The Index of the Complex Eigenvalues of a Parity Progressive Population Operator

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In this paper we discuss the index of the complex eigenvalue of a parity progressive population operator. Under certain conditions, we first prove that all the complex eigenvalues of this operator, except at most finitely many ones, are of index 1, and then, as an application of this result, we obtain the asymptotic expansion of

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1. INTRODUCTION AND MAIN RESULT

Population evolution processes can be described by either age structure models or parity structure models. With these models, one can quantitatively study population evolution processes by virtue of mathematical methods and get more profound results. For age-structured population systems, useful results have been obtained by many investigators (see [9–14, 16]).

The idea of parity progressive population was first put forward by Henry [5] in 1953. Afterwards, scholars used the concept for population analysis many times [2–4]. During the initial stage, papers on parity research focused on the analysis of the relationship between parity patterns and population parameters, but were not based on a strict theory. Recently, population modelling based on parity structure has developed rapidly. Yu and Zhu [17] and other scholars have developed a parity progressive population model dependent on birth interval and time. Yu et al. [18] set up a dynamic population model of parity progression with age structure. By applying the spectral theory of linear operators and the $C_0$-semigroup theory of functional analysis, the parity progressive population operator [17] (or age-specific parity progressive population operator [18]) and the stability of population systems were studied quite deeply. It was proved in [17] that the spectral set of the parity progressive population operator consists of infinitely many isolated eigenvalues with finite algebraic multiplicity, involving a unique real eigenvalue with both geometric and algebraic multiplicity 1. However, in order to get the asymptotic expansion of the solution of the corresponding population equation and obtain more profound stability results for the population system, one should investigate the index of the complex eigenvalues of the corresponding population operator. As far as we know, the results in this respect are scarce and our present paper is just an effort to solve this problem. Under certain conditions, we prove that all the complex eigenvalues of the parity progressive population operator, except at most finitely many, are of index 1. As an application of this result, we obtain the asymptotic expansion of the solution of the corresponding parity progressive population equation. These are the new results in parity population dynamics.
The parity progressive dynamic equation of a population can be written as [17]

\[
\frac{\partial x_n(t, s)}{\partial t} + \frac{\partial x_n(t, s)}{\partial s} = -\lambda_n(s)x_n(t, s), \quad n = 0, 1, 2, \ldots, N,
\]

\[
x_n(t, 0) = \int_0^M \lambda_{n-1}(s)x_{n-1}(t, s) \, ds, \quad n = 1, 2, \ldots, N,
\]

\[
x_0(t, 0) = \sum_{n=1}^N k_{n-1}x_n(t, 0),
\]

\[
x_n(0, s) = x_n^{(0)}(s), \quad n = 0, 1, 2, \ldots, N,
\]

where \( t \) represents time, \( s \) is interval, \( \lambda_n(s) \) \((n \geq 1)\) is the ratio of the number of the women who give their \((n + 1)\)th birth to interval \( s \) after they have given their \( n \)th birth to the number of the women who give their \( n \)th birth, \( x_n(t, s) \) \((n \geq 1)\) is the number of the women who give their \( n \)th birth at time \( t \) and interval \( s \), \( x_0(t, 0) \) is the number of the female babies who are given birth to by all the women, \( x_0(t, s) \) is the number of the women who give birth to a female baby at time \( t \) and interval \( s \), and \( \lambda_0(s) \) is the ratio of the number of the female babies who have grown up and given birth to a female baby at interval \( s \) to the number of all the female babies. \( M \) indicates the length of the birth interval of women, and \( k_{n-1} \) indicates the ratio of female babies to babies of the \( n \)th delivery. \( x_n(0, s) = x_n^{(0)}(s) \) \((n = 0, 1, \ldots, N)\) is the initial state.

We say some words about the model (1.1). In a sense, the parity progressive population model (1.1) is suitable only for human demography. The methods to establish this model do not fit animal systems, because their birthing intervals are not deliberately delayed through conscious decisions of reproductively active individuals.

For simplicity, let

\[
x(t, s) = \begin{pmatrix} x_0(t, s) \\ x_1(t, s) \\ \vdots \\ x_N(t, s) \end{pmatrix}, \quad D(s) = \begin{pmatrix} \lambda_0(s) & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N(s) \end{pmatrix}.
\]

System (1.1) can be written as

\[
\frac{\partial x(t, s)}{\partial t} + \frac{\partial x(t, s)}{\partial s} = -D(s)x(t, s), \quad x(t, 0) = \int_0^M B(s)x(t, s) \, ds,
\]

\[
x(0, s) = x^{(0)}(s),
\]

(1.2)
where \( x^{(0)}(s) = (x_0^{(0)}(s), x_1^{(0)}(s), \ldots, x_N^{(0)}(s))^T \) and

\[
B(s) = \begin{pmatrix}
    k_0\lambda_0(s) & k_1\lambda_1(s) & \cdots & k_{N-1}\lambda_{N-1}(s) & 0 \\
    \lambda_0(s) & 0 & \cdots & 0 & 0 \\
    0 & \lambda_1(s) & \cdots & 0 & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & \lambda_{N-1}(s) & 0
\end{pmatrix}.
\]

Considering the practical meaning of \( \lambda_n(s) \) and \( k_n \), we may assume that \( 0 < \lambda_n(s) \leq 1, \ 0 < k_n < 1, \ n = 0, 1, \ldots, N, \) and \( \lambda_n(s) \) are measurable functions.

Now, we define the inner-product and norm in space \( L^2[0, M] \) as

\[
(x(r), y(r)) = \int_0^M x(r)y(r) \, dr, \quad \forall x(r), y(r) \in L^2[0, M],
\]

\[
\|x(r)\|_{L^2[0, M]} = \left( \int_0^M |x(r)|^2 \, dr \right)^{\frac{1}{2}}.
\]

and we choose the following product space \( H \) as the state space of system (1.1):

\[
H = L^2[0, M] \times L^2[0, M] \times \cdots \times L^2[0, M], \quad N + 1 \text{ in total.}
\]

In space \( H \), the inner-product and norm are as follows:

\[
(x(r), y(r))_0 = \sum_{k=0}^N (x_k(r), y_k(r))
\]

\[
= \sum_{k=0}^N \int_0^M x_k(r)y_k(r) \, dr, \quad \forall x(r), y(r) \in H,
\]

\[
\|x(r)\|_0 = (x(r), y(r))_0^{\frac{1}{2}}.
\]

It is obvious that \( H \) is a Hilbert space. We define a matrix operator \( A \) with domain \( D(A) \) as

\[
Ax(s) = -\frac{dx(s)}{ds} - D(s)x(s),
\]

\[
D(A) = \left\{ x(s) \in H \left| -\frac{dx(s)}{ds} - D(s)x(s) \in H \right. \right\}
\]

and \( x(0) = \int_0^M B(s)x(s) \, ds \).

and we call it a parity progressive population operator.
Then we can rewrite system (1.2) as an abstract evolution equation in Hilbert space $H$:

\[
\frac{dx(t)}{dt} = Ax(t),
\]

\[
x(0) = x^{(0)}(s).
\]

(1.3)

Let $\sigma_p(A)$, $\rho(A)$, and $R(\lambda, A)$ denote the point spectral set of $A$, the resolvent set of $A$, and the resolvent of $A$, respectively.

For any eigenvalue $\lambda$ of $A$, let $N^k(A, \lambda) = \{f \in D(A^k) \mid (\lambda I - A)^k f = 0\}$. It is clear that

$$N^1(A, \lambda) \subseteq N^2(A, \lambda) \subseteq N^3(A, \lambda) \subseteq \cdots,$$

for $\lambda \in \sigma_p(A)$.

We call the smallest integer $k$ such that $N^k(A, \lambda) = N^{k+1}(A, \lambda)$ the index of $\lambda$.

It is easy to see that if both the geometric multiplicity of $\lambda$ and the index of $\lambda$ are 1, then the algebraic multiplicity of $\lambda$ is 1 also.

(H) Let $[0, M] = \sum_{i=1}^{P-1} I_i$, where $I_i = [r_i, r_{i+1})$ ($1 \leq i < P - 1$), $I_{P-1} = [r_{P-1}, r_P]$, $r_1 = 0$, $r_P = M$. Assume that $\lambda_n(s) = \mu_{ni}$ for $s \in I_i$, $\mu_{ni} \in (0, 1]$, $i = 1, 2, \ldots, P - 1$, are positive constants.

It is easy to see that, for a nation or a district, the above assumption is of practical significance. In fact, the dynamics of parity progressive population is based on the idea that births are a series of interrelated progressive events; i.e., a woman’s own birth, the birth of her first child, second child, and so forth, are a series of such events. In order to measure the birth rate, we must study how an event progresses to the next. Among all the women, how many of them would give birth to a first baby and how many to a second one? These results can be demonstrated by percentage. For instance, 85% of the women who give birth to a first baby give birth to a second denoted by $\lambda_1 = 85\%$, 45% of the women who give birth to a second baby give birth to a third denoted by $\lambda_2 = 45\%$, and so on. If the interval between two births of a woman is considered, then the meaning of $\lambda_i(s)$ can be understood similarly. In view of these facts, it is not difficult to comprehend the rationality of Assumption (H).

The main result of this paper is the following:

**Theorem 1.1.** Assume that (H) is satisfied. Then all the complex eigenvalues of $A$, except at most finitely many, are of index 1.

The message about the geometric multiplicity, algebraic multiplicity, and, index of the eigenvalues of the parity progressive population operator is closely related to the quantitative analysis of the population behavior, especially the asymptotic expansion of the solution of the corresponding population equation. In Section 3, we shall obtain the result in this respect according to Theorem 1.1.
2. THE PROOF OF THE MAIN RESULT

In order to prove Theorem 1.1 four lemmas are in order.

**Lemma 2.1.** (1) The spectral set of $A$ consists of countably infinite isolated eigenvalues with finite algebraic multiplicity. $\lambda \in \sigma_p(A)$ if and only if $\lambda$ is the zero-point of the function

$$F(\lambda) = 1 - \sum_{n=0}^{N-1} k_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda),$$

where $\beta_k(\lambda) = \int_0^M \lambda_k(s) e^{-\lambda s} \lambda_k(\rho) \, ds$, $k = 0, 1, \ldots, N$. For $\lambda \in \sigma_p(A)$, the geometric multiplicity of $\lambda$ equals 1 and the corresponding eigenfunction is $x(s) = (x_0(s), x_1(s), \ldots, x_N(s))^T$, where

$$x_0(s) = \exp \left[-\lambda s - \int_0^s \lambda_0(\rho) \, d\rho\right],$$

$$x_n(s) = \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \times \exp \left[-\lambda s - \int_0^s \lambda_n(\rho) \, d\rho\right], \quad n = 1, 2, \ldots, N.$$  

(2) $A$ has a unique real eigenvalue $\lambda_0$ with algebraic multiplicity 1; the real parts of other eigenvalues of $A$ are less than $\lambda_0$.

(3) There exist at most finitely many eigenvalues of $A$ in any strip region $\{\lambda \in \mathbb{C} \mid -\infty < a_1 \leq \Re \lambda \leq a_2 < \infty\}$ of the complex plane.

**Proof.** (1) and (2) have been obtained in [17]. In the following we prove (3) by reduction to absurdity.

If the assertion is not true, then there must be infinitely many eigenvalues of $A$, $\{\lambda_1 \mid \lambda_n = \alpha_n + i\tau_n\}_{n=1}^{\infty}$, $\alpha_1 \leq \alpha_n \leq \alpha_2$. Then a subsequence $\{\lambda_l \mid \lambda_l = \alpha_l + i\tau_l\}_{l=1}^{\infty}$ can be selected such that $\alpha_l \to \alpha_0$ and $|\tau_l| \to \infty$ as $l \to \infty$ and $F(\lambda_l) = 1$; i.e.,

$$\sum_{n=0}^{N-1} k_n \beta_0(\lambda_l) \beta_1(\lambda_l) \cdots \beta_n(\lambda_l) = 1.$$

Since

$$1 = \sum_{n=0}^{N-1} k_n \beta_0(\lambda_l) \beta_1(\lambda_l) \cdots \beta_n(\lambda_l)$$

$$= \sum_{n=0}^{N-1} k_n \int_0^M \cdots \int_0^M \lambda_0(s_0) \cdots \lambda_n(s_n)$$

$$\times \exp \left[-\alpha_l \sum_{k=0}^n s_k - \sum_{k=0}^n \int_0^{s_k} \lambda_k(\rho) \, d\rho\right] \exp \left[-\tau_l \sum_{k=0}^n s_k\right] ds_0 \cdots ds_n,$$
therefore
\[ 1 = \sum_{n=0}^{N-1} k_n \int_0^M \cdots \int_0^M \lambda_0(s_0) \cdots \lambda_n(s_n) \]
\[ \times \exp \left[ -\alpha_l \sum_{k=0}^n s_k - \sum_{k=0}^n \int_0^{s_k} \lambda_k(\rho) d\rho \right] \cos \left( \tau_l \sum_{k=0}^n s_k \right) ds_0 \cdots ds_n \]
\[ \leq \sum_{n=0}^{N-1} k_n \int_0^M \cdots \int_0^M \lambda_0(s_0) \cdots \lambda_n(s_n) \exp \left[ -\sum_{k=0}^n \int_0^{s_k} \lambda_k(\rho) d\rho \right] \]
\[ \times \exp \left[ -\alpha_l \sum_{k=0}^n s_k - \sum_{k=0}^n \int_0^{s_k} \lambda_k(\rho) d\rho \right] \cos \left( \tau_l \sum_{k=0}^n s_k \right) ds_0 \cdots ds_n \]
\[ + \sum_{n=0}^{N-1} k_n \int_0^M \cdots \int_0^M \lambda_0(s_0) \cdots \lambda_n(s_n) \]
\[ \times \exp \left[ -\alpha_l \sum_{k=0}^n s_k - \sum_{k=0}^n \int_0^{s_k} \lambda_k(\rho) d\rho \right] \cos \left( \tau_l |\sum_{k=0}^n s_k| \right) ds_0 \cdots ds_n. \quad (2.2) \]

Since \( \alpha_l \to \alpha_0 \) as \( l \to \infty \), the first term on the right hand side of (2.2) converges to zero, and since \( |\tau_l| \to \infty \) as \( l \to \infty \), the second term also converges to zero, which is impossible. Hence (3) is proved.

**Lemma 2.2** [15]. Let \( H \) be a Hilbert space and let \( T \) be a closed linear operator on \( H \). Suppose that \( \lambda \) is an eigenvalue of \( T \) and that, for any \( f \in N^1(T, \lambda) \), there is a \( g \in N^1(T^*, \overline{\lambda}) \), \( g \neq 0 \), such that \( (f, g) \neq 0 \). Then the index of \( \lambda \) is 1.

**Lemma 2.3.** (1) The adjoint operator \( A^* \) of \( A \) has the form
\[ A^* y(s) = \frac{dy(s)}{ds} - D(s)y(s) + D(s)B, \]
where $B$ is given by

$$B = (k_0 y_0(0) + y_1(0), k_1 y_0(0) + y_2(0), \ldots, k_{N-1} y_0(0) + y_N(0), 0)^T,$$

$$D(A^*) = \{ y(s) \in H \mid A^* y(s) \in H, \ y(M) = 0 \}. $$

(2) $\lambda \in \sigma_p(A^*)$ if and only if $F(\lambda) = 0$.

(3) If $\lambda \in \sigma_p(A^*)$ then the eigenfunction corresponding to $\lambda$ is $y(s) = (y_0(s), y_1(s), \ldots, y_N(s))^T$, where

$$y_n(s) = \left[ c_n - \int_0^s \left( k_n \lambda_n(\tau) + c_{n+1} \lambda_n(\tau) \right) \exp \left( -\lambda_1 - \int_0^\tau \lambda_n(p) \, dp \right) \, d\tau \right] \cdot \exp \left( \lambda_1 + \int_0^s \lambda_n(\tau) \, d\tau \right), \quad n = 0, 1, 2, \ldots, N - 1,$$

$$y_N(s) = c_N \exp \left( \lambda_1 + \int_0^s \lambda_n(\tau) \, d\tau \right).$$

Here,

$$c_i = \sum_{n=i}^{N-1} k_n \beta_i(\lambda) \beta_{i+1}(\lambda) \cdots \beta_n(\lambda), \quad i = 1, 2, \ldots, N - 1,$$

$$c_0 = 1, \quad c_N = 0.$$

**Proof.** (1) For any $x(s) \in D(A)$, $y(s) \in D(A^*)$, since

$$(Ax, y)_0 = \sum_{n=0}^{N} \int_0^M \left( - \frac{dx_n(s)}{ds} - \lambda_n(s) x_n(s) \right) \overline{y_n(s)} \, ds$$

$$= - \sum_{n=0}^{N} \int_0^M y_n(s) \overline{dx_n(s)} \, ds - \sum_{n=0}^{N} \int_0^M \lambda_n(s) x_n(s) \overline{y_n(s)} \, ds$$

$$= - \sum_{n=0}^{N} \left| y_n(s) \overline{x_n(s)} \right|_0^M + \sum_{n=0}^{N} \int_0^M x_n(s) \frac{dy_n(s)}{ds} \, ds$$

$$- \sum_{n=0}^{N} (x_n(s), \lambda_n(s) y_n(s))$$

$$= x_0(0) \overline{y_0(0)} + \sum_{n=1}^{N} x_n(0) \overline{y_n(0)}$$

$$+ \sum_{n=0}^{N} \left( x_n(s), \frac{dy_n(s)}{ds} - \lambda_n(s) y_n(s) \right)$$

(2.3)
and
\[
x_0(0) = \sum_{n=1}^{N} k_{n-1} \int_0^M \lambda_{n-1}(s) x_{n-1}(s) \, ds = \sum_{n=0}^{N-1} (x_n(s), k_n \lambda_n(s)), \tag{2.4}
\]
\[
x_n(0) = \int_0^M \lambda_{n-1}(s) x_{n-1}(s) \, ds = (x_{n-1}(s), \lambda_{n-1}(s)), \quad n = 1, 2, \ldots, N, \tag{2.5}
\]
substituting (2.4) and (2.5) into (2.3) obtains
\[
(Ax, y)_0 = \sum_{n=0}^{N-1} (x_n(s), k_n \lambda_n(s)y_0(0)) + \sum_{n=0}^{N-1} (x_n(s), \lambda_n(s)y_{n+1}(0))
\]
\[
+ \sum_{n=0}^{N-1} \left( x_n(s), \frac{dy_n(s)}{ds} - \lambda_n(s)y_n(s) \right)
\]
\[
= \sum_{n=0}^{N-1} \left( x_n(s), \frac{dy_n(s)}{ds} - \lambda_n(s)y_n(s) + k_n \lambda_n(s)y_0(0) \right.
\]
\[
+ \lambda_n(s)y_{n+1}(0) + \left( x_N(s), \frac{dy_N(s)}{ds} - \lambda_N(s)y_N(s) \right)
\]
\[
= \left( x(s), \frac{dy(s)}{ds} - D(s)y(s) + D(s)B \right)_0 = (x, A^*y)_0.
\]

(2) can be verified directly.

(3) We consider the equation \((\lambda I - A^*)y(s) = 0\); i.e.,
\[
\frac{dy(s)}{ds} - (\lambda + D(s))y(s) + D(s)B = 0, \quad y(M) = 0.
\]

This is equivalent to
\[
\frac{dy_0(s)}{ds} = (\lambda + \lambda_0(s))y_0(s) - (k_0 y_0(0) + y_1(0)) \lambda_0(s)
\]
\[
\frac{dy_1(s)}{ds} = (\lambda + \lambda_1(s))y_1(s) - (k_1 y_0(0) + y_2(0)) \lambda_1(s)
\]
\[
\ldots \ldots \ldots
\]
\[
\frac{dy_{N-1}(s)}{ds} = (\lambda + \lambda_{N-1}(s))y_{N-1}(s) - (k_{N-1} y_0(0) + y_N(0)) \lambda_{N-1}(s)
\]
\[
\frac{dy_N(s)}{ds} = (\lambda + \lambda_N(s))y_N(s).
\]
By solving Eqs. (2.7) we have
\[
y_n(s) = \left[ c_n - \int_0^s \left( k_n \lambda_n(\tau) y_n(0) + \lambda_n(\tau) y_{n+1}(0) \right) e^{-\lambda_n \int_0^\tau \lambda_n(\rho) d\rho} d\tau \right] e^{\lambda_n \int_0^s \lambda_n(\rho) d\rho}, \quad n = 0, 1, 2, \ldots, N - 1, \tag{2.8}
\]
and
\[
y_N(s) = c_N e^{\lambda_N \int_0^s \lambda_N(\rho) d\rho}. \tag{2.9}
\]

From \( y(M) = 0 \) and \( c_n = y_n(0) \) it follows that
\[
c_n = \int_0^M \left( k_n \lambda_n(\tau) c_0 + \lambda_n(\tau) c_{n+1} - \int_0^\tau \lambda_n(\rho) d\rho \right) d\tau
= k_n c_0 \int_0^M \lambda_n(\tau) e^{-\lambda_n \int_0^\tau \lambda_n(\rho) d\rho} d\tau
+ \int_0^M \lambda_n(\tau) e^{-\lambda_n \int_0^\tau \lambda_n(\rho) d\rho} d\tau
= k_n c_0 \beta_n(\lambda) + c_{n+1} \beta_n(\lambda), \quad n = 0, 1, 2, \ldots, N - 1,
\]
\[
c_N = 0.
\]

Since the value of the determinant of the coefficient of (2.9) is equal to \( F(\lambda) \), we have by assumption of Lemma 2.1 that the coefficient determinant of linear equations (2.9) is equal to zero. It is easy to see that the rank of the coefficient matrix of Eqs. (2.9) is \( N \). We know that the system of (2.9) has a solution \( c_0, c_1, \ldots, c_N \) and \( c_0, c_1, \ldots, c_N \) would not be zero at the same time, and we also know that the dimension of the space of solution is 1.

Without loss of generality, we choose \( c_0 = 1 \). Then we have
\[
c_1 = \sum_{n=1}^{N-1} k_n \beta_1(\lambda) \beta_2(\lambda) \cdots \beta_n(\lambda),
\]
\[
c_2 = \sum_{n=2}^{N-1} k_n \beta_2(\lambda) \beta_3(\lambda) \cdots \beta_n(\lambda), \quad \ldots,
\]
\[
c_{N-1} = k_{N-1} \beta_{N-1}(\lambda),
\]
\[
c_N = 0. \tag{2.10}
\]

Inserting it into (2.8), we get a non-zero solution, and the vector corresponding to \( \lambda \) is \( y(s) = (y_0(s), y_1(s), \ldots, y_N(s))^T \), where, \( y_i, i = 0, 1, 2, \ldots, N, \) are given by (2.8). The proof of this lemma is complete.
LEMMA 2.4. Let \( q_n = (k_n + c_{n+1})\beta_{n-1}\beta_0(\lambda)\beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \) (where \( \beta_{-1} = 1, n = 0, 1, \ldots, N - 1 \)). Suppose that \( \lambda, \lambda' \) are eigenvalues of \( A \) and \( A' \), respectively. If

\[
\sum_{n=0}^{N-1} q_n \int_0^M \tau \lambda_n(\tau) \exp \left[-\lambda \tau - \int_0^\tau \lambda_n(\rho) d\rho \right] d\tau \neq 0,
\]

then the index of \( \lambda \) is 1.

Proof. From Lemmas 2.1 and 2.3 it follows that

\[
(x(s), y(s))_0
= \sum_{n=0}^N (x_n(s), y_n(s))
= \int_0^M \left[ 1 - \int_0^s (k_0 + c_1) \lambda_0(\tau) \exp \left(-\lambda \tau - \int_0^\tau \lambda_0(\rho) d\rho \right) d\tau \right] sds
+ \sum_{n=1}^{N-1} \int_0^M \beta_0(\lambda)\beta_1(\lambda) \cdots \beta_{n-1}(\lambda)
\times \left[ c_n - \int_0^s (k_n + c_{n+1}) \lambda_n(\tau) \exp \left(-\lambda \tau - \int_0^\tau \lambda_n(\rho) d\rho \right) d\tau \right] ds
= \int_0^M \left[ 1 + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \cdots \beta_{n-1}(\lambda) \right] ds
- \int_0^M \left[ \int_0^s (k_0 + c_1) \lambda_0(\tau) \exp \left(-\lambda \tau - \int_0^\tau \lambda_0(\rho) d\rho \right) + \sum_{n=1}^{N-1} (k_n + c_{n+1})
\times \int_0^s \beta_0(\lambda)\beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \lambda_n(\tau) \exp \left(-\lambda \tau - \int_0^\tau \lambda_n(\rho) d\rho \right) d\tau \right] ds
= \left[ 1 + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \right] M
- \left( k_0 + c_1 \right) \int_0^M (M - \tau) \lambda_0(\tau) e^{-\lambda \tau - \int_0^\tau \lambda_0(\rho) d\rho} d\tau
+ \sum_{n=1}^{N-1} (k_n + c_{n+1}) \beta_0(\lambda)\beta_1(\lambda) \cdots \beta_{n-1}(\lambda)
\times \int_0^M (M - \tau) \lambda_n(\tau) \exp \left(-\lambda \tau - \int_0^\tau \lambda_n(\rho) d\rho \right) d\tau
\right]
= \left[ 1 + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \right] M - (k_0 + c_1) M \beta_0(\lambda)
\]
\[ + (k_0 + c_1) \int_0^M \tau \lambda_0(\tau) e^{-\lambda \tau - \int_0^\lambda \lambda_0(\rho) d\rho} d\tau \]

\[ - \sum_{n=1}^{N-1} M(k_n + c_{n+1}) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \beta_n(\lambda) \]

\[ + \sum_{n=1}^{N-1} (k_n + c_{n+1}) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \int_0^M \tau \lambda_n(\tau) e^{-\lambda \tau - \int_0^\lambda \lambda_0(\rho) d\rho} d\tau \]

\[ = \left[ 1 + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \right] M - (k_0 + c_1) M \beta_0(\lambda) \]

\[ - \sum_{n=1}^{N-1} (k_n + c_{n+1}) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda) M \]

\[ + \sum_{n=0}^{N-1} (k_n + c_{n+1}) \beta_{-1}(\lambda) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \]

\[ \times \int_0^M \tau \lambda_n(\tau) e^{-\lambda \tau - \int_0^\lambda \lambda_0(\rho) d\rho} d\tau. \]

Noting that

\[ \left[ 1 + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \right] M \]

\[ - (k_0 + c_1) M \beta_0(\lambda) - \sum_{n=1}^{N-1} (k_n + c_{n+1}) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda) M \]

\[ = \left[ M + \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) M \right] \]

\[ - M \left[ k_0 \beta_0(\lambda) + \sum_{n=1}^{N-1} k_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda) \right] \]

\[ - \left[ c_1 \beta_0(\lambda) + \sum_{n=1}^{N-1} c_{n+1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda) \right] M \]

\[ = \sum_{n=1}^{N-1} c_n \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) M \]

\[ - \left[ c_1 \beta_0(\lambda) + \sum_{n=1}^{N-1} c_{n+1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_n(\lambda) \right] M \]

\[ = 0, \]
therefore (2.11) becomes

\[
(x(s), y(s))_0 = \sum_{n=0}^{N-1} (k_n + c_{n+1}) \beta_{-1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{n-1}(\lambda) \\
\times \int_{0}^{M} \tau \lambda_n(\tau) e^{-\lambda \tau - \int_{\tau}^{0} \lambda_n(\rho) d\rho} d\tau \\
= \sum_{n=0}^{N-1} q_n \int_{0}^{M} \tau \lambda_n(\tau) e^{-\lambda \tau - \int_{\tau}^{0} \lambda_n(\rho) d\rho} d\tau.
\]

By virtue of Lemma 2.2 we know that the proof of this lemma is complete.

**Proof of Theorem 1.1.** We prove the theorem by reduction to absurdity. Assuming that there exist infinitely many \( \lambda_l \in \sigma_p(A) \) with indexes more than 1, it follows from Lemma 2.4 that

\[
N - 1 \sum_{n=0}^{P-1} q_n \int_{0}^{M} \tau \lambda_n(\tau) e^{-\lambda \tau - \int_{\tau}^{0} \lambda_n(\rho) d\rho} d\tau = 0.
\]

Recalling (H) we have

\[
N - 1 \sum_{n=0}^{P-1} q_n \int_{0}^{M} \tau \lambda_n(\tau) e^{-\lambda \tau - \int_{\tau}^{0} \lambda_n(\rho) d\rho} d\tau
\]

\[
= \sum_{n=0}^{N-1} \sum_{i=1}^{P-1} \frac{q_n \mu_{n_{i+1}}}{\lambda_l + \mu_{n_i}} \left\{ \tau e^{-\left(\lambda_l + \mu_{n_i}\right) \tau} \left\vert_{r_{i+1}}^{r_i} - \int_{r_{i+1}}^{r_i} e^{-\left(\lambda_l + \mu_{n_i}\right) \tau} d\tau \right\} \right\} \tag{2.12}
\]

\[
= \sum_{n=0}^{N-1} \sum_{i=1}^{P-1} \frac{q_n \mu_{n_{i+1}}}{\lambda_l + \mu_{n_i}} \left\{ \left( r_i + \frac{1}{\lambda_l + \mu_{n_i}} \right) e^{-\left(\lambda_l + \mu_{n_i}\right) r_i} - \left( r_{i+1} + \frac{1}{\lambda_l + \mu_{n_i}} \right) e^{-\left(\lambda_l + \mu_{n_i}\right) r_{i+1}} \right\}.
\]

\[
= 0.
\]

Equation (2.12) can be rewritten as

\[
q_{N-1} \mu_{N-1, P-1} \frac{M + \frac{1}{\lambda_l + \mu_{N-1, P-1}}}{\lambda_l + \mu_{N-1, P-1}} e^{-\left(\lambda_l + \mu_{N-1, P-1}\right) M}
\]

\[
= q_{N-1} \mu_{N-1, P-1} \left( r_{P-1} + \frac{1}{\lambda_l + \mu_{N-1, P-1}} \right) e^{-\left(\lambda_l + \mu_{N-1, P-1}\right) r_{P-1}}
\]
\begin{align}
&+ \sum_{n=0}^{N-2} \frac{q_n \mu_{n,p-1}}{\lambda_l + \mu_{n,p-1}} \left\{ \left( r_{p-1} + \frac{1}{\lambda_l + \mu_{n,p-1}} \right) e^{-\left( \lambda_l + \mu_{n,p-1} \right) r_{p-1}} \right. \\
&\quad - \left( M + \frac{1}{\lambda_l + \mu_{n,p-1}} \right) e^{-\left( \lambda_l + \mu_{n,p-1} \right) M} \left\} \right. \\
&+ \sum_{n=0}^{N-1} \sum_{i=1}^{P-2} \frac{q_n \mu_{n,i}}{\lambda_l + \mu_{n,i}} \left\{ \left( r_i + \frac{1}{\lambda_l + \mu_{n,i}} \right) e^{-\left( \lambda_l + \mu_{n,i} \right) r_i} \\
&\quad - \left( r_{i+1} + \frac{1}{\lambda_l + \mu_{n,i}} \right) e^{-\left( \lambda_l + \mu_{n,i} \right) r_{i+1}} \right\}. \tag{2.13}
\end{align}

We multiply

\[
\frac{\lambda_l + \mu_{N-1,p-1}}{q_{N-1}} \left( M + \frac{1}{\lambda_l + \mu_{N-1,p-1}} \right)^{-1}
\]

from both sides in (2.13) and then take the modulus to obtain

\[
\mu_{N-1,p-1} e^{-\left( \text{Re} \, \lambda_l + \mu_{N-1,p-1} \right) M}
\]

\[
\leq \mu_{N-1,p-1} \left| \frac{r_{p-1} + \left( \lambda_l + \mu_{N-1,p-1} \right)^{-1}}{M + \left( \lambda_l + \mu_{N-1,p-1} \right)^{-1}} \right| e^{-\left( \text{Re} \, \lambda_l + \mu_{N-1,p-1} \right) r_{p-1}} \\
+ \sum_{n=0}^{N-2} \mu_{n,p-1} \left| \frac{q_n}{q_{N-1}} \cdot \frac{\lambda_l + \mu_{n,p-1}}{\lambda_l + \mu_{n,p-1}} \right| \\
\times \left\{ \left( \frac{r_{p-1} + \left( \lambda_l + \mu_{n,p-1} \right)^{-1}}{M + \left( \lambda_l + \mu_{n,p-1} \right)^{-1}} \right) e^{-\left( \text{Re} \, \lambda_l + \mu_{n,p-1} \right) r_{p-1}} \\
+ \left( \frac{M + \left( \lambda_l + \mu_{n,p-1} \right)^{-1}}{M + \left( \lambda_l + \mu_{N-1,p-1} \right)^{-1}} \right) e^{-\left( \text{Re} \, \lambda_l + \mu_{n,p-1} \right) M} \right\} \\
+ \sum_{n=0}^{N-1} \sum_{i=1}^{P-2} \mu_{ni} \left| \frac{q_n}{q_{N-1}} \cdot \frac{\lambda_l + \mu_{N-1,p-1}}{\lambda_l + \mu_{n,i}} \right| \\
\times \left\{ \left( \frac{r_i + \left( \lambda_l + \mu_{n,i} \right)^{-1}}{M + \left( \lambda_l + \mu_{n,i} \right)^{-1}} \right) e^{-\left( \text{Re} \, \lambda_l + \mu_{n,i} \right) r_i} \\
+ \left( \frac{r_{i+1} + \left( \lambda_l + \mu_{n,i} \right)^{-1}}{M + \left( \lambda_l + \mu_{N-1,p-1} \right)^{-1}} \right) e^{-\left( \text{Re} \, \lambda_l + \mu_{n,i} \right) r_{i+1}} \right\}. \tag{2.14}
\]

By Lemma 2.1 we know that \( \text{Re} \, \lambda_l \to -\infty \) as \( l \to \infty \). Noting that the limits

\[
\lim_{l \to \infty} \left| \frac{\lambda_l + \mu_{N-1,p-1}}{\lambda_l + \mu_{n,i}} \right| \text{ and } \lim_{l \to \infty} \left| \frac{r_i + \left( \lambda_l + \mu_{n,i} \right)^{-1}}{M + \left( \lambda_l + \mu_{N-1,p-1} \right)^{-1}} \right|,
\]

\( n = 0, 1, \ldots, N - 1; \quad i = 1, 2, \ldots, P - 1 \).
exist and
\[
\lim_{l \to \infty} e^{(\Re \lambda_i + \mu_{N-1,P-1})M} \cdot e^{- (\Re \lambda_i + \mu_i)} r_i = 0,
\]
\[
n = 0, 1, 2, \ldots, N - 1; \quad i = 1, 2, \ldots, P - 1,
\]
if
\[
\lim_{l \to \infty} \left| \frac{q_n}{q_{N-1}} \right| = 0, \quad n = 0, 1, 2, \ldots, N - 2, \quad (2.15)
\]
then multiplying (2.14) by \( e^{(\Re \lambda_i + \mu_{N-1,P-1})M} \) from both sides and letting \( l \to \infty \), we obtain that \( \mu_{N-1,P-1} \leq 0 \). This fact is a contradiction to \( \mu_{N-1,P-1} > 0 \) (see (H)). Eventually, we get that there are at most finitely many eigenvalues in \( \sigma_p(A) \) with the index not being 1. Thus we will complete the proof of Theorem 1.1 if we can show that (2.15) is valid.

In fact, from Lemma 2.1, (2.10), and the representation formula of \( q_n \) (see Lemma 2.4) we obtain
\[
q_0 = \frac{1}{\beta_0(\lambda)},
\]
\[
q_n = \frac{1 - k_0 \beta_0(\lambda) - k_1 \beta_0(\lambda) \beta_1(\lambda) - \cdots - k_{N-1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{N-1}(\lambda)}{\beta_n(\lambda)}, \quad n = 1, 2, \ldots, N - 2,
\]
\[
q_{N-1} = k_{N-1} \beta_0(\lambda) \beta_1(\lambda) \beta_2(\lambda) \cdots \beta_{N-2}(\lambda).
\]
It is easy to see that
\[
\lim_{l \to \infty} |\beta_n(\lambda)| = +\infty, \quad n = 0, 1, 2, \ldots, N,
\]
which together with (2.16) implies that
\[
\lim_{l \to \infty} \left| \frac{q_0}{q_{N-1}} \right| = \lim_{l \to \infty} \left| \frac{1}{\beta_0(\lambda) k_{N-1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{N-1}(\lambda)} \right| = 0,
\]
\[
\lim_{l \to \infty} \left| \frac{q_n}{q_{N-1}} \right| = \lim_{l \to \infty} \left| \frac{1 - k_0 \beta_0(\lambda) - k_1 \beta_0(\lambda) \beta_1(\lambda) - \cdots - k_{N-1} \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{N-1}(\lambda)}{k_{N-1} \beta_n(\lambda) \beta_0(\lambda) \beta_1(\lambda) \cdots \beta_{N-2}(\lambda)} \right| = 0, \quad n = 1, 2, \ldots, N - 2.
\]
This completes our proof.
3. THE APPLICATION OF THE MAIN RESULT

In this section, based on the result obtained in the above sections, we get an accurate and explicit asymptotic expansion of the solution of the corresponding parity population system which was not obtained in [1, 17, 18]. This result can be used to analyze the stability and the oscillatory behavior of the population system.

Let \( \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}, \ldots \) be eigenvalues of \( A \). We arrange them according to the size of their real parts, if their real parts are the same, and then arrange them according to the size of their imaginary parts; that is,

\[
\lambda_0 \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \cdots \rightarrow \lambda_k \rightarrow \lambda_{k+1} \rightarrow \cdots
\]

According to Lemma 2.1, there exists a positive integer \( k \) such that the inequality \( \text{Re} \, \lambda_k > \text{Re} \, \lambda_{k+1} \) holds. We denote by \( P_{\lambda_j} \) the projective operator corresponding to \( \lambda_j \); that is,

\[
P_{\lambda_j} f(s) = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_j}} R(\lambda, A) f(s) d\lambda \quad \text{for} \quad \forall f(s) \in H,
\]

where \( \Gamma_{\lambda_j} = \{ \lambda : |\lambda - \lambda_j| = r_{\lambda_j} \} \), and \( r_{\lambda_j} \) is sufficiently small so that

\[
\{ \lambda : |\lambda - \lambda_j| \leq r_{\lambda_j} \} \cap \sigma(A) = \{ \lambda_j \}.
\]

If \( \lambda_j \) is the eigenvalue of \( A \) with the algebraic multiplicity 1, then by [6] we have

\[
(P_{\lambda_j} f)(s) = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j) R(\lambda, A) f(s).
\]

Recalling the expression for \( R(\lambda, A) \) (see [17, P.28]) and computations similar to [17, P.30] we obtain

\[
(P_{\lambda_j} f)(s) = \frac{1}{F'(\lambda_j)} \begin{pmatrix}
F_0(\lambda_j) \exp \left( -\lambda_j s - \int_0^s \lambda_0(\rho) \, d\rho \right) \\
F_1(\lambda_j) \exp \left( -\lambda_j s - \int_0^s \lambda_1(\rho) \, d\rho \right) \\
\vdots \\
F_N(\lambda_j) \exp \left( -\lambda_j s - \int_0^s \lambda_N(\rho) \, d\rho \right)
\end{pmatrix},
\]

where the definitions of \( F_n(\lambda_j) \) \( (n = 0, 1, 2, \ldots, N) \) are given in [17, P.27].
By Theorem 1.1, similarly to [7, P 69; 8, P 235], we obtain the following asymptotic expansion of the solution of (1.1):

**Theorem 3.1.** Let \((H)\) be satisfied and let the possible finite eigenvalues of \(A\) with algebraic multiplicity not equal to 1 lie in the left half plane \(\text{Re} \lambda < \text{Re} \lambda_{k+1}\). Then the solution \(x(t, s)\) of system (1.1) can be expressed as

\[
x(t, s) = T(t)x^{(0)}(s) = \left( T(t) \sum_{j=0}^{k} P_{\lambda_j} x^{(0)}(s) \right)(s) + O(e^{(\text{Re} \lambda_{k} - \epsilon)t})
\]

\[
= \sum_{j=0}^{k} e^{\lambda_{j} t} \left\{ \begin{array}{l}
F_{0}(\lambda_{j}) \exp \left( -\lambda_{j} s - \int_{0}^{s} \lambda_{0}(\rho) \, d\rho \right) \\
F_{1}(\lambda_{j}) \exp \left( -\lambda_{j} s - \int_{0}^{s} \lambda_{1}(\rho) \, d\rho \right) \\
\vdots \\
F_{N}(\lambda_{j}) \exp \left( -\lambda_{j} s - \int_{0}^{s} \lambda_{N}(\rho) \, d\rho \right)
\end{array} \right\} + O(e^{(\text{Re} \lambda_{k} - \epsilon)t}),
\]

where \(T(t)\) is the positive semigroup generated by \(A\) (see [17]) and \(\epsilon\) is an arbitrary small positive number such that \(\text{Re} \lambda_{k} - \epsilon > \text{Re} \lambda_{k+1}\).

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**REFERENCES**


