda_1(G) \leq \max\{\sqrt{d_id_j} : 1 \leq i, j \leq n, \ v_i, v_j \in E\},$$
with equality holds iff $G$ is a regular or bipartite semiregular graph.

(9) (Favaron et al. [6]). (i) For any graph without isolated vertices,
$$\lambda_1(G) \leq \max\{m_i : v_i \in V\}.$$
(ii) For any graph,
$$\lambda_1(G) \leq \max\{d_im_i : v_i \in V\}.$$

The upper bounds from (1)–(4) apply to some particular graphs. Hong [11] has pointed out that the upper bound in (6) is an improvement on the upper bound in (5) while the upper bound (4) is a special case of the upper bound (5). Now we will give some upper bounds for simple connected graphs.

It is a result of Perron–Frobenius in matrix theory (see [7, p. 66]) which states that a non-negative matrix $B$ always has a non-negative eigenvalue $r$ such that the moduli of all the eigenvalues of $B$ do not exceed $r$. To this ‘maximal’ eigenvalue $r$ there corresponds a non-negative eigenvector
$$BY = rY \ (Y \geq 0, \ Y \neq 0).$$

**Lemma 2.1** (Horn and Johnson [13]). Let $M = (m_{ij})$ be an $n \times n$ irreducible non-negative matrix with spectral radius $\lambda_1(M)$, and let $R_i(M)$ be the $i$th row sum of $M$, i.e., $R_i(M) = \sum_{j=1}^{n} m_{ij}$. Then
$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \lambda_1(M) \leq \min\{R_i(M) : 1 \leq i \leq n\}.$$  \text{(11)}
Moreover, if the row sums of $M$ are not all equal, then the both inequalities in (11) are strict.

**Lemma 2.2.** Let $G$ be a bipartite graph with bipartition $V = U \cup W$. Then $\lambda_1(G) = \sqrt{m_1m_2}$ if each vertex $u$ of $U$ has same average degree (average degree of the vertices adjacent to $u$) $m_1$ and each vertex $w$ of $W$ has same average degree (average degree of the vertices adjacent to $w$) $m_2$.

**Proof.** Let $\lambda_1(G)$ be the largest eigenvalue of $A(G)$. Therefore, $\lambda_1(G)$ is also the largest eigenvalue of $M = K^{-1}(D^{-1}A(G)D)K$, where $D$ is the diagonal matrix with the degrees of the vertices as the diagonal entries and $K$ is the diagonal matrix with the square root of average degree of the vertices as the diagonal entries.
Now the \((i,j)\)th element of \(M\) is
\[
\begin{cases}
\sqrt{\frac{m_2}{m_1}} \frac{d_j}{d_i} & \text{if } v_i v_j \in E, \ v_i \in U,
\sqrt{\frac{m_1}{m_2}} \frac{d_j}{d_i} & \text{if } v_i v_j \in E, \ v_i \in W,
0 & \text{otherwise}.
\end{cases}
\]

Using Lemma 2.1 on \(M\), we get \(\lambda_1(G) = \sqrt{m_1 m_2}\).

**Theorem 2.3.** If \(G\) is a simple connected graph and \(\lambda_1(G)\) is the spectral radius, then
\[
\lambda_1(G) \leq \max\{\sqrt{m_i m_j} : 1 \leq i, j \leq n, \ v_i v_j \in E\},
\]
where \(m_i\) is the average degree of the vertices adjacent to \(v_i \in V\). Moreover, the equality in (12) holds if and only if \(G\) is either a graph with all the vertices of equal average degree or a bipartite graph with vertices of same set having equal average degree.

**Proof.** Let \(X = (x_1, x_2, \ldots, x_n)^T\) be an eigenvector of \(D(G)^{-1}A(G)D(G)\) corresponding to an eigenvalue \(\lambda_1(G)\). Also let one eigencomponent (say \(x_i\)) be equal to 1 and the other eigencomponents be less than or equal to 1, that is, \(x_i = 1\) and \(0 < x_k \leq 1, \forall k\).

Let \(x_j = \max \{x_k : v_i v_k \in E\}\).

Now the \((i,j)\)th element of \(D(G)^{-1}A(G)D(G)\) is
\[
\begin{cases}
\frac{d_j}{d_i} & \text{if } v_i v_j \in E,
0 & \text{otherwise}.
\end{cases}
\]

We have
\[
\{D(G)^{-1}A(G)D(G)\}X = \lambda_1(G)X.
\]

From the \(i\)th equation of (13),
\[
\lambda_1(G)x_i = \sum_k \{d_k x_k/d_i : v_i v_k \in E\}, \text{ i.e., } \lambda_1(G) \leq m_i x_i.
\]

From the \(j\)th equation of (13),
\[
\lambda_1(G)x_j = \sum_k \{d_k x_k/d_j : v_j v_k \in E\}, \text{ i.e., } \lambda_1(G)x_j \leq m_j.
\]

From (14) and (15), we get
\[
\{\lambda_1(G)\}^2 \leq m_i \lambda_1(G)x_j \leq m_i m_j.
\]

Therefore, \(\lambda_1(G) \leq \sqrt{m_i m_j}, \ 1 \leq i, j \leq n, \ v_i v_j \in E\).

Hence, \(\lambda_1(G) \leq \max\{\sqrt{m_i m_j} : 1 \leq i, j \leq n, \ v_i v_j \in E\}\).

Now suppose that equality in (12) holds. Then all inequalities in the above argument must be equalities. In particular, we have from (14) that \(x_k = x_j \ \forall k\) such that \(v_i v_k \in E\).

Also from (15) that \(x_k = 1 \ \forall k\) such that \(v_j v_k \in E\).
Case 1: $x_j = 1$. Let $V_1 = \{v_k, x_k = 1\}$. If $V_1 \neq V(G)$, there exist vertices $v_r, v_p \in V_1$, $v_q \notin V_1$ such that $v_r v_p \in E$ and $v_p v_q \in E$ since $G$ is connected. Therefore, from $\lambda_1(G) x_r = 1 \{d_j x_j/d_z: v_r v_j \in E\} \leq m_r$ and $\lambda_1(G) x_p = 1 \{d_j x_j/d_p: v_p v_j \in E\} < m_p$, we have $\lambda_1(G) < \sqrt{m_r m_p}$, which contradicts that the equality holds in (12). Thus $V_1 = V(G)$ and $G$ is a graph with all the vertices of equal average degree.

Case 2: $x_j < 1$. Then $x_k = 1$ for $v_k \in N_G(v_j)$ and $x_k = x_j$ for $v_k \in N_G(v_i)$. Let $U = \{v_k, x_k = 1\}$ and $W = \{v_k, x_k = x_j\}$. So $N_G(v_j) \subseteq U$ and $N_G(v_i) \subseteq W$. Further, for any vertex $v_r \in N_G(N_G(v_j))$ there exists a vertex $v_p \in N_G(v_i)$ such that $v_r v_p \in E, v_r, v_p \in E$. Therefore, $x_p = x_j$ and $\lambda_1(G) x_p = 1 \{d_k x_k/d_p: v_p v_k \in E\} \leq m_p$. Using (14), we get $\lambda^2_l(G) \leq m_i m_p$. We have $\lambda^2_l(G) \geq m_i m_p$, therefore $\lambda^2_l(G) = m_i m_p$, which shows that $x_r = 1$. Hence $N_G(N_G(v_j)) \subseteq U$. By a similar argument, we can show that $N_G(N_G(v_i)) \subseteq W$. Continuing the procedure, it is easy to see, since $G$ is connected, that $V = U \cup W$ and that the subgraphs induced by $U$ and $W$, respectively are empty graphs. Hence $G$ is bipartite. Moreover, the average degree of vertices in $U$ are the same and the average degree of vertices in $W$ are also the same.

Conversely, if $G$ is a graph with all the vertices of equal average degree then the equality is satisfied. Let $G$ be a bipartite graph with bipartition $V = U \cup W$ and any two vertex of same set ($U$ or $W$) have same average degree. Using Lemma 2.2 we can show that equality holds in (12).

Remark 2.4. (i) Eq. (12) is always better than (9). Let $t = \max\{m_i: 1 \leq i \leq n\}$, then we get $m_i \leq t, v_i \in V$. Using this we can easily get

$$\max\{\sqrt{m_i m_j}: 1 \leq i, j \leq n, v_i v_j \in E\} \leq \max\{m_i: 1 \leq i \leq n\}.$$  

(ii) Eq. (10) is always better than (8). For this, let $t = \max\{\sqrt{d_j d_j}: 1 \leq i \leq n, v_i v_j \in E\}$. This implies $d_j \leq t^2/d_i, v_i v_j \in E$. Using this fact we get

$$\max\{\sqrt{d_i d_j}: 1 \leq i \leq n\} \leq t = \max\{\sqrt{d_j d_j}: 1 \leq i \leq n, v_i v_j \in E\}.$$  

(iii) Now we show that Eq. (10) improves Eq. (6). Let (10) gives the maximum at the $i$th vertex we wish to prove that $d_i m_i \leq 2e - n + 1,$

i.e., \[\frac{1}{n} \sum_{j=1}^{n} d_j: v_i v_j \in E\] \[= \frac{1}{n} \sum_{i=1}^{n} d_i - (n - 1),\]

i.e., \[\frac{1}{n} \sum_{j=1}^{n} d_j: v_i v_j \notin E\] \[= (n - 1) \geq 0.\]

For connected graph it is always true. Hence (10) is better than (6).

For the path graphs, the value given by (6) increases with the number of nodes but the value obtained from (10) remains constant.

Example 2.5. Values of $\lambda_1$ and of the various mentioned bounds for the graphs shown in Fig. 1 give (up to three decimal places) the following results:

<table>
<thead>
<tr>
<th></th>
<th>(6)</th>
<th>(7)</th>
<th>(10)</th>
<th>(12)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>2.903</td>
<td>3.742</td>
<td>3.372</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>G2</td>
<td>4.409</td>
<td>5.099</td>
<td>5.099</td>
<td>4.69</td>
<td>4.623</td>
</tr>
</tbody>
</table>
Here we see that (12) is better than (6), (7), and (10) but it is not always true. For instance, for the following graph $G_3$ (see Fig. 2), the use of (12) gives $\lambda_1(G_3) \leq 4.416$, while the use of (6), (7), and (10) gives $\lambda_1(G_3) \leq 4.359$, $\lambda_1(G_3) \leq 4$, and $\lambda_1(G_3) \leq 4.243$, respectively.

Now we give another bound which is always better than Hong (6) formula but not always better than (7), (10), and (12).

Lemma 2.6 (Papendieck and Recht [15]). If $q_1, q_2, \ldots, q_n$ are positive numbers, then

$$\min_i \frac{p_i}{q_i} \leq \frac{p_1 + p_2 + \cdots + p_n}{q_1 + q_2 + \cdots + q_n} \leq \max_i \frac{p_i}{q_i}$$

for any real numbers $p_1, p_2, \ldots, p_n$. Equality holds on either side if and only if all the ratios $p_i/q_i$ are equal.

Theorem 2.7. Let $G$ be a simple connected graph with $n$ vertices and $e$ edges. Also let $d_1$ and $d_n$ be the maximum degree and the minimum degree of vertices of $G$, respectively. If $\lambda_1(G)$ is the spectral radius of $A(G)$, then

$$\lambda_1(G) \leq \sqrt{2e - (n - 1)d_n + (d_n - 1)d_1}.$$  \hspace{1cm} (16)

Moreover, the equality in (16) holds if and only if $G$ is a regular graph or a star graph.

Proof. Let $A_i$ denote the $i$th row of $A(G)$ and $d_i$ the $i$th row sum of $A(G)$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit length eigenvector of $A(G)$ corresponding to the eigenvalue
For $i = 1, 2, \ldots, n$, let $X(i)$ denote the vector obtained from $X$ by replacing those components $x_j$ by 0 such that $a_{ij} = 0$. Since $A(G)X = \lambda_1(G)X$, we have $A_iX(i) = A_iX = \lambda_1(G)x_i$.

By the Cauchy–Schwartz inequality, for $i = 1, 2, \ldots, n$, we have

$$\lambda_1^2(G)x_i^2 = |A_iX(i)|^2 \leq |A_i|^2|X(i)|^2 = d_i \sum_j \{x_j^2: v_i v_j \in E\}.$$ (16)

Summing the above inequalities we obtain

$$\lambda_1^2(G) \leq \sum_{i=1}^n d_i \sum_j \{x_j^2: v_i v_j \in E\}$$

$$= \sum_{i=1}^n x_i^2 \sum_j \{d_j: v_i v_j \in E\}$$

$$= \sum_{i=1}^n d_im_ix_i^2$$

$$\leq \sum_{i=1}^n [2e - d_i - (n - 1 - d_i)d_n]x_i^2$$

$$= 2e - (n - 1)d_n + (d_n - 1) \sum_{i=1}^n d_i x_i^2 \left(\text{using } \sum_{i=1}^n x_i^2 = 1\right)$$

$$\leq 2e - (n - 1)d_n + (d_n - 1)d_1. \quad (17)$$

Now suppose that equality in (16) holds. Then all inequalities in the above argument must be equalities. In particular, from (17) we have that

$$d_im_i = 2e - d_i - (n - 1 - d_i)d_n,$$

for all $v_i \in V$. Therefore, either $d_i = n - 1$ or $d_j = d_n$, for all $v_i \in V$, $v_iv_j \notin E$, which implies that either (a) $G$ is a regular graph or (b) $G$ is a bi-degreed graph in which each vertex is of degree either $d_n$ or $n - 1$.

If $d_n > 1$ then from (18), we get

$$\sum_{i=1}^n d_ix_i^2 = d_1. \quad (19)$$

It follows from (19) and using Lemma 2.6, $d_1 = d_2 = \cdots = d_n$. Therefore, $G$ is a regular graph if $d_n > 1$.

Hence $G$ is a regular graph or a star graph.

Conversely, if $G$ is a regular graph or a star graph the equality is satisfied. \qed

**Corollary 2.8.** Let $G$ be a simple connected graph with $n$ vertices and $e$ edges. Then

$$\lambda_1(G) \leq \sqrt{2e - n + 1}.$$ (18)

Equality holds if and only if $G$ is the star $K_{1, n-1}$ or the complete graph $K_n$. 


Proof. The result follows by \( d_n \geq 1 \) and Theorem 2.7. □

3. Lower bound for spectral radius

We now give some known lower bounds for the spectral radius \( \lambda_1(G) \).

(1) (Collatz and Sinogowitz [4]). If \( G \) is a connected graph of order \( n \), then
\[
\lambda_1(G) \geq \lambda_1(P_n) = 2 \cos(\pi/(n + 1)).
\]
(20)
The lower bound occurs only when \( G \) is the path \( P_n \).

(2) (Hong [8]). If \( G \) is a connected unicyclic graph, then
\[
\lambda_1(G) \geq \lambda_1(C_n) = 2,
\]
(21)
where \( C_n \) denotes the cycle on \( n \) vertices. The lower bound occurs only when \( G \) is the cycle \( C_n \).

(3) (Favaron et al. [6]) For any simple graph,
\[
\lambda_1(G) \geq \sqrt{d_1}.
\]
(22)

Now we find the lower bound of the largest eigenvalue of a graph \( G \), we need the following Theorem 3.1.

Theorem 3.1. Let \( G \equiv (V,E) \) be a graph with vertex subset \( V' = \{v_1,v_2,\ldots,v_k\} \) having same set of neighbors \( \{v_{k+1},v_{k+2},\ldots,v_s\} \), where \( V = \{v_1,\ldots,v_k,\ldots,v_s,\ldots,v_n\} \). Then this graph \( G \) has at least \((k-1)\) equal eigenvalues 0. Also the corresponding \((k-1)\) eigenvectors are
\[
(1,-1,0,\ldots,0)^T, (1,0,-1,0,\ldots,0)^T, \ldots, (1,0,\ldots,-1,0,\ldots,0)^T.
\]

Proof. Let \( X = (x_1,x_2,\ldots,x_n)^T \) be an eigenvector corresponding to an eigenvalue \( \lambda \) of \( A(G) \). Therefore, \( \lambda x_i = \sum_j x_j (v_i,v_j \in E) \), \( i = 1,2,\ldots,n \).

We can easily see that the eigenvalue 0 with corresponding eigenvectors
\[
(1,-1,0,\ldots,0)^T, (1,0,-1,0,\ldots,0)^T, \ldots, (1,0,\ldots,-1,0,\ldots,0)^T
\]
satisfy the above relation. Since these \((k-1)\) eigenvectors are linearly independent, therefore 0 is an eigenvalue of \( A(G) \) of multiplicity at least \((k-1)\) with the above mentioned \((k-1)\) eigenvectors. □

Now we define graphs \( H_1, H_2, H_3 \) and \( H_4 \) in Fig. 3. For each graph, the highest degree vertex \( v_1 \) has degree \( d_1 \) and is connected to a vertex \( v_j \) of degree \( d_j \) which is the maximum degree of the neighboring vertices of \( v_1 \). Let \( c_{1j} \) be the cardinality of common neighbor between \( v_1 \) and \( v_j \). Therefore, the number of pendant vertices that connect to \( v_1 \) and \( v_j \) are \((d_1 - c_{1j} - 1)\) and \((d_j - c_{1j} - 1)\), respectively and the common neighbor vertices of \( v_1 \) and \( v_j \) are all degree 2.
Theorem 3.2. Let $G$ be a simple graph with at least one edge and $d_1$ be the highest degree of $G$. Then

$$\lambda_1(G) \geq \sqrt{\frac{(d_1 + d_j - 1) + \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1) + 4c_{ij}^2 + 8c_{ij}\sqrt{d_1}}}{2}},$$

(23)

where $d_j = \max\{d_k : v_1v_k \in E\}$ and $c_{ij}$ is the cardinality of the common neighbor between $v_1$ and $v_j$.

Proof. It is well known that the index of a graph is monotone, in the sense that if $H$ is a subgraph of $G$, then $\lambda_1(H) \leq \lambda_1(G)$.

Now we are to find out the eigenvalues of $H_1$. Since $(d_1 - c_{ij} - 1)$, $(d_j - c_{ij} - 1)$ and $c_{ij}$ number of vertices of $H_1$ separately have same set of neighbors, therefore using Theorem 3.1, $(d_1 + d_j - c_{ij} - 5)$ number of eigenvalues are 0.

Let $\lambda(H_1)$ be a non-zero eigenvalue of $H_1$. Also let $x_2$ and $x_4$ be the eigencomponents corresponding to vertices $v_1$ and $v_j$ of an eigenvalue $\lambda(H_1)$. Since $\lambda(H_1) \neq 0$, therefore all the eigencomponents corresponding to pendant vertices connected to $v_1$ are equal, say $x_1$. Similarly, all the eigencomponents corresponding to the common neighbor vertices of $v_1$ and $v_j$ are equal, say $x_3$. Also, the eigencomponents corresponding to
pendant vertices connected to \( v_j \) are equal, say \( x_5 \). Therefore, the non-zero eigenvalues of \( H_1 \) satisfies the following system of equations:

\[
\begin{align*}
\hat{\lambda}(H_1)x_1 &= x_2, \\
\hat{\lambda}(H_1)x_2 &= (d_1 - c_{1j} - 1)x_1 + c_{1j}x_3 + x_4, \\
\hat{\lambda}(H_1)x_3 &= x_2 + x_4, \\
\hat{\lambda}(H_1)x_4 &= (d_j - c_{1j} - 1)x_5 + c_{1j}x_3 + x_2, \\
\hat{\lambda}(H_1)x_5 &= x_4.
\end{align*}
\]

Eliminating \( x_1, x_2, x_3, x_4, x_5 \) from above system of equations, the non-zero eigenvalues of \( H_1 \) are obtained from the following equation:

\[
\hat{\lambda}^4(H_1) - (d_1 + d_j - 1)\hat{\lambda}^2(H_1) + (d_1 - 1)(d_j - 1) - c_{1j}^2 - 2c_{1j}\hat{\lambda}(H_1) = 0.
\]

From the well-known property that the sum of the eigenvalues of the adjacency matrix is 0. Since \( \hat{\lambda}_1(H_1) \) is the spectral radius and there is at least one edge in \( G \), therefore

\[
\hat{\lambda}_1(H_1) \geq \sqrt{d_1}.
\]

We can easily show that the spectral radius of \( H_2, H_3 \) or \( H_4 \) are also satisfies (24). Let \( G \) be a graph with highest degree vertex \( v_1 \) of degree \( d_1 \) and connected to a vertex \( v_j \) of degree \( d_j \), which is maximum degree of the neighboring vertices of \( v_1 \). Then \( G \) is a super graph of any one of the graphs \( H_1, H_2, H_3 \) or \( H_4 \) except path graph \( P_2 \) and star graph.

If \( G \) is a path graph \( P_2 \) or a star graph, then equality holds in (23). Therefore,

\[
\hat{\lambda}_1(G) \geq \sqrt{\frac{(d_1 + d_j - 1) + \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1) + 4c_{1j}^2 + 8c_{1j}\sqrt{d_1}}}{2}}.
\]

**Lemma 3.3** (Marcus and Minc [14]). Let \( G \) be a graph and \( v_j \) be any vertex of \( G \). Then

\[
\hat{\lambda}_i(G) \geq \hat{\lambda}_i(G - v_j) \geq \hat{\lambda}_{i+1}(G), \quad i = 1, 2, \ldots, (n - 1).
\]
Theorem 3.4. Let \( T \) be a tree with \( n > 2 \) and \( \lambda_2(T) \) be the second highest eigenvalue of \( T \). Then

\[
\lambda_2(T) \geq \sqrt{\frac{(d_1 + d_j - 1) - \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1)}}{2}},
\]

where \( d_j = \max\{d_k: v_1v_k \in E\} \) and \( v_1 \) is the highest degree vertex of degree \( d_1 \).

**Proof.** Let us consider the tree \( H_3 \) in Fig. 3. Let \( \lambda_2(H_3) \) be the second largest eigenvalue of \( H_3 \). The non-zero eigenvalues of \( H_3 \) can be obtained from the following equation:

\[
\lambda^4(H_3) - (d_1 + d_j - 1)\lambda^2(H_3) + (d_1 - 1)(d_j - 1) = 0.
\]

Therefore,

\[
\lambda_2(H_3) = \sqrt{\frac{(d_1 + d_j - 1) - \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1)}}{2}}.
\]

If tree \( T \) is a star graph with \( n \geq 3 \), then equality holds in (25). Otherwise \( T \) will be a super tree of \( H_3 \). Using Lemma 3.3, we get

\[
\lambda_2(T) \geq \sqrt{\frac{(d_1 + d_j - 1) - \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1)}}{2}}.
\]

Theorem 3.5. Let \( T \) be a tree with \( n > 2 \) and \( \lambda_1(T) \) be the spectral radius of \( T \). Then

\[
\lambda_1(T) \leq \sqrt{n - 1 - \frac{(d_1 + d_j - 1) - \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1)}}{2}},
\]

where \( d_j = \max\{d_k: v_1v_k \in E\} \) and \( v_1 \) is the highest degree vertex of degree \( d_1 \).

**Proof.** The spectrum of \( T \) is symmetric, since trees are bipartite. Hence it follows with an adjacency matrix \( A \) of \( T \) that

\[
2(n - 1) = \text{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2 \geq 2\lambda_1^2(T) + 2\lambda_2^2(T).
\]

Hence

\[
\lambda_1(T) \leq \sqrt{n - 1 - \lambda_2^2(T)} \leq \sqrt{n - 1 - \frac{(d_1 + d_j - 1) - \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1)}}{2}}.
\]

(by Theorem 3.4).

**Remark 3.6.** The upper bounds given by (1), (2), (6), (7) and (16) are same for the case of trees and is equal to \( \sqrt{n - 1} \). But for \( n > 3 \), the upper bound obtained by (27) is strictly less than \( \sqrt{n - 1} \), except for star graphs.
4. Lower bound for smallest eigenvalue

(1) (Brigham and Dutton [2]).

\[ \lambda_n(G) \geq -\sqrt{2e(n-1)/n}. \] (28)

(2) (Constantine [5], Hong [9], Powers [16]). Let \( G \) be a simple graph with \( n \) vertices. Then

\[ \lambda_n(G) \geq -\sqrt{(n/2)((n+1)/2)}, \] (29)

where \([x]\) denotes the largest integer not greater than \( x \). The inequality holds iff \( G \) is the complete bipartite graph \( K_{[n/2],[n+1]/2} \).

Due to Perron–Frobenius it is known that the magnitude of every eigenvalue is less than or equal to the spectral radius of an adjacency matrix \( A(G) \). By this theorem we can obtain a lower bound for the lowest eigenvalue.

Theorem 4.1. Let \( G \) be a simple connected graph with \( n \) vertices and \( e \) edges. Then

\[ \lambda_n(G) \geq -\sqrt{2e - (n-1)d_n + (d_n - 1)d_1}, \] (30)

where \( d_1 \) and \( d_n \) are the highest and the lowest degree of \( G \).

Remark 4.2. (i) Eq. (30) is always better than (28). For this we are to show that

\[ 2e/n \leq (n-1)d_n - (d_n - 1)d_1, \] i.e., \( 2e \leq n(n-1) + n(d_n - 1)(n-1 - d_1) \), which is always true for connected graphs.

(ii) Eq. (30) is always better than (29) for trees. For trees the bound (30) is \(-\sqrt{n-1}\). We are to prove that \( n-1 \leq n/2[\frac{n+1}{2}] \). We can easily show that it is true for all \( n \).

Acknowledgements

Finally, the authors would like to thank the referees for their comments and careful reading of the original paper helping us to make a number of improvements on it.

References