Journal of Mathematical Analysis and Applications **253**, 651–670 (2001)
doi:10.1006/jmaa.2000.7187, available online at http://www.idealibrary.com on IDEAL®

Inequalities for Derivatives of Functions in Harmonic Hardy Spaces

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Submitted by David J. Hallenbeck

Received December 15, 1997

1. INTRODUCTION AND RESULTS

Let $\mathscr{M}(f,r)$ denote the mean value of an integrable function f on the sphere $S(r)$ of radius r centred at the origin 0 of $\mathbb{R}^N$, where $N \geq 2$; that is,

$$\mathscr{M}(f,r) = \int_{S(r)} f\,d\sigma,$$

where σ is $(N-1)$-dimensional surface measure on $S(r)$, normalized so that $\sigma(S(r)) = 1$. Let u be a (real-valued) harmonic function on the unit ball B of $\mathbb{R}^N$. Define

$$\|u\|_p = \lim_{r\to 1-} \left(\mathscr{M}(|u|^p, r)\right)^{1/p},$$

where $1 \leq p < +\infty$. (The (possibly infinite) limit exists, since $\mathscr{M}(|u|^p, \cdot\,)$ is increasing on $(0,1)$.) Similarly, define

$$\|u\|_\infty = \lim_{r\to 1-} \left(\sup_{S(r)} |u|\right).$$

The function u is said to belong to the *harmonic Hardy space* h^p, where $1 \leq p \leq \infty$, if $\|u\|_p < \infty$. (For facts about h^p spaces, we refer to [4, Chap. 6]. Note that in [4] harmonic functions are generally complex-valued.) We also write h^+ for the cone of non-negative harmonic functions on B. Note that $h^1 = h^+ - h^+$; that is $u \in h^1$ if and only if $u = u_1 - u_2$ for some $u_1, u_2 \in h^+$. If $u \in h^+$, then by the mean value property of harmonic functions $\|u\|_1 = u(0)$.

0022-247X/01 $35.00
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A typical point of $\mathbb{R}^N$ is denoted by $x = (x_1, \dots, x_N)$ and we write $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$. The space of all homogeneous harmonic polynomials of degree m, where $m \geq 0$, on $\mathbb{R}^N$ is denoted by $\mathscr{H}_m$. We define

$$d_m = \dim \mathscr{H}_m$$

and note that

$$d_0 = 1, \qquad d_1 = N, \qquad d_m = \binom{N+m-1}{N-1} - \binom{N+m-3}{N-1} \quad (m \geq 2)$$

(see, e.g., [4, p. 82]). Some particular values, with $m \geq 1$, are

$$d_m = 2 \quad (N = 2), \qquad d_m = 2m + 1 \quad (N = 3),$$
$$d_m = (m+1)^2 \quad (N = 4),$$

and in general

$$d_m / m^{N-2} \to 2/(N-2)! \qquad (m \to \infty) \tag{1.1}$$

(see, e.g., [4, p. 94]). We write $\mathbb{N}$ for the set of all positive integers.

The following result is essentially due to Goldstein and Kuran [6, Theorem 2].

THEOREM A. *If $u \in h^1$ and $m \in \mathbb{N}$, then*

$$\left| \frac{\partial^m u}{\partial x_1^m}(0) \right| \leq m!\, d_m \|u\|_1. \tag{1.2}$$

There exist non-zero functions in h^1 for which equality holds in (1.2) *for each $m \in \mathbb{N}$.*

In fact, Theorem A is proved in [6] only for functions in h^+, but the stated result is an easy corollary (see [2, p. 168]). An alternative proof of Theorem A is given in [2, Theorem 2]. The functions u for which equality holds in (1.2) are known explicitly ([2, 6]; see Section 4.3 below). In the case $m = 1$ Theorem A is equivalent to a classical inequality for the norm of the gradient of u; namely, $|\nabla u(0)| \leq N \|u\|_1$ when $u \in h^1$ (since, by a rotation, we can align the x_1-axis with $\nabla u(0)$).

Theorem A has an analogue for functions in h^2. Although it appears not to have been stated explicitly, it is an easy corollary of a result of Brelot and Choquet [5] on harmonic polynomials. We state the result for h^2 here and give a short proof in Section 4.4.

PROPOSITION. *If $u \in h^2$ and $m \in \mathbb{N}$, then*

$$\left| \frac{\partial^m u}{\partial x_1^m}(0) \right| \le m! \sqrt{d_m} \, \|u\|_2. \tag{1.3}$$

Equality holds if and only if u is an element of $\mathscr{H}_m$ that is symmetric about the x_1-axis.

Axially symmetric elements of $\mathscr{H}_m$ are briefly discussed in Section 2.2.

Theorems 1 and 2 below generalize Theorem A in two ways. Theorem 1 gives an inequality corresponding to (1.2) for an arbitrary partial derivative of u evaluated at an arbitrary point of B. Theorem 2 is an extension of Theorem A to functions of class h^p, where $1 \le p \le \infty$. The inequality in Theorem 2 is equivalent to (1.2) and (1.3) when $p = 1, 2$ respectively.

If $\alpha = (\alpha_1, \dots, \alpha_N)$ is an ordered N-tuple of non-negative integers (which we refer to as a *multi-index*), then we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and

$$D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}.$$

THEOREM 1. *If $u \in h^1$, $x \in B$, and $|\alpha| = m \in \mathbb{N}$, then*

$$|D^\alpha u(x)| \le m!(1 - |x|)^{1-m-N} (d_m + c_m |x|) \|u\|_1, \tag{1.4}$$

where $c_m = 0$ if $N = 2$ and

$$c_m = \binom{m + N - 3}{m} \qquad \text{if } N \ge 3.$$

In the case where $x = 0$, inequality (1.4) is equivalent to [2, formula (3.5)] which itself includes (1.2) as a special case. The cases of equality in (1.4) when $x = 0$ are given explicitly in [2, Theorem 3]. Further remarks about the sharpness of (1.4) are given in Section 3.3, where it is shown that the function of $|x|$ on the right-hand side of (1.4) is best possible when $D^\alpha = \partial^m / \partial x_1^m$. Corresponding to (1.4) in the case $m = 0$ there is the classical inequality

$$|u(x)| \le (1 - |x|)^{1-N} (1 + |x|) \|u\|_1,$$

which for functions of class h^+ is Harnack's inequality (see, e.g., [4, p. 47]).

Throughout the sequel, we use the notation

$$\gamma = \frac{1}{2}(N - 3)$$

and use the standard notation $P_m^{(\gamma,\gamma)}$ for Jacobi polynomials, as given in Szegö's book [10]. When $N \geq 3$, the polynomial $P_m^{(\gamma,\gamma)}$ is proportional to the ultraspherical polynomial $P_m^{(\gamma+1/2)}$,

$$P_m^{(\gamma,\gamma)} = \frac{\Gamma(m+\gamma+1)\Gamma(2\gamma+1)}{\Gamma(m+2\gamma+1)\Gamma(\gamma+1)} P_m^{(\gamma+1/2)}$$

[10, formula (4.7.1)]. In particular, corresponding to $N = 3$, we have $P_m^{(0,0)} = P_m^{(1/2)} = P_m$, the Legendre polynomial of degree m.

THEOREM 2. *If $u \in h^p$, where $1 \leq p \leq \infty$, and $m \in \mathbb{N}$, then*

$$\left| \frac{\partial^m u}{\partial x_1^m}(0) \right| \leq m!\, d_m C_{m,N,p} \|u\|_p, \tag{1.5}$$

where $C_{m,N,1} = 1$ and

$$C_{m,N,p} = \binom{m+\gamma}{m}^{-1} \left(\left(\mathrm{B}\left(\gamma+1, \frac{1}{2}\right) \right)^{-1} \times \int_0^\pi \sin^{2\gamma+1}\theta \, |P_m^{(\gamma,\gamma)}(\cos\theta)|^{p/(p-1)} \, d\theta \right)^{(p-1)/p} \tag{1.6}$$

when $1 < p \leq \infty$ (with the convention that $\frac{p}{p-1} = \frac{p-1}{p} = 1$ when $p = \infty$).

In every case equality holds in (1.5) for some non-zero function u in h^p.

Details of the cases of equality are given in Section 4.3. As well as the case $p = 1$ of Theorem 2 (that is, Theorem A) the cases $m = 1$, $p = \infty$ and $m = 1$, $p = 2$ are also known (see [4, pp. 108, 123]). By estimating the constants $C_{m,N,p}$, we will deduce the following result.

COROLLARY 1. *If $u \in h^p$ and $m \in \mathbb{N}$, then*

$$\left| \frac{\partial^m u}{\partial x_1^m}(0) \right| \leq \begin{cases} m!\, d_m^{1/p} \|u\|_p & (1 \leq p < 2) \\ m!\, d_m^{1/2} \|u\|_p & (2 \leq p \leq \infty). \end{cases}$$

For $p = 1, 2$ respectively, Theorem A and the proposition show that the corollary is sharp. We shall also see that in the case where $2 < p \leq \infty$ the exponent $1/2$ cannot be improved. The proof of the corollary is given in Section 7, together with explicit evaluations of some of the constants $C_{m,N,p}$. In particular, we will verify by direct calculation that $C_{m,N,2} = 1/\sqrt{d_m}$ and thus confirm that Theorem 2 includes inequality (1.3) in the proposition.

Another result essentially due to Goldstein and Kuran [6, Theorem 1] may be stated as follows. Here $|\nabla_m \phi|$ denotes the norm of the mth gradient of a C^∞ function ϕ on $\mathbb{R}^N$:

$$|\nabla_m \phi| = \left(m! \sum_{|\alpha|=m} \frac{(D^\alpha \phi)^2}{\alpha!} \right)^{1/2}.$$

THEOREM B. *If $u \in h^+$ and $m \in \mathbb{N}$, then*

$$\{N(N+2)\cdots(N+2m-2)\}^{-1} |\nabla_m u(0)|^2 \le m!\, d_m (u(0))^2. \quad (1.7)$$

There exist non-zero functions u in h^+ for which equality holds in (1.7) *for each $m \in \mathbb{N}$.*

Details of the cases of equality are given in [6]. With $m = 1$, inequality (1.7) is classical: $|\nabla u(0)| \le N u(0)$ when $u \in h^+$.

Our next result extends Theorem B to functions in h^1 and links it with Theorem A. Differentiation in the direction of a unit vector y in $\mathbb{R}^N$ is denoted by $\partial/\partial r_y$. We write S for $S(1)$, the unit sphere in $\mathbb{R}^N$.

THEOREM 3. *If $u \in h^1$ and $m \in \mathbb{N}$, then*

$$\{(N(N+2)\cdots(N+2m-2)\}^{-1} |\nabla_m u(0)|^2 \le \sup_{y \in S} \left| \frac{\partial^m u}{\partial r_y^m}(0) \right| \|u\|_1$$
$$\le m!\, d_m \|u\|_1^2. \quad (1.8)$$

There exist non-zero functions u in h^1 for which equality holds throughout (1.8) *for each $m \in \mathbb{N}$.*

Since harmonic functions are invariant under rotations of the axes, the second inequality in (1.8) follows immediately from inequality (1.2) in Theorem A. The key to the first inequality in (1.8) is a special case of the corollary to the following theorem. Some preliminary remarks are required. Recall that a harmonic function u on B has a unique expansion of the form $\sum_{j=0}^\infty H_j$, where $H_j \in \mathscr{H}_j$; the series converges to u on B and $\sum_{j=0}^\infty |H_j|$ converges locally uniformly on B (see, e.g., [4, p. 84]). We call $\sum_{j=0}^\infty H_j$ the *polynomial expansion* of u. If (j_k) is any strictly increasing (finite or infinite) sequence of non-negative integers, then we call $\sum_k H_{j_k}$ a *part sum* for u. The convergence properties of the polynomial expansion ensure that any part sum is also harmonic on B.

THEOREM 4. *Let u be harmonic on B and let v be a part sum for u. If $r, R \in (0,1)$, then*

$$\mathscr{M}(v^2, \sqrt{rR}) \le \sup_{S(r)} |v| \mathscr{M}(|u|, R). \quad (1.9)$$

Equality holds in (1.9) *if and only if either* (i) $v = 0$ *or* (ii) $v = u(0) \neq 0$ *and* u *is of constant sign* (*in the wide sense*) *on* $S(R)$.

If $u \in h^+$ and $v = u$, then letting $R \to 1-$ in (1.9) we obtain

$$\mathscr{M}\left(u^2, \sqrt{r}\right) \leq \sup_{S(r)} |u|\, u(0),$$

an inequality due to Goldstein and Kuran [6, Theorem 3]. In the general case, we can let $r, R \to 1-$ in (1.9) to obtain the inequality in the following corollary.

COROLLARY 2. *If* u *is harmonic on* B *and* v *is a part sum for* u, *then*

$$\|v\|_2^2 \leq \|v\|_\infty \|u\|_1. \tag{1.10}$$

There exists a positive non-polynomial harmonic function u *on* B *for which* (*finite*) *equality holds in* (1.10) *whenever* v *is a polynomial part sum for* u.

Our proof of Theorem 3 requires only the special case of Corollary 2 in which v consists of a single term H_m from the polynomial expansion of u. Some inequalities related to (1.9) and (1.10) are discussed in Section 5.

2. PREREQUISITES FOR THE PROOFS OF THEOREMS 1 AND 2

2.1. We first recall some facts about representations of functions belonging to h^p spaces as Poisson integrals. The *Poisson kernel* K of B is defined on $B \times S$ by

$$K(x, y) = \left(1 - |x|^2\right)|x - y|^{-N}.$$

Note that $K(\cdot, y) \in h^+$ for each $y \in S$. If μ is a finite signed measure on S, then the *Poisson integral* J_μ is defined by

$$J_\mu(x) = \int_S K(x, y)\, d\mu(y) \qquad (x \in B).$$

In the case where $\mu = f\sigma$ for some integrable function f on S, we write J_f for $J_{f\sigma}$. For each finite signed measure μ on S, we have $J_\mu \in h^1$, and if $f \in L^p(S)$, where $1 \leq p \leq \infty$, then $J_f \in h^p$. Conversely, if $u \in h^1$, then $u = J_\mu$ for some signed measure μ such that $|\mu|(S) = \|u\|_1$, and if $u \in h^p$, where $1 < p \leq \infty$, then $u = J_f$ of some $f \in L^p(S)$ with

$$\|u\|_p = \left(\int_S |f|^p \, d\sigma\right)^{1/p} \quad (1 < p < \infty); \qquad \|u\|_\infty = \sup_S |f| \ (p = \infty). \quad (2.1)$$

See, e.g., [4, Chap. 6].

Since every partial derivative of $K(x, y)$ (with respect to the coordinates of x) is bounded on $E \times S$ for every compact subset E of B, we have

$$D^\alpha J_\mu(x) = \int_S D^\alpha K(x, y) \, d\mu(y) \qquad (x \in B) \qquad (2.2)$$

for every finite signed measure μ on S and every operator D^α.

2.2. Here we list some facts about harmonic polynomials and polynomial expansions of functions harmonic on B. Recall first that the spaces $\mathscr{H}_j$ are mutually orthogonal in the sense that

$$\int_{S(r)} GH \, d\sigma = 0 \qquad \left(G \in \mathscr{H}_j,\, H \in \mathscr{H}_k,\, j \neq k,\, r > 0\right). \qquad (2.3)$$

(see, e.g., [4, p. 75]). If u is harmonic on B with polynomial expansion $\sum_{j=0}^{\infty} H_j$, then it follows from the convergence properties of the polynomial expansion and (2.3) that

$$\mathscr{M}(u^2, r) = \sum_{j=0}^{\infty} \mathscr{M}\left(H_j^2, r\right) \qquad (0 < r < 1). \qquad (2.4)$$

Hence if $u \in h^2$, then for each non-negative integer m

$$\|H_m\|_2 \leq \|u\|_2 \qquad (2.5)$$

with equality if and only if $u \in \mathscr{H}_m$.

We are especially interested in the polynomial expansion of the Poisson kernel $K(\cdot, y)$ and therefore need briefly to discuss axially symmetric homogeneous harmonic polynomials. For each $y \in S$ and each non-negative integer j there exists a unique element $I_{y,j}$ of $\mathscr{H}_j$ such that $I_{y,j}(y) = 1$ and $I_{y,j}$ is axially symmetric with axis Oy (that is, $I_{y,j}(x)$ depends only on $|x|$ and the inner product $x \cdot y = x_1 y_1 + \cdots + x_N y_N$), and we have

$$\|I_{y,j}\|_\infty = 1 = I_{y,j}(y) \qquad (2.6)$$

and

$$\|I_{y,j}\|_2 = 1/\sqrt{d_j} \qquad (2.7)$$

(see [5 or 4, Chap. 5]; the latter uses a different notation and normalization). Note that $I_{y,j}$ is given in terms of a Jacobi polynomial by the equation

$$I_{y,j}(x) = \binom{j+\gamma}{j}^{-1} |x|^j P_j^{(\gamma,\gamma)}\left(\frac{x \cdot y}{|x|}\right) \qquad (x \neq 0,\ y \in S) \tag{2.8}$$

(see, e.g., [3, p. 477]). The polynomial expansion of $K(\cdot, y)$ is

$$K(x,y) = \sum_{j=0}^{\infty} d_j I_{y,j}(x) \qquad (x \in B,\ y \in S). \tag{2.9}$$

See [9, p. 30; 6, Sect. 2].

3. PROOF OF THEOREM 1

3.1. The key to the proof of Theorem 1 is an estimate for $D^\alpha K(x,y)$ ((3.8) below) which ultimately depends upon the following lemma.

LEMMA 1. *Let P be a homogeneous polynomial of degree m on $\mathbb{R}^N$. If $|\alpha| = k \leq m$, then*

$$|D^\alpha P(x)| \leq \frac{m!}{(m-k)!}|x|^{m-k} \sup_S |P| \qquad (x \in \mathbb{R}^N).$$

Lemma 1 follows easily by induction on k from a classical result of Kellogg [7, Theorem IV] which states that if P is a homogeneous polynomial of degree m on $\mathbb{R}^N$ and $|P| \leq 1$ on S, then $|\nabla P| \leq m$ on S.

3.2. Here we prove inequality (1.4) in Theorem 1. Since for each $y \in S$ the polynomial $I_{y,j}$ is homogeneous of degree j and satisfies (2.6) and since $d_j = O(j^{N-2})$ as $j \to \infty$ (see (1.1)), it follows from Lemma 1 that for each multi-index α the series $\sum_{j=0}^\infty d_j |D^\alpha I_{y,j}|$ converges locally uniformly on B. Hence, writing $m = |\alpha|$, we may differentiate term by term in (2.9) and use Lemma 1 to obtain

$$\begin{aligned} |D^\alpha K(x,y)| &= \left| \sum_{j=m}^{\infty} d_j D^\alpha I_{y,j}(x) \right| \qquad (x \in B, y \in S) \\ &\leq \sum_{j=m}^{\infty} d_j \frac{j!}{(j-m)!}|x|^{j-m}. \end{aligned} \tag{3.1}$$

We wish to evaluate the sum in (3.1). Note that

$$\sum_{j=m}^{\infty} d_j \frac{j!}{(j-m)!} t^{j-m} = \frac{d^m}{dt^m} \sum_{j=0}^{\infty} d_j t^j = \frac{d^m}{dt^m}\left((1+t)(1-t)^{1-N}\right) \qquad (|t|<1) \quad (3.2)$$

(see, e.g., [9, p. 30]). We prove by induction that

$$\frac{d^m}{dt^m}\left((1+t)(1-t)^{1-N}\right) = m!(1-t)^{1-m-N}(d_m + c_m t). \quad (3.3)$$

The case $m = 1$ is easy; note that $d_1 = N$ and $c_1 = N - 2$. Supposing that (3.3) is true for some value of m, we find that

$$\frac{d^{m+1}}{dt^{m+1}}\left((1+t)(1-t)^{1-N}\right)$$
$$= m!(1-t)^{-m-N}\left(c_m + (m+N-1)d_m + (m+N-2)c_m t\right).$$

It is easy to check that $(m+N-2)c_m = (m+1)c_{m+1}$, so to complete the induction it remains only to show that

$$c_m + (m+N-1)d_m = (m+1)d_{m+1}. \quad (3.4)$$

With $N = 2$, (3.4) is trivial. For the case where $N \geq 3$ note that

$$d_m = \frac{2m+N-2}{m+N-2}\binom{m+N-2}{m}, \quad (3.5)$$

whence

$$c_m + (m+N-1)d_m = \frac{(2m+N)(m+N-2)!}{m!(N-2)!} = (m+1)d_{m+1}.$$

This completes the proof of (3.3).

From (3.1)–(3.3) it follows that

$$|D^\alpha K(x,y)| \leq m!(1-|x|)^{1-m-N}(d_m + c_m|x|) \qquad (x \in B,\ y \in S). \quad (3.6)$$

If $u \in h^1$, then $u = J_\mu$ for some signed measure μ with $|\mu|(S) = \|u\|_1$. From (2.2) and (3.6) we obtain

$$|D^\alpha u(x)| \le \int_S |D^\alpha K(x,y)|\, d|\mu|(y)$$
$$\le m!(1-|x|)^{1-m-N}(d_m + c_m|x|)\|u\|_1.$$

3.3. We show that equality is possible in (1.4) in the case where $\alpha = (m, 0, \ldots, 0)$ and x is on the x_1-axis. Here and in the sequel we write

$$y_o = (1, 0, \ldots, 0).$$

Let $x = (x_1, 0, \ldots, 0)$, where $0 \le x_1 < 1$. Let $u = K(\cdot, y_o)$. Then $u \in h^+$ and $\|u\|_1 = u(0) = 1$. We have

$$u(x) = (1 + x_1)(1 - x_1)^{1-N},$$

and from (3.3) it follows that

$$\left| \frac{\partial^m u}{\partial x_1^m}(x) \right| = m!(1-|x|)^{1-m-N}(d_m + c_m|x|). \tag{3.7}$$

4. PROOF OF THEOREM 2 AND THE PROPOSITION

4.1. In the proof of Theorem 2 we use the following elementary lemma.

LEMMA 2. *Let P: $\mathbb{R} \to \mathbb{R}$ be a polynomial of degree m such that P is either even or odd. If F is defined by*

$$F(x) = |x|^m P\left(\frac{x \cdot y}{|x|}\right) \qquad (x \in \mathbb{R}^N \setminus \{0\})$$

for some y in $\mathbb{R}^N$, then

$$\frac{\partial^m F}{\partial x_1^m} = m! P(y_1).$$

In proving Lemma 2, we suppose first that P is even and write $P(t) = \sum_{j=0}^{m/2} a_{2j} t^{2j}$. Then

$$F(x) = \sum_{j=0}^{m/2} a_{2j}(x \cdot y)^{2j} |x|^{m-2j}.$$

By Leibniz' theorem,

$$\frac{\partial^m}{\partial x_1^m}\left((x\cdot y)^{2j}|x|^{m-2j}\right) = \sum_{k=0}^{m}\binom{m}{k}\frac{\partial^k}{\partial x_1^k}(x\cdot y)^{2j}\frac{\partial^{m-k}}{\partial x_1^{m-k}}|x|^{m-2j}. \quad (4.1)$$

Since the functions $x \mapsto (x\cdot y)^{2j}$ and $x \mapsto |x|^{m-2j}$ are polynomials of degrees $2j$ and $m-2j$ respectively, all the summands in (4.1) with $k \neq 2j$ vanish. With $k = 2j$ the summand is

$$\binom{m}{2j}(2j)!y_1^{2j}(m-2j)! = m!y_1^{2j}.$$

Hence

$$\frac{\partial^m F}{\partial x_1^m} = \sum_{j=0}^{m/2} a_{2j} m! y_1^{2j} = m!P(y_1).$$

Now suppose that P is odd. Then $P(t) = tQ(t)$ for some even polynomial Q of degree $m-1$, and

$$F(x) = |x|^{m-1}(x\cdot y)Q\left(\frac{x\cdot y}{|x|}\right) = (x\cdot y)G(x),$$

say. By the result of the previous paragraph,

$$\frac{\partial^m F}{\partial x_1^m} = my_1\frac{\partial^{m-1}G}{\partial x_1^{m-1}} = my_1(m-1)!Q(y_1) = m!P(y_1).$$

4.2. Here we prove inequality (1.5) in Theorem 2. With $p = 1$ the result is known [2, Theorem 2]. If $1 < p \le \infty$, then $u = J_f$ for some $f \in L^p(S)$ satisfying (2.1). Since we may pass differential operators under the integral sign as in (2.2), we obtain from (2.9) that

$$\frac{\partial^m u}{\partial x_1^m}(x) = \int_S \frac{\partial^m}{\partial x_1^m}\left(\sum_{j=0}^{\infty} d_j I_{y,j}(x)\right) f(y)\, d\sigma(y). \quad (4.2)$$

As explained in Section 3.2, we may differentiate the series in (4.2) term by term. Thus

$$\frac{\partial^m u}{\partial x_1^m}(0) = d_m\int_S \frac{\partial^m I_{y,m}}{\partial x_1^m} f(y)\, d\sigma(y).$$

In view of (2.8) and the fact that the Jacobi polynomial $P_m^{(\gamma,\gamma)}$ is even or odd according as m is even or odd, it follows from Lemma 2 that

$$\frac{\partial^m I_{y,m}}{\partial x_1^m} = \binom{m+\gamma}{m}^{-1} m! P_m^{(\gamma,\gamma)}(y_1).$$

Hence

$$\left|\frac{\partial^m u}{\partial x_1^m}(0)\right| \le \binom{m+\gamma}{m}^{-1} m! d_m \int_S \left|P_m^{(\gamma,\gamma)}(y_1) f(y)\right| d\sigma(y). \quad (4.3)$$

By Hölder's inequality and (2.1), the integral in (4.3) does not exceed

$$\left(\int_S \left|P_m^{(\gamma,\gamma)}(y_1)\right|^{p/(p-1)} d\sigma(y)\right)^{(p-1)/p} \|u\|_p \quad (4.4)$$

(with the convention stated in Theorem 2 for the case $p = \infty$). The integral in (4.4) equals

$$\int_0^\pi \sin^{2\gamma+1}\theta \left|P_m^{(\gamma,\gamma)}(\cos\theta)\right|^{p/(p-1)} d\theta \Big/ \int_0^\pi \sin^{2\gamma+1}\theta \, d\theta$$

and the integral in the denominator here is $\mathrm{B}(\gamma+1, \frac{1}{2})$. Inequality (1.5) now follows.

4.3. For $p = 1$ the cases of equality in (1.5) are given in [2, Theorem 2], and for the sake of completeness we quote them here. It is convenient to identify $\mathbb{R}^2$ with the complex plane in the usual way. With $N = 2$ and $p = 1$ equality holds in (1.5) if and only if

$$u = \pm \sum_{j=0}^{m-1} \left(\alpha_j K(\cdot, e^{2j\pi i/m}) - \beta_j K(\cdot, e^{(2j+1)\pi i/m})\right),$$

where α_j, β_j are non-negative numbers. With $N \ge 3$ and $p = 1$ equality holds in (1.5) if and only if

$$u = \pm\left(\alpha K(\cdot, y_o) + (-1)^m \beta K(\cdot, -y_o)\right),$$

where α, β are non-negative numbers. (In [2] u was normalized to have $\|u\|_1 = 1$, so the statements there differ slightly from those here.)

Next we consider cases of equality in (1.5) when $1 < p \le \infty$. Fix $m \in \mathbb{N}$, define a function f on S by

$$f(y) = \begin{cases} |P_m^{(\gamma,\gamma)}(y_1)|^{1/(p-1)} \operatorname{sign}\left(P_m^{(\gamma,\gamma)}(y_1)\right) & (1 < p < \infty) \\ \operatorname{sign}\left(P_m^{(\gamma,\gamma)}(y_1)\right) & (p = \infty), \end{cases}$$

and let $u = J_f$. Clearly $u \in h^\infty \subseteq h^p$. Since $f(y)P_m^{(\gamma,\gamma)}(y_1) \ge 0$ for all $y \in S$, equality holds in (4.3) and it follows from the case of equality in

Hölder's inequality that the integral in (4.3) is equal to that in (4.4). Hence equality holds in (1.5). For fixed p with $1 < p < \infty$ and $m \in \mathbb{N}$, the function u for which equality holds in (1.5) is unique up to a multiplicative constant.

4.4. Here we give a short proof of the proposition independently of Theorem 2. Our proof depends on the following lemma.

LEMMA 3. *If $H \in \mathscr{H}_m$, where $m \in \mathbb{N}$, then*

$$|H(y_o)| \leq \sqrt{d_m}\,\|H\|_2$$

with equality if and only if H is proportional to $I_{y_o, m}$.

With $N \geq 3$ (a slightly stronger version of) this result is given by Brelot and Choquet [5, Proposition 4]. To prove Lemma 3 with $N = 2$, write $H(r, \theta) = ar^m \cos(m\theta + \delta)$ and note that

$$|H(1,0)| = |a \cos \delta| \leq |a| = \sqrt{2}\,\|H\|_2$$

with equality if and only if $H(r, \theta) = \pm ar^m \cos(m\theta)$.

Suppose now that $u \in h^2$ and let the polynomial expansion of u be $\sum_{j=0}^{\infty} H_j$. Then using Lemma 3 and (2.5) we find that

$$\left|\frac{\partial^m u}{\partial x_1^m}(0)\right| = \left|\frac{\partial^m H_m}{\partial x_1^m}\right| = m!\,|H_m(y_o)| \leq m!\sqrt{d_m}\,\|H_m\|_2 \leq m!\sqrt{d_m}\,\|u\|_2, \tag{4.5}$$

and there is equality throughout (4.5) if and only if $u = H_m$ and H_m is proportional to $I_{y_o, m}$.

5. PROOF OF THEOREM 4 AND SOME COMMENTS

5.1. Inequality (1.9) is an immediate consequence of the equation

$$\mathscr{M}(v^2, \sqrt{rR}) = \int_S v(rx)u(Rx)\,d\sigma(x). \tag{5.1}$$

Let the polynomial expansion of u be $\sum_{j=0}^{\infty} H_j$ and let the part sum v be $\sum_k H_{j_k}$. The following proof of (5.1) uses the homogeneity of the polynomials H_j, the convergence properties of the polynomial expansion, and the relations (2.4) and (2.3):

$$
\begin{aligned}
\mathscr{M}\left(v^2, \sqrt{rR}\right) &= \sum_k \mathscr{M}\left(H_{j_k}^2, \sqrt{rR}\right) \\
&= \sum_k (rR)^{j_k} \int_S H_{j_k}^2 \, d\sigma \\
&= \sum_k \sum_{j=0}^{\infty} r^{j_k} R^j \int_S H_{j_k} H_j \, d\sigma \\
&= \sum_k \sum_{j=0}^{\infty} \int_S H_{j_k}(rx) H_j(Rx) \, d\sigma(x) \\
&= \int_S \left(\sum_k H_{j_k}(rx) \right) \left(\sum_{j=0}^{\infty} H_j(Rx) \right) d\sigma(x) \\
&= \int_S v(rx) u(Rx) \, d\sigma(x).
\end{aligned}
$$

5.2. By (5.1) equality holds in (1.9) if and only if

$$
v(rx)u(Rx) = \left(\sup_{S(r)} |v| \right) |u(Rx)| \qquad (x \in S).
$$

Hence the conditions stated in Theorem 4 are sufficient for equality in (1.9). Suppose now that equality holds in (1.9). Then either $u = 0$ on a relatively open subset of $S(R)$ and hence, by real-analyticity, everywhere on $S(R)$ or, by continuity, v is constant on $S(r)$. In the former case $u = v = 0$ on B. In the latter case either $v = 0$ or $v = H_o = u(0)$. If $v = u(0) \neq 0$, then $u(0)u \geq 0$ on $S(R)$.

5.3. Inequality (1.10) follows from (1.9). Now let $u = K(\cdot, y_o)$ and let v be a part sum for u. Then, by (2.9), v has the form

$$
\sum_{k \in \Lambda} d_k I_{y_o, k}
$$

for some non-empty set Λ of non-negative integers. By (2.4), (2.6), and (2.7)

$$
\|v\|_2^2 = \sum_{k \in \Lambda} d_k = \|v\|_\infty,
$$

and since $\|u\|_1 = u(0) = 1$, we have equality in (1.10). Obviously $\|v\|_2$ and $\|v\|_\infty$ are finite if and only if v is a polynomial.

5.4. If p, q are numbers such that $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$, then Hölder's inequality applied to (5.1) gives a companion inequality to (1.9):

$$\mathscr{M}\left(v^2, \sqrt{rR}\right) \le \left(\mathscr{M}(|v|^p, r)\right)^{1/p}\left(\mathscr{M}(|u|^q, R)\right)^{1/q}. \tag{5.2}$$

Also by (5.1),

$$\mathscr{M}\left(v^2, \sqrt{rR}\right) \le \mathscr{M}(|v|, r) \sup_{S(R)} |u|. \tag{5.3}$$

If p, q are Hölder conjugates (that is, p, q are as above or $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$), then letting $r, R \to 1-$ in (1.9), (5.2), and (5.3), we obtain

$$\|v\|_2^2 \le \|v\|_p \|u\|_q. \tag{5.4}$$

5.5. The question arises as to whether (5.4) is really more general than a more obvious consequence of Hölder's inequality, viz. $\|v\|_2^2 \le \|v\|_p \|v\|_q$. Thus we ask: is $\|u\|_q < \|v\|_q$ possible when v is a part sum for u?

With $q = 2$ the answer is clearly "no." With $q = \infty$ the answer is "yes." In the case where $q = \infty$ and v consists of a single term H_m from the polynomial expansion of u some quite precise information is given in [1]: $\|H_m\|_\infty \ge cm^{(N-2)/2}\|u\|_\infty$ is possible, where c is positive and depends only on N (and the exponent $(N-2)/2$ cannot be improved). It seems likely that $\|u\|_q < \|v\|_q$ is possible for every q with $1 \le q \le \infty$ and $q \ne 2$, and calculations, which we omit, confirm this when $N = 2$ and $q \in [1, 1.76] \cup [2.5, \infty)$.

6. PROOF OF THEOREM 3

6.1. We need the following lemma.

LEMMA 4. *Let $\sum_{j=0}^{\infty} H_j$ be the polynomial expansion of a harmonic function u on B. Then*

$$|\nabla_m u(0)|^2 = m!N(N+2)\cdots(N+2m-2)\|H_m\|_2^2 \tag{6.1}$$

for each $m \in \mathbb{N}$.

To prove Lemma 4, note first that, since the polynomial expansion can be differentiated term by term,

$$|\nabla_m u(0)| = |\nabla_m H_m|. \tag{6.2}$$

If α is a multi-index, we write $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, and if P is a homogeneous polynomial of degree m on $\mathbb{R}^N$ given by $P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then we denote the operator $\sum_{|\alpha|=m} a_\alpha D^\alpha$ by D_P. We then have

$$|\nabla_m P|^2 = m! \sum_{|\alpha|=m} \frac{(D^\alpha P)^2}{\alpha!} = m! \sum_{|\alpha|=m} a_\alpha^2 \alpha! = m! D_P P. \tag{6.3}$$

If $G, H \in \mathscr{H}_m$, then

$$D_G H = N(N+2) \cdots (N+2m-2) \int_S GH \, d\sigma \tag{6.4}$$

(Kuran [8, Lemma 1]). Taking $P = H_m$ in (6.3) and $G = H = H_m$ in (6.4), we obtain

$$|\nabla_m H_m|^2 = m! N(N+2) \cdots (N+2m-2) \|H_m\|_2^2,$$

which together with (6.2) yields (6.1).

6.2. We can now complete the proof of (1.8) in Theorem 3. It remains to prove only the first inequality in (1.8). Again let the polynomial expansion of u be $\sum_{j=0}^{\infty} H_j$. Taking $v = H_m$ in Corollary 2, we obtain

$$\|H_m\|_2^2 \leq \|H_m\|_\infty \|u\|_1.$$

If $y \in S$, then

$$\frac{\partial^m u}{\partial r_y^m}(0) = \left(\frac{d^m}{dt^m} \sum_{j=0}^{\infty} t^j H_j(y) \right)_{t=0} = m! H_m(y).$$

Hence

$$m! \|H_m\|_2^2 \leq \sup_{y \in S} \left| \frac{\partial^m u}{\partial r_y^m}(0) \right| \|u\|_1. \tag{6.5}$$

The first inequality in (1.8) follows from (6.5) and Lemma 4.

6.3. Let $u = K(\cdot, y_o)$. Then $u \in h^+$. We verify that for this function equality holds throughout (1.8) for each $m \in \mathbb{N}$. From the polynomial expansion (2.9) it follows that

$$\frac{\partial^m u}{\partial r_y^m}(0) = d_m \frac{\partial^m I_{y_o, m}}{\partial r_y^m} = m! d_m I_{y_o, m}(y)$$

for each $y \in S$. Hence by (2.6)

$$\sup_{y \in S} \left| \frac{\partial^m u}{\partial r_y^m}(0) \right| = m! d_m.$$

Also by Lemma 4 and (2.7),

$$\begin{aligned} |\nabla_m u(0)|^2 &= m! N(N+2) \cdots (N+2m-2) d_m^2 \|I_{y_o, m}\|^2 \\ &= m! N(N+2) \cdots (N+2m-2) d_m. \end{aligned}$$

Since $\|u\|_1 = u(0) = 1$, it now follows that equality holds throughout (1.8).

7. VALUES AND ESTIMATES FOR THE CONSTANTS $C_{m,N,p}$

7.1 (the case $N = 2$). When $N = 2$, we have $\gamma = -1/2$ and the Jacobi polynomial $P_m^{(-1/2,-1/2)}$ is proportional to the Chebyshev polynomial of the first kind T_m,

$$P_m^{(-1/2,-1/2)}(\cos\theta) = \binom{m - \frac{1}{2}}{m} T_m(\cos\theta) = \binom{m - \frac{1}{2}}{m} \cos(m\theta)$$

(see [10, Sect. 4.1]). Hence, when $1 < p < \infty$,

$$\begin{aligned} C_{m,2,p} &= \left(\frac{1}{\pi} \int_0^{\pi} |\cos(m\theta)|^{p/(p-1)} \, d\theta \right)^{(p-1)/p} \\ &= \left(\frac{2}{\pi} \int_0^{\pi/2} \cos^{p/(p-1)} \phi \, d\phi \right)^{(p-1)/p} \\ &= \left(\frac{1}{\pi} \mathrm{B}\left(\frac{2p-1}{2p-2}, \frac{1}{2} \right) \right)^{(p-1)/p}. \end{aligned}$$

A similar calculation gives $C_{m,2,\infty} = 2/\pi$.

7.2 (the case $m = 1$). We have $P_1^{(\gamma,\gamma)}(t) = (\gamma+1)t$ (see [10, formula (4.5.1)]). Hence, if $1 < p < \infty$, then

$$\begin{aligned} C_{1,N,p} &= \left(\left(\mathrm{B}\left(\gamma+1, \frac{1}{2} \right) \right)^{-1} \int_0^{\pi} \sin^{2\gamma+1}\theta |\cos\theta|^{p/(p-1)} \, d\theta \right)^{(p-1)/p} \\ &= \left(\mathrm{B}\left(\gamma+1, \frac{2p-1}{2p-2} \right) \Big/ \mathrm{B}\left(\gamma+1, \frac{1}{2} \right) \right)^{(p-1)/p}. \end{aligned}$$

Similarly,

$$C_{1,N,\infty} = \mathrm{B}(\gamma+1,1)/\mathrm{B}\left(\gamma+1,\frac{1}{2}\right) = \frac{1}{\pi}\mathrm{B}\left(\frac{N}{2},\frac{1}{2}\right).$$

7.3 (the case $p = 2$). Here we verify that $C_{m,N,2} = 1/\sqrt{d_m}$ and thus confirm that Theorem 2 implies (1.3) in the proposition. With $p = 2$ the integral in (1.6) is

$$\int_0^{\pi} \sin^{2\gamma+1}\theta\left(P_m^{(\gamma,\gamma)}(\cos\theta)\right)^2 d\theta = \int_{-1}^{1}(1-t^2)^{\gamma}\left(P_m^{(\gamma,\gamma)}(t)\right)^2 dt$$
$$= \frac{2^{2\gamma+1}(\Gamma(m+\gamma+1))^2}{(2m+2\gamma+1)m!\,\Gamma(m+2\gamma+1)} \tag{7.1}$$

(see [10, formula (4.3.3)]). Hence, after some simplification we find that

$$C_{m,N,2} = \left(\frac{m!\,2^{2\gamma+1}\Gamma(\gamma+1)\Gamma\left(\gamma+\frac{3}{2}\right)}{(2m+2\gamma+1)\Gamma(m+2\gamma+1)\sqrt{\pi}}\right)^{1/2}$$
$$= \left(\frac{m!\,\Gamma(2\gamma+2)}{(2m+2\gamma+1)\Gamma(m+2\gamma+1)}\right)^{1/2}$$

by the duplication formula for the Γ-function. Substituting $N-3$ for 2γ gives

$$C_{m,N,2} = \left(\frac{2m+N-2}{m+N-2}\binom{m+N-2}{m}\right)^{-1/2} = 1/\sqrt{d_m},$$

by (3.5).

7.4 *Proof of Corollary* 1. The cases $p = 1, 2$ are covered by Theorem A and the proposition. Suppose now that $1 < p < 2$. Since

$$|P_m^{(\gamma,\gamma)}(t)| \le \binom{m+\gamma}{m} \qquad (-1 \le t \le 1)$$

(see [10, formula (7.32.2)]), we have

$$\int_0^{\pi}\sin^{2\gamma+1}\theta\,|P_m^{(\gamma,\gamma)}(\cos\theta)|^{p/(p-1)}\,d\theta$$
$$\le \binom{m+\gamma}{m}^{(2-p)/(p-1)}\int_0^{\pi}\sin^{2\gamma+1}\theta\left(P_m^{(\gamma,\gamma)}(\cos\theta)\right)^2 d\theta.$$

Using the value of the latter integral given in (7.1), we obtain

$$C_{m,N,p} \le \left(\left(\mathrm{B}\left(\gamma+1, \frac{1}{2}\right) \right)^{-1} \binom{m+\gamma}{m}^{-2} \times \frac{2^{2\gamma+1}(\Gamma(m+\gamma+1))^2}{(2m+2\gamma+1)m!\Gamma(m+2\gamma+1)} \right)^{(p-1)/p}. \quad (7.2)$$

When $N = 2$ (and $\gamma = -\frac{1}{2}$) we readily simplify (7.2) to obtain $C_{m,2,p} \le 2^{-(p-1)/p}$. When $N \ge 3$ we simplify (7.2) by writing the B-function and the binomial coefficient in terms of the Γ-function and then using the duplication formula for the Γ-function: the result is

$$\begin{aligned} C_{m,N,p} &\le \left(\frac{m!\Gamma(2\gamma+2)}{(2m+2\gamma+1)\Gamma(m+2\gamma+1)} \right)^{(p-1)/p} \\ &= \left(\frac{m!(N-2)!}{(2m+N-2)(m+N-3)!} \right)^{(p-1)/p} \\ &= d_m^{-(p-1)/p}, \end{aligned}$$

by (3.5). It now follows, for all N, that $d_m C_{m,N,p} \le d_m^{1/p}$. This completes the proof for $1 < p < 2$.

Since

$$m!d_m C_{m,N,p} = \sup\left\{ \left| \frac{\partial^m u}{\partial x_1^m}(0) \right| : u \in h^p, \|u\|_p = 1 \right\},$$

it is easy to see that $C_{m,N,p}$ is a decreasing function of p when m and N are fixed. Hence the case of the corollary where $2 < p \le \infty$ follows from the case $p = 2$. It remains to justify our claim that the exponent $1/2$ cannot be improved when $2 < p \le \infty$. In view of the monotonicity of $C_{m,N,p}$, it is enough to work with $p = \infty$. Thus we wish to show that $C_{m,N,\infty} > c d_m^{-1/2}$, where c is a positive constant depending only on N. When $N = 2$ we have $d_m = 2$ and $C_{m,2,\infty} = 2/\pi$ for all $m \in \mathbb{N}$. Now suppose that $N \ge 3$ (so $\gamma \ge 0$). Writing c for a positive constant depending only on N but possibly varying from line to line and using Stirling's formula to estimate $\binom{m+\gamma}{m}$, we have

$$C_{m,N,\infty} > cm^{-\gamma}\int_0^{\pi}\sin^{2\gamma+1}\theta\,|P_m^{(\gamma,\gamma)}(\cos\theta)|\,d\theta$$

$$= cm^{-\gamma}\int_0^1(1-t^2)^{\gamma}|P_m^{(\gamma,\gamma)}(t)|\,dt$$

$$> cm^{-\gamma}\int_0^1(1-t)^{\gamma}|P_m^{(\gamma,\gamma)}(t)|\,dt$$

$$> cm^{-\gamma-1/2}$$

(see [10; formula (7.34.1)]). Since $\gamma+\frac{1}{2}=(N-2)/2$, it now follows from (1.1) that $C_{m,N,\infty} > cd_m^{-1/2}$, as required.

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