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New integral representations of nth order convex functions

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ABSTRACT

In this paper we give an integral representation of an n-convex function f in general case without additional assumptions on function f. We prove that any n-convex function can be represented as a sum of two $(n+1)$-times monotone functions and a polynomial of degree at most n. We obtain a decomposition of n-Wright-convex functions which generalizes and complements results of Maksa and Páles (2009) [13]. We define and study relative n-convexity of n-convex functions. We introduce a measure of n-convexity of f. We give a characterization of relative n-convexity in terms of this measure, as well as in terms of nth order distributional derivatives and Radon–Nikodym derivatives. We define, study and give a characterization of strong n-convexity of an n-convex function f in terms of its derivative $f^{(n+1)}(x)$ (which exists a.e.) without additional assumptions on differentiability of f. We prove that for any two n-convex functions f and g, such that f is n-convex with respect to g, the function g is the support for the function f in the sense introduced by Wąsowicz (2007) [29], up to polynomial of degree at most n.

1. Introduction

The notion of *nth order convexity* (or *n-convexity*) was defined in terms of divided differences by Popoviciu [22] (cf. also [24,23,11]), however, we will not state it here. Instead we list some definitions of nth order convexity which are equivalent to Popoviciu's definition.

Proposition 1.1. *A function $f(x)$ is n-convex on (a,b) $(n \geqslant 1)$ if and only if its derivative $f^{(n-1)}(x)$ exists and is convex on (a,b) (with the convention $f^{(0)}(x) = f(x)$).*

This fact first was proved by Hopf [10, p. 24] and by Popoviciu [22, p. 38] (see also [12,24]). Many results on n-convex functions one can found, among others, in [11,1,2,7,12,24,16,19,28–30,6].

Recall that convex functions satisfy various smoothness properties. A convex function defined on (a,b) is continuous and has both right and left derivatives $f'_R(x)$ and $f'_L(x)$ at each point of (a,b). In addition both these derivatives are non-decreasing and satisfy inequality $f'_L(x) \leqslant f'_R(x)$ for all $x \in (a,b)$ (see [24,12]). Thus we have

Proposition 1.2. *A function $f(x): (a,b) \to \mathbb{R}$ is nth order convex $(n \geqslant 1)$ if and only if its right derivative $f_R^{(n)}(x)$ (or left derivative $f_L^{(n)}(x)$) exists and is non-decreasing on (a,b).*

If $f(x)$ is sufficiently smooth on $[a,b]$, then from Taylor's Theorem we have

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$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)(x-a)^k}{k!}+\frac{1}{n!}\int_a^b (x-t)_+^n f^{(n)}(t)\,dt,$$

where $(x-t)_+^{n-1}=\max\{(x-t)^{n-1},0\}$.

Now assume $f(x)$ is nth order convex on (a,b) $(n\geqslant 1)$. Then the left and right derivatives $f_L^{(n)}(x)$ and $f_R^{(n)}(x)$ exist on (a,b). In addition, both these functions are non-decreasing. With each such f we associate the measure μ defined on (a,b) by

$$\mu\big([x,y]\big)=f_R^{(n)}(y)-f_L^{(n)}(x),$$

for $a<x\leqslant y<b$. This is a non-negative Borel measure on (a,b). If $f_R^{(n)}(a)$ is finite then μ can be extended to a bounded (finite) measure on the whole $[a,c]$, for all $c<b$. In this case $f(x)$ has the representation

$$f(x)=\sum_{k=0}^{n}\frac{f_R^{(k)}(a)(x-a)^k}{k!}+\frac{1}{n!}\int_a^b (x-t)_+^n\,d\mu(t),$$

for $x\in(a,b)$. If we cannot extend μ to the endpoint a, then we will have this representation only on closed subintervals of (a,b). The converse also holds. These results can be found in Popoviciu [22] (see also Karlin and Studden [11], Bullen [4], Brown [3], Granata [7], Pinkus and Wulbert [19]). In other words, the above integral representation is valid for all $x\in(a,b)$ if μ is of bounded variation on (a,b), otherwise we have this representation only on closed subinterval of (a,b).

In this paper we give an analogue of the integral representation above in general case. The representation we obtain deals with measures μ with not necessarily bounded variations. Our characterization is constructive. We give explicit formulas for n-spectral measures corresponding to an n-convex function in this representation (see Section 2).

The strength of the representation developed in Section 2 is exploited in the rest of the paper. It is used to further study of n-convexity, and to obtain complete characterizations of strong n-convexity, n-Wright-convexity, and relative n-convexity of functions, among other. Finally, the representation is employed to examine support-type properties of n-convex functions.

In Section 3 we prove that an n-convex function can be represented as a sum of two $(n+1)$-times monotone functions and a polynomial of degree at most n. This result generalizes the well-known theorem on representation of a convex function as a sum of non-increasing and non-decreasing functions, and a polynomial of degree at most 1 (see Roberts and Varberg [24]). Using our decomposition we obtain the decomposition of n-Wright-convex functions, which generalizes and complements results of Maksa and Páles [13].

In Section 4 we define and study relative n-convexity of n-convex functions. Relative n-convexity induces the partial ordering in the set of n-convex functions. We define a measure of n-convexity of an n-convex function f using n-spectral measures in our representation. We give a characterization of relative n-convexity in terms of the measure of n-convexity, as well as in terms of nth order distributional derivatives, and in terms of Radon–Nikodym derivatives. Using the Lebesgue decomposition of n-spectral measures corresponding to an n-convex function f, we consider the corresponding decomposition of the function f. This decomposition is applied to derive some useful characterizations of the relative n-convexity.

We define and study the notion of strong n-convexity that generalizes the strong convexity. It is well known that the strong convexity of a function f can be characterized in terms of its second derivative $f''(x)$ for twice differentiable f. We give a characterization of the strong n-convexity of an n-convex function f in terms of only derivative $f^{(n+1)}(x)$ (which exists almost everywhere with respect to Lebesgue measure), without any additional assumptions on differentiability of f.

In Section 5 we obtain a generalization of Wąsowicz [29] results. We prove, that for any two n-convex functions f and g, such that f is n-convex with respect to g, the function g is the support for the function f in the sense introduced by Wąsowicz [29], up to a polynomial of degree at most n.

2. Integral representation

In this chapter we give an integral representation of an n-convex function f without additional assumptions on f. We derive explicit formulas for n-spectral measures corresponding to f that can be applied to measures of not necessary bounded variation on (a,b).

By λ we denote the Lebesgue measure. Let Π_n be the family of all polynomials of degree at most n. Let $f:(a,b)\to\mathbb{R}$ be an nth order convex function on the interval (a,b), where $-\infty\leqslant a<b\leqslant\infty$, $n=1,2,\ldots$. Then $f_R^{(n)}(x)$ is non-decreasing and right-continuous on (a,b). Henceforth $f^{(n)}(x)$ will be used to denote $f_R^{(n)}(x)$. A function $f^{(n)}(x)$ must satisfy one of the following three conditions:

A. There exist $x_1,x_2\in(a,b)$ such that $f^{(n)}(x_1)<0$ and $f^{(n)}(x_2)>0$,
B. $f^{(n)}(x)\geqslant 0$ for all $x\in(a,b)$,
C. $f^{(n)}(x)\leqslant 0$ for all $x\in(a,b)$.

Theorem 2.1. *For $n \geqslant 1$ each nth order convex function $f : (a,b) \to \mathbb{R}$ satisfying the property* A *admits the representation of the form*

$$f(x) = \int\limits_{(a,\xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!}\, dg_{(n)-}(u) + \int\limits_{[\xi,b)} \frac{(x-u)_+^n}{n!}\, dg_{(n)+}(u) + Q(x), \tag{2.1}$$

where $\xi \in (a,b)$, $g_{(n)-} \leqslant 0$ is a non-decreasing right-continuous function on (a,b), $g_{(n)+}$ is a non-decreasing left-continuous function on (a,b) such that $g_{(n)+}(\xi) = g_{(n)-}(\xi) = 0$, and $Q \in \Pi_{n-1}$. Moreover, the functions $g_{(n)-}$, $g_{(n)+}$ and Q are determined uniquely, $g_{(n)+} = f_+^{(n)}$ a.e., $g_{(n)-} = f_-^{(n)}$ a.e.

Notation 2.2. The quantities

$$\Psi_{(n)-}(x) = \Psi_{(n)-}\big(x; a, \xi, dg_{(n)-}(u)\big) = \int\limits_{(a,\xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!}\, dg_{(n)-}(u), \tag{2.2}$$

$$\Psi_{(n)+}(x) = \Psi_{(n)+}\big(x; \xi, b, dg_{(n)+}(u)\big) = \int\limits_{[\xi,b)} \frac{(x-u)_+^n}{n!}\, dg_{(n)+}(u) \tag{2.3}$$

appear frequently and hence from now we will be using the above notation.

Remark 2.3. A straightforward calculation shows that

$$\frac{d^n}{dx^n} \Psi_{(n)-}\big(x; a, \xi, dg_{(n)-}(u)\big) = \int\limits_{(a,\xi]} \big[-\chi_{(-\infty,0)}(x-u)\big]\, dg_{(n)-}(u) = g_{(n)-}(x) \quad \text{a.e. } \big(x \in (a,\xi)\big), \tag{2.4}$$

$$\frac{d^n}{dx^n} \Psi_{(n)+}\big(x; \xi, b, g_{(n)+}(u)\big) = \int\limits_{[\xi,b)]} \chi_{(0,\infty)}(x-u)\, dg_{(n)+}(u) = g_{(n)+}(x) \quad \text{a.e. } \big(x \in (\xi,b)\big). \tag{2.5}$$

Proof of Theorem 2.1. Let f be an nth order convex function satisfying the property A. Then there exists $\xi \in (a,b)$ such that $f^{(n)}(\xi+) \geqslant 0$ and $f^{(n)}(\xi-) \leqslant 0$. Let $g_{(n)-}(x)$ and $g_{(n)+}(x)$ be right-continuous and left-continuous functions, respectively, and such that

$$g_{(n)-}(x) = \min\{0, f^{(n)}(x)\}, \qquad g_{(n)+}(x) = \max\{0, f^{(n)}(x)\} \quad \text{a.e.} \tag{2.6}$$

Then $g_{(n)-}(\xi) = g_{(n)+}(\xi) = 0$. From (2.4), (2.5) and (2.6) we obtain that the functions $f(x)$ and $\Psi_{(n)-}(x) + \Psi_{(n)+}(x)$ differ on (a,b) by a polynomial of degree at most $n-1$. Thus (2.1) is satisfied. Conversely, assume f is of the form (2.1). By Remark 2.3, $f_-^{(n)}(x) = g_{(n)-}(x)$ and $f_+^{(n)}(x) = g_{(n)+}(x)$ a.e. Thus $f^{(n)}(x)$ is non-decreasing and right-continuous on (a,b). This implies that $f(x)$ is nth order convex on (a,b). The proof is completed. □

Remark 2.4. Note that since $[-(x-u)]_+^n = 0$ for $u < x$, and $(x-u)_+^n = 0$ for $u > x$, the integral (2.2) is over $[x,\xi]$ and the integral (2.3) is over $[\xi,x]$. Since $dg_{(n)-}(u)$ and $dg_{(n)+}(u)$ are of bounded variations on (x,ξ) and (ξ,x), respectively, the integrals in (2.2) and (2.3) are well defined.

Remark 2.5. If $g_{(n)-}(b-) = 0$, then in (2.4) we set $\xi = b$. Similarly if $g_{(n)+}(a+) = 0$, then we put $\xi = a$ in (2.5).

Theorem 2.6. *For $n \geqslant 1$ each n-convex function $f : (a,b) \to \mathbb{R}$ satisfying the property* B *admits the representation*

$$f(x) = \int_a^b \frac{(x-u)_+^n}{n!}\, dg_{(n)}(u) + Q(x), \tag{2.7}$$

where $Q(x) = c_n x^n/n! + \cdots + c_0$, $c_n \geqslant 0$, and $g_{(n)}(x)$ is a non-negative non-decreasing left-continuous function on (a,b) satisfying $g_{(n)}(a+) = 0$. Moreover, $Q(x)$ and $g_{(n)}(x)$ are uniquely determined, $c_n = f^{(n)}(a+)$, $g_{(n)}(x) = f^{(n)}(x) - c_n$ a.e., and $Q(x) = f(x) - \psi_{(n)+}(x; a, b, dg_{(n)}(u))$.

Proof. Assume f is n-convex function such that $f^{(n)}(x) \geqslant 0$ $(x \in (a,b))$. Taking into account that $f^{(n)}(x)$ is non-negative and non-decreasing on (a,b), $f^{(n)}(a+) = c_n$ exists and is finite. Let $g_{(n)}(x)$ be a left-continuous function such that $g_{(n)}(x) = f^{(n)}(x) - c_n$ a.e. $(x \in (a,b))$. Then $g_{(n)}(x)$ is non-negative, non-decreasing, and satisfies $g_{(n)}(a+) = 0$. In view of Remark 2.5, by (2.5) with a in place of ξ and $g_{(n)}(x)$ in place of $g_{(n)+}(x)$, we have

$$\frac{d^n}{dx^n} \psi_{(n)+}\big(x; a, b, dg_{(n)}(u)\big) = g_{(n)}(x) \quad \text{a.e. } \big(x \in (a,b)\big).$$

Consequently

$$\frac{d^n}{dx^n}\psi_{(n)+}\big(x;a,b,dg_{(n)}(u)\big)=f^{(n)}(x)-c_n \quad \text{a.e. } \big(x\in(a,b)\big).$$

Thus the functions $\psi_{(n)+}(x;a,b,dg_{(n)}(u))$ and $f(x)-c_nx^n/n!$ differ on (a,b) by a polynomial of degree at most $(n-1)$. The theorem is proved. □

Theorem 2.7. *For $n\geqslant 1$ each nth order convex function $f:(a,b)\to\mathbb{R}$ satisfying the property* C *admits the representation of the form*

$$f(x)=\int_a^b(-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\,dg_{(n)}(u)+Q(x), \tag{2.8}$$

where $Q(x)=c_nx^n/n!+\cdots+c_0$, $c_n\leqslant 0$, and $g_{(n)}(x)$ is a non-positive non-decreasing right-continuous function on (a,b) such that $g_{(n)}(b-)=c_n$. Moreover, $g_{(n)}$ and Q are uniquely determined, $c_n=f^{(n)}(b-)$, $g_{(n)}(x)=f^{(n)}(x)-c_n$ a.e., and $Q(x)=f(x)-\psi_{(n)-}(x;a,b,dg_{(n)}(u))$.

Proof. The proof is similar to the proof of Theorem 2.6 and hence it is omitted. □

Remark 2.8. The representations (2.1), (2.7) and (2.8) can be rewritten in equivalent forms using the following two measures associated with the distribution functions $g_{(n)-}(x)$ and $g_{(n)+}(x)$, defined as

$$\mu_{(n)-}(du)=dg_{(n)-}(u),\qquad \mu_{(n)+}(du)=dg_{(n)+}(u).$$

We will call $\mu_{(n)-}$ and $\mu_{(n)+}$ the *n-spectral measures* of an n-convex function f.

The following theorem summarizes Theorems 2.1, 2.6 and 2.7.

Theorem 2.9.

a) *For $n\geqslant 1$ each n-convex function $f:(a,b)\to\mathbb{R}$ admits the representation*

$$f(x)=\int_{(a,\xi]}(-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\mu_{(n)-}(du)+\int_{[\xi,b)}\frac{(x-u)_+^n}{n!}\mu_{(n)+}(du)+Q(x), \tag{2.9}$$

where $\xi\in[a,b]$. Moreover, if $f^{(n)}(x)$ satisfies the condition B *(or* C*), then $\xi=a$, $\mu_{(n)-}=0$ and $\mu_{(n)+}(du)=d(f^{(n)}(u)-f^{(n)}(a+))$ (or $\xi=b$, $\mu_{(n)+}=0$ and $\mu_{(n)-}(du)=d(f^{(n)}(u)-f^{(n)}(b-))$), and if $f^{(n)}(x)$ satisfies the condition* A*, then $\xi\in(a,b)$, $f^{(n)}(\xi-)\leqslant 0$, $f^{(n)}(\xi+)\geqslant 0$, $\mu_{(n)-}(du)=df_-^{(n)}(u)$, $\mu_{(n)+}(du)=df_+^{(n)}(u)$ and $Q(x)\in\Pi_n$.*

b) *If $f^{(n)}(a+)=\alpha$ exists and is finite, then $f(x)$ can be rewritten in the form*

$$f(x)=\int_a^b\frac{(x-u)_+^n}{n!}\mu_{(n)a+}(du)+Q_a(x),$$

where $\mu_{(n)a+}(du)=d(f^{(n)}(u)-\alpha)_+$, $Q_a(x)\in\Pi_n$.

c) *If $f^{(n)}(b-)=\beta$ exists and is finite, then $f(x)$ can be rewritten in the form*

$$f(x)=\int_a^b(-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\mu_{(n)b-}(du)+Q_b(x),$$

where $\mu_{(n)b-}(du)=d(f^{(n)}(u)-\beta)_-$, $Q_b(x)\in\Pi_n$.

Denoting

$$\psi_f(x)=\psi_{(n)f}(x)=\int_{(a,\xi]}(-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\mu_{(n)-}(du)+\int_{[\xi,b)}\frac{(x-u)_+^n}{n!}\mu_{(n)+}(du),$$

(2.9) can be rewritten in the form

$$f(x)=\psi_f(x)+Q(x).$$

Note, that every function $f(x)$ can be trivially written as $f(x) = f(x) - cx^n/n! + cx^n/n!$ ($c \in \mathbb{R}$). Thus $f(x)$ can be also written in the form

$$f(x) = \psi_{f-cx^n/n!}(x) + Q_c(x), \tag{2.10}$$

where $Q_c(x) \in \Pi_n$.

Another representation is given in the following theorem. This representation is important in applications of the theory to study relative n-convexity.

Theorem 2.10. *Let $f : (a,b) \to \mathbb{R}$ be an n-convex function. For every $\xi \in (a,b)$ the function $f(x)$ has the representation*

$$f(x) = \int\limits_{(a,\xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)\xi-}(du) + \int\limits_{[\xi,b)} \frac{(x-u)_+^n}{n!} \mu_{(n)\xi+}(du) + Q_\xi(x), \tag{2.11}$$

where

$$\begin{aligned} &\mu_{(n)\xi-}(du) = d\big[f^{(n)}(u) - f^{(n)}(\xi+)\big]_-, \\ &\mu_{(n)\xi+}(du) = d\big[f^{(n)}(u) - f^{(n)}(\xi+)\big]_+, \\ &Q_\xi \in \Pi_n. \end{aligned} \tag{2.12}$$

Moreover, we have

$$\mu_{(n)\xi-} + \mu_{(n)\xi+} = \mu_{(n)-} + \mu_{(n)+}, \tag{2.13}$$

where $\mu_{(n)-}$ and $\mu_{(n)+}$ are the n-spectral measures corresponding to f.

Proof. Let $a < \xi < b$. Put $c = f^{(n)}(\xi+)$ and denote $g_c(x) = f(x) - cx^n/n!$. Then $g_c^{(n)}(\xi-) \leqslant 0$ and $g_c^{(n)}(\xi+) \geqslant 0$, and consequently the function $g_c^{(n)}(x)$ satisfies the condition A. By Theorem 2.9 with $g_c(x)$ in place of $f(x)$, and taking into account (2.10), we obtain the representation (2.11) with the measures $\mu_{(n)\xi-}$ and $\mu_{(n)\xi+}$ satisfying (2.12). It implies that $f^{(n)}(x) - f^{(n)}(\xi+)$ is the distribution function corresponding to the sum of measures $\mu_{(n)\xi-} + \mu_{(n)\xi+}$. By Theorem 2.9, the distribution function corresponding to the sum of n-spectral measures $\mu_{(n)-} + \mu_{(n)+}$ equals $f^{(n)}(x)$ up to a constant. Thus the distribution functions of measures $\mu_{(n)\xi-} + \mu_{(n)\xi+}$ and $\mu_{(n)-} + \mu_{(n)+}$ differ on (a,b) by a constant. Consequently these measures coincide, so (2.13) is proved. □

3. *n*-Convexity and multiple monotonicity

From Theorem 2.9 on the representation of an n-convex function f we obtain that f can be represented by the sum of two $(n+1)$-times monotone functions and a polynomial of degree at most n. Applying this we obtain a theorem on decomposition of an n-Wright-convex function, which complements and generalizes results of Maksa and Páles [13].

By the standard definition (cf. Williamson [31]) a function $f : (a,b) \to \mathbb{R}$ is called *n-times monotone non-increasing* ($n \geqslant 2$) if $(-1)^k f^{(k)}(x)$ is non-negative, non-increasing, and convex for $x \in (a,b)$ and $k = 0, 1, \ldots, n-2$. When $n = 1$, $f(x)$ is simply non-negative and non-increasing. The well-known representation for n-times monotone non-increasing functions on $(0,\infty)$ states that

$$f(x) = \int\limits_0^\infty (1-ux)_+^{n-1}\, d\beta(u) \quad (x > 0), \tag{3.1}$$

with $\beta(u)$ being non-decreasing (see Williamson [31]).

A function f is called *n-times monotone non-decreasing* (briefly *n-times monotone*) ($n \geqslant 2$) if $f^{(k)}(x)$ is non-negative, non-decreasing, and convex for $x \in (a,b)$ and $k = 0, 1, 2, \ldots, n-2$. When $n = 1$, $f(x)$ is simply non-negative and non-decreasing. From (3.1) we derive the following representations of functions $f : (a,b) \to \mathbb{R}$, which are $(n+1)$-times monotone non-increasing and $(n+1)$-times monotone non-decreasing on (a,b), respectively:

$$f(x) = \int\limits_a^b \frac{(x-u)_+^n}{n!}\, d\beta(u), \tag{3.2}$$

$$f(x) = \int\limits_a^b \frac{[-(x-u)]_+^n}{n!}\, d\beta(u), \tag{3.3}$$

where $\beta(u)$ is non-decreasing.

We point out that the representation (3.2) has a short proof. Also, without loss of generality we may assume that $a=-\infty$ and $b=\infty$.

Theorem 3.1. *Let $n \geqslant 1$. A function $f:\mathbb{R}\to\mathbb{R}$ satisfying the condition $f(-\infty)=0$ is n-times monotone non-decreasing if and only if it admits the representation*

$$f(x)=\int_{-\infty}^{\infty}\frac{(x-u)_+^{n-1}}{(n-1)!}\,d\beta(u), \tag{3.4}$$

where $\beta(u)$ is non-decreasing and $\beta(-\infty)=0$. Moreover, $\beta(u)$ is unique at its points of continuity and $\beta(u)=f^{(n-1)}(u)$ a.e.

Proof. The sufficiency is evident by differentiating (3.4), for

$$f^{(k)}(x)=\int_{-\infty}^{\infty}\frac{(x-u)_+^{n-1-k}}{(n-1-k)!}\,d\beta(u)$$

$(k=0,1,\dots,n-1)$ is evidently non-negative and non-decreasing.

To see the necessity let us consider $\beta(u)=f^{(n-1)}(u)$ (with the convention $f^{(0)}(x)=f(x)$). Then (3.4) can be rewritten as

$$f(x)=\int_{-\infty}^{x}\frac{(x-u)^{n-1}}{(n-1)!}\,df^{(n-1)}(u). \tag{3.5}$$

We prove it by induction.

Let $n=1$. Since $f(x)$ is non-decreasing and $f(-\infty)=0$, we have

$$f(x)=\int_{-\infty}^{x}df(u). \tag{3.6}$$

This proves (3.5) for $n=1$.

Assume that (3.5) holds for some $n\geqslant 1$, i.e. n-times monotone non-decreasing functions $f(x)$ such that $f(-\infty)=0$ are of the form (3.5). Let $f(x)$ (such that $f(-\infty)=0$) be an $(n+1)$-times monotone non-decreasing, i.e. $f(x)$ is n-times monotone non-decreasing and $f^{(n)}(x)$ is non-decreasing. It is not difficult to show that $f^{(n)}(-\infty)=0$. By (3.5) and (3.6), with $f^{(n)}(x)$ in place of $f(x)$, we have

$$\begin{aligned}f(x)&=\int_{-\infty}^{x}\frac{(x-u)^{n-1}}{(n-1)!}\,df^{(n-1)}(u)=\int_{-\infty}^{x}\frac{(x-u)^{n-1}}{(n-1)!}f^{(n)}(u)\,du=\int_{-\infty}^{x}\frac{(x-u)^{n-1}}{(n-1)!}\int_{-\infty}^{u}df^{(n)}(v)\,du\\&=\int_{-\infty}^{x}\int_{v}^{x}\frac{(x-u)^{n-1}}{(n-1)!}\,du\,df^{(n)}(v)=\int_{-\infty}^{x}\frac{(x-v)^{n}}{n!}\,df^{(n)}(v).\end{aligned}$$

This proves (3.5), with n replaced by $n+1$, so the induction is complete. □

Let $\mathcal{M}_{n+}((a,b))$ ($\mathcal{M}_{n-}((a,b))$) be the class of all n-times monotone non-decreasing (non-increasing) functions on (a,b). Taking into account (3.2) and (3.3), by Theorem 2.9 we obtain the following decomposition.

Theorem 3.2. *Let $n\geqslant 1$ and $f:(a,b)\to\mathbb{R}$. Then f is nth order convex if and only if f is of the form*

$$f(x)=M_1(x)+M_2(x)+Q(x),$$

where $(-1)^{n+1}M_1(x)\in\mathcal{M}_{(n+1)-}((a,\xi))$, $M_2(x)\in\mathcal{M}_{(n+1)+}((\xi,b))$, with $a\leqslant\xi\leqslant b$, $Q(x)=c_nx^n/n!+\dots+c_0\in\Pi_n$. Moreover, if $M_1\equiv 0$ and $M_2\not\equiv 0$ then $c_n\geqslant 0$, if $M_2\equiv 0$ and $M_1\not\equiv 0$ then $c_n\leqslant 0$, and if $M_1\not\equiv 0$ and $M_2\not\equiv 0$ then $c_n=0$ and $M_1(\xi-)=M_2(\xi+)$.

Remark 3.3. Note that if $g(x)\in\mathcal{M}_{(n+1)+}((a,b))$, then $\varphi(x)=g(-x)\in\mathcal{M}_{(n+1)-}((-b,-a))$. Thus $M_2(x)=g(x)$ is nth order convex on (a,b) and $M_1(x)=(-1)^{n+1}\varphi(x)$ is nth order convex on $(-b,-a)$.

The following proposition gives a characterization of n-times monotone non-decreasing functions in terms of difference operators (see McNeil [14]).

Proposition 3.4. *Let $n \geqslant 1$ and $f : (a,b) \to \mathbb{R}$. Then the following statements are equivalent.*

(i) *f is n-times monotone on (a,b).*
(ii) *f is non-negative and for any $k = 1, \dots, n$, any $x \in (a,b)$, and any $h_i > 0$, $i = 1, \dots, k$ such that $x + h_1 + \dots + h_k \in (a,b)$ the function f satisfies*

$$\Delta_{h_k} \dots \Delta_{h_1} f(x) \geqslant 0, \tag{3.7}$$

where $\Delta_{h_k} \dots \Delta_{h_1}$ denote sequential applications of the first-order difference operator Δ_h given by $\Delta_h f(x) = f(x+h) - f(x)$ whenever $x, x+h \in (a,b)$.
(iii) *f is non-negative and satisfies, for any $k = 1, \dots, n$, any $x \in (a,b)$, and any $h > 0$ such that $x + kh \in (a,b)$*

$$(\Delta_h)^k f(x) \geqslant 0, \tag{3.8}$$

where $(\Delta_h)^k$ denote the k-monotone sequential iterations of the operator Δ_h.

Note that in Gilányi and Páles [6] functions satisfying (3.7) with $k = n+1$ are called *Wright-convex of order n* (or simply *n-Wright-convex*). As it is extensively discussed in [12] ($n \geqslant 1$), the functions satisfying (3.8) with $k = n+1$ are called *Jensen-convex of order n*. It is well known that, under the assumption of continuity, the Jensen convexity of order n and nth order convexity are equivalent. In the study of inequalities (3.7) and (3.8), functions that satisfy (3.7) and (3.8) with equality play a crucial role. For $n \in \mathbb{N}$, a function $P : \mathbb{R} \to \mathbb{R}$ is called a *polynomial function of degree at most n* if it satisfies the Fréchet equation, i.e. if

$$(\Delta_h)^{n+1} P(x) = 0 \quad (h, x \in \mathbb{R}).$$

Polynomials are exactly the continuous polynomial functions, however, in terms of Hamel bases, one can construct non-continuous polynomial functions (see [12]). Maksa and Páles [13] proved that any n-Wright-convex function can be represented as the sum of a continuous n-convex function and a polynomial function.

Proposition 3.5. *Let $n \geqslant 1$ and $f : (a,b) \to \mathbb{R}$. Then f is an n-Wright-convex function if and only if f is of the form*

$$f(x) = C(x) + P(x) \quad \bigl(x \in (a,b)\bigr), \tag{3.9}$$

where $C : (a,b) \to \mathbb{R}$ is a continuous n-convex function and $P : \mathbb{R} \to \mathbb{R}$ is a polynomial function of degree at most n with $P(\mathbb{Q}) = \{0\}$. Furthermore, under the assumption $P(\mathbb{Q}) = 0$, the decomposition (3.9) is unique.

From Theorem 3.2 and Proposition 3.5 we obtain the following decomposition of n-Wright-convex functions.

Theorem 3.6. *Let $n \geqslant 1$ and $f : (a,b) \to \mathbb{R}$. Then f is an n-Wright-convex if and only if f is of the form*

$$f(x) = M_1(x) + M_2(x) + Q(x) + P(x),$$

where $(-1)^{n+1} M_1(x)$ is an $(n+1)$-times monotone non-increasing on (a,ξ), $M_2(x)$ is an $(n+1)$-times monotone non-decreasing on (ξ,b), $a \leqslant \xi \leqslant b$, $M_1(x) + M_2(x)$ is continuous on (a,b), $Q(x)$ is a polynomial of degree at most n (as in Theorem 3.2) and $P(x)$ is a polynomial function of degree at most n.

4. Relative n-convexity. Strong n-convexity

Let $g : (a,b) \to \mathbb{R}$ be an n-convex function. We say that a function $f : (a,b) \to \mathbb{R}$ is *n-convex with respect to g* if $f - g$ is n-convex, and denote it by $f \succcurlyeq_n g$.

Various other generalizations of convexity via related convexity properties have been proposed. The relative n-convexity defined above is a generalization of the relative convexity (for $n = 1$) studied in Karlin and Studden [11] (cf. [5,8,17,18]).

Remark 4.1. If f is n-convex with respect to g, then both $f - g$ and g are n-convex. Writing $f = g + (f - g)$, we obtain that f necessarily must be n-convex.

Functions $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ that are decreasing and increasing on the same intervals will be called *isotonic*, and we say that they are members of the same isotonic class.

Theorem 4.2. *Let $f, g : (a,b) \to \mathbb{R}$ be n-convex functions with n-spectral measures $\mu_{(n)-}$, $\mu_{(n)+}$ and $\nu_{(n)-}$, $\nu_{(n)+}$, respectively. Then f is n-convex with respect to g if and only if*

$$\mu_{(n)} \geqslant \nu_{(n)}, \tag{4.1}$$

where

$$\mu_{(n)} = \mu_{(n)-} + \mu_{(n)+}, \qquad \nu_{(n)} = \nu_{(n)-} + \nu_{(n)+}.$$

Proof. Let f and g satisfy the assumptions of the theorem. Fix $a < \xi < b$. By Theorem 2.10, $f(x)$ and $g(x)$ can be written in the form (2.11) with the measures $\mu_{(n)\xi-}$, $\mu_{(n)\xi+}$, $\nu_{(n)\xi-}$ and $\nu_{(n)\xi+}$, and the polynomials $Q_\xi(x)$ and $R_\xi(x)$, respectively. In other words, the functions $f(x) - Q_\xi(x)$ and $g(x) - R_\xi(x)$ are isotonic. Moreover, by (2.13), we have

$$\mu_{(n)-} + \mu_{(n)+} = \mu_{(n)\xi-} + \mu_{(n)\xi+}, \qquad \nu_{(n)-} + \nu_{(n)+} = \nu_{(n)\xi-} + \nu_{(n)\xi+}. \tag{4.2}$$

Therefore $f(x) - g(x)$ is of the form (2.11), with the measures $\mu_{(n)\xi-} - \nu_{(n)\xi-}$ and $\mu_{(n)\xi+} - \nu_{(n)\xi+}$ in the place of $\mu_{(n)\xi-}$ and $\mu_{(n)\xi+}$, respectively, and $Q_\xi(x) - R_\xi(x)$ in the place of $Q_\xi(x)$. By Theorem 2.10, $f - g$ is n-convex if and only if

$$\mu_{(n)\xi-} \geqslant \nu_{(n)\xi-}, \qquad \mu_{(n)\xi+} \geqslant \nu_{(n)\xi+}.$$

From (4.2) we conclude (4.1). The theorem is proved. □

Theorem 4.2 suggests that we can define a *measure of nth order convexity* of an n-convex function by the operator

$$K : f \to \mu^f_{(n)} = \mu_{(n)} = \mu_{(n)-} + \mu_{(n)+}.$$

In the sequel we will call $\mu^f_{(n)}$ the *measure of n-convexity of f*, or shortly the *n-convexity measure*. From Theorem 4.2 we have

Theorem 4.3.

$$f \succcurlyeq_n g \quad \textit{if and only if} \quad \mu^f_{(n)} \geqslant \mu^g_{(n)}.$$

We shall say that functions $f, g : (a, b) \to \mathbb{R}$ are of *modulo Π_n*, or that they are members of the same *modulo Π_n class*, if they differ by a polynomial $Q \in \Pi_n$. The relation modulo Π_n is an equivalence relation and hence it defines equivalence classes. For n-convex f and $g : (a, b) \to \mathbb{R}$ that are members of the same modulo Π_n class we therefore have that $f^{(n)}(x)$ and $g^{(n)}(x)$ differ on (a, b) by a constant. Consequently, by Theorem 2.10, we have the following theorem.

Theorem 4.4.

$$f = g \pmod{\Pi_n} \quad \textit{if and only if} \quad \mu^f_{(n)} = \mu^g_{(n)}.$$

We now show that this relation induces a partial ordering.

Theorem 4.5. *The relative n-convexity relation induces a partial ordering on modulo Π_n equivalence classes of n-convex functions.*

Proof. We will show that the relation is reflective, antisymmetric, and transitive.

Reflectivity. For all f we have $f - f \equiv 0 \in \Pi_n$. Thus $f \succcurlyeq_n f$.

Antisymmetry. Suppose $f \succcurlyeq_n g$ and $g \succcurlyeq_n f$. Then $f - g$ and $g - f$ are n-convex. Thus, both functions $(f - g)^{(n)}(x)$ and $[-(f - g)]^{(n)}(x)$ are non-decreasing. Consequently, $(f - g)^{(n)}(x) = 0$ $(x \in (a, b))$. This implies that $f - g \in \Pi_n$, that is $f = g \pmod{\Pi_n}$.

Transitivity. Suppose $f \succcurlyeq_n g$ and $g \succcurlyeq_n h$. Then both $f - g$ and $g - h$ are n-convex. Writing $f - h$ in the form $f - h = (f - g) + (g - h)$, we obtain that $f - h$ is n-convex as the sum of the n-convex functions. The theorem is proved. □

As a simple example of the use of Theorem 2.10 we prove the following theorem. We denote by $d\mu/d\nu = \varphi$ the Radon–Nikodym derivative of a measure μ with respect to a measure ν (see [25]).

Theorem 4.6. *Let $f, g : (a, b) \to \mathbb{R}$ be n-convex. Then*

a) *there exists an n-convex function f_{max} such that*

$$f_{max} \succcurlyeq_n f, \qquad f_{max} \succcurlyeq_n g,$$

and for every n-convex function h

$$(h \succcurlyeq_n f \textit{ and } h \succcurlyeq_n g) \quad \Rightarrow \quad h \succcurlyeq_n f_{max},$$

b) *there exists an n-convex function* f_{min} *such that*

$$f \succcurlyeq_n f_{min}, \qquad g \succcurlyeq_n f_{min},$$

and for every n-convex function h

$$(f \succcurlyeq_n h \text{ and } g \succcurlyeq_n h) \quad \Rightarrow \quad f_{min} \succcurlyeq_n h,$$

c) *if* $f \succcurlyeq_n g$ *and* $f \neq g \pmod{\Pi_n}$, *then there exists an n-convex function w such that* $f \neq w \pmod{\Pi_n}$, $g \neq w \pmod{\Pi_n}$ *and*

$$f \succcurlyeq_n w \succcurlyeq_n g.$$

Proof. Let f and g be n-convex. By Theorem 2.10 we may assume that f and g admit representations given by (2.11) with the same $\xi \in (a,b)$, the measures $\mu^f_{(n)\xi-}$, $\mu^f_{(n)\xi+}$, $\mu^g_{(n)\xi-}$, $\mu^g_{(n)\xi+}$, and with the polynomials Q^f_ξ and Q^g_ξ, respectively. Consider the Radon–Nikodym derivatives

$$\begin{aligned}
\varphi_1 &= d\mu^f_{(n)\xi-}/d\big(\mu^f_{(n)\xi-}+\mu^g_{(n)\xi-}\big),\\
\psi_1 &= d\mu^g_{(n)\xi-}/d\big(\mu^f_{(n)\xi-}+\mu^g_{(n)\xi-}\big),\\
\varphi_2 &= d\mu^f_{(n)\xi+}/d\big(\mu^f_{(n)\xi+}+\mu^g_{(n)\xi+}\big),\\
\psi_2 &= d\mu^g_{(n)\xi+}/d\big(\mu^f_{(n)\xi+}+\mu^g_{(n)\xi+}\big).
\end{aligned}$$

It is not difficult to see that it suffices to take the functions f_{max} and f_{min} of the form (2.11) with the measures

$$\begin{aligned}
\mu^{max}_{(n)\xi-} &= \max(\varphi_1,\psi_1)\big(\mu^f_{(n)\xi-}+\mu^g_{(n)\xi-}\big),\\
\mu^{max}_{(n)\xi+} &= \max(\varphi_2,\psi_2)\big(\mu^f_{(n)\xi+}+\mu^g_{(n)\xi+}\big),\\
\mu^{min}_{(n)\xi-} &= \max(\varphi_1,\psi_1)\big(\mu^f_{(n)\xi-}+\mu^g_{(n)\xi-}\big),\\
\mu^{min}_{(n)\xi+} &= \max(\varphi_2,\psi_2)\big(\mu^f_{(n)\xi+}+\mu^g_{(n)\xi+}\big),
\end{aligned}$$

to prove parts a) and b).

To prove c) assume $f \succcurlyeq_n g$ and $f \neq g \pmod{\Pi_n}$. Then $f-g$ is n-convex and $f-g \neq 0 \pmod{\Pi_n}$. Thus it suffices to take $w = g + \frac{1}{2}(f-g)$. The theorem is proved. □

As usual we denote distributional derivatives by f' (see [26,27]), pointwise derivatives by $f'(x)$, nth order distributional derivatives by $f^{(n)}$, and nth order pointwise derivatives by $f^{(n)}(x)$. Theorem 4.2 suggests that we can use distributional derivatives and the Radon–Nikodym derivatives to derive simple criteria for the relative n-convexity $f \succcurlyeq_n g$.

Theorem 4.7. *Let* $f, g : (a,b) \to \mathbb{R}$ *be n-convex functions with the n-convexity measures* $\mu^f_{(n)}$ *and* $\mu^g_{(n)}$, *respectively. Then the following conditions are equivalent*:

a) $f \succcurlyeq_n g$,
b) $\mu^f_{(n)} \geqslant \mu^g_{(n)}$,
c) $f^{(n+1)} \geqslant g^{(n+1)}$,
d) $d\mu^g_{(n)}/d\mu^f_{(n)} \leqslant 1$.

Via Lebesgue's decomposition theorem and the decomposition of a singular measure, every σ-finite measure μ can be decomposed into a sum of an absolutely continuous measure (with respect to the Lebesgue measure), a singular continuous measure, and a discrete measure, i.e.

$$\mu = \mu_{cont} + \mu_{sing} + \mu_{pp},$$

where μ_{cont} is the absolutely continuous part, μ_{sing} is the singular continuous part and μ_{pp} is the pure point part (a discrete measure) (see Royden [25]). These three measures are uniquely determined.

Remark 4.8. The following decomposition yields an analogous decomposition of an n-convex function. Namely, any n-convex function f with the n-spectral measures $\mu_{(n)-}$ and $\mu_{(n)+}$ can be represented as a sum

$$f = f_{cont} + f_{sing} + f_{pp}, \tag{4.3}$$

where f_{cont}, f_{sing} and f_{pp} correspond to the absolutely continuous parts the singular continuous parts, and the pure point parts of the n-spectral measures $\mu_{(n)-}$ and $\mu_{(n)+}$, respectively ($\mu_{(n)-} = \mu_{(n)-cont} + \mu_{(n)-sing} + \mu_{(n)-pp}$, $\mu_{(n)+} = \mu_{(n)+cont} + \mu_{(n)+sing} + \mu_{(n)+pp}$). Note, that f_{cont}, f_{sing} and f_{pp} are n-convex. Moreover, they are unique, up to a polynomial of degree at most n.

It is not difficult to prove the following lemma.

Lemma 4.9. *Let μ and ν be two σ-finite measures having the following decompositions into a sum of an absolutely continuous measure, a singular continuous measure and a discrete measure*

$$\mu = \mu_{cont} + \mu_{sing} + \mu_{pp} \quad \text{and} \quad \nu = \nu_{cont} + \nu_{sing} + \nu_{pp}.$$

Then $\mu \geqslant \nu$ if and only if $\mu_{cont} \geqslant \nu_{cont}$, $\mu_{sing} \geqslant \nu_{sing}$ and $\mu_{pp} \geqslant \nu_{pp}$.

Taking into account the decomposition (4.3) of an n-convex function, by Lemma 4.9 we immediately obtain the following three theorems useful in studying relative n-convexity.

Theorem 4.10. *Let f and $g : (a, b) \to \mathbb{R}$ be n-convex functions having the decompositions $f = f_{cont} + f_{sing} + f_{pp}$, $g = g_{cont} + g_{sing} + g_{pp}$ (see (4.3)). Then $f \succcurlyeq_n g$ if and only if $f_{cont} \succcurlyeq_n g_{cont}$ and $f_{sing} \succcurlyeq_n g_{sing}$ and $f_{pp} \succcurlyeq_n g_{pp}$.*

Theorem 4.11.

$$f_{cont} \succcurlyeq_n g_{cont} \quad \text{iff} \quad f_{cont}^{(n+1)}(x) \geqslant g_{cont}^{(n+1)}(x).$$

Theorem 4.12.

$$f_{pp} \succcurlyeq_n g_{pp} \quad \text{iff} \quad f_{pp}^{(n+1)} \geqslant g_{pp}^{(n+1)},$$

where $f_{pp}^{(n+1)} = \sum_k a_k \delta_{x_k}$, $g_{pp}^{(n+1)} = \sum_k b_k \delta_{y_k}$.

The notion of convexity can be extended not only to the case when the order of convexity is of higher dimension, but also in several other ways. One of the most important generalizations is the notion of strong convexity. A function $f : (a, b) \to \mathbb{R}$ is called *strongly convex with modulus $c > 0$* if

$$f\big(tx + (1-t)y\big) \leqslant tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in (a, b)$ and $t \in [0, 1]$. Strongly convex functions were introduced by Polyak [21]. Some properties of them can be found, among other, in [24,9,20] and [15]. Not attempting to be complete, we just recall here two results concerning strong convexity which play crucial roles in further investigations (see [24]). The first one characterizes strong convexity in terms of convexity, while the second one characterizes twice differentiable strongly convex function in terms of its second derivative $f''(x)$.

Proposition 4.13. *A function $f : (a, b) \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if the function $f(x) - cx^2$ is convex.*

Proposition 4.14. *Assume that $f : (a, b) \to \mathbb{R}$ is twice differentiable and $c > 0$. Then f is strongly convex with modulus c if and only if $f''(x) \geqslant 2c$ ($x \in (a, b)$).*

As a generalization of strong convexity with modulus c, we define strong n-convexity with modulus c. We say that a function f is *strongly n-convex with modulus c* ($n \geqslant 1$, $c > 0$) if f is n-convex with respect to the function $g(x) = cx^{(n+1)}/(n+1)!$. By Proposition 4.13 the strong convexity with modulus $2c$ (cf. Roberts and Varberg [24]) coincides with our strong 1-convexity with modulus c. Writing $f(x) = (f(x) - cx^{n+1}/(n+1)!) + cx^{n+1}/(n+1)!$, we obtain that if f is strongly n-convex with modulus $c > 0$, then f is n-convex.

The following theorem gives a characterization of a strongly n-convex function f with modulus c without additional assumptions on differentiability of f. This generalizes Proposition 4.14.

Theorem 4.15. *Let $f : (a, b) \to \mathbb{R}$ be an n-convex function and $c > 0$. Then f is strongly n-convex with modulus c if and only if*

$$f^{(n+1)}(x) \geqslant c \quad \text{for } x \in (a, b)\ \lambda\ a.e.$$

Proof. Note that the function $g(x) = cx^{n+1}/(n+1)!$ is n-convex with the n-convexity measure $\mu^g_{(n)}(dx) = dg^{(n)}(x) = c\,dx$, where $g^{(n)}(x) = cx$. Writing $g(x)$ in the form (4.3), we have $g = g_{cont}$, $g_{sing} = 0$ and $g_{pp} = 0$. As f is n-convex, we look at its integral representation given by (2.7), with the n-convexity measure $\mu^f_{(n)}(dx) = df^{(n)}(x)$. Since $f^{(n)}(x)$ is non-decreasing its derivative $f^{(n+1)}(x)$ exists for $x \in (a,b)$ λ a.e. By Remark 4.8 $f(x)$ can be represented as the sum

$$f = f_{cont} + f_{sing} + f_{pp}.$$

From Lemma 4.9, and taking into account that $g_{cont} = g$, $g_{sing} = g_{pp} = 0$, we obtain that

$$f \succcurlyeq_n g \quad \text{iff} \quad f_{cont} \succcurlyeq_n g_{cont}.$$

By Theorem 4.10, $f_{cont} \succcurlyeq_n g_{cont}$ iff $f^{(n+1)}_{cont}(x) \geqslant g^{(n+1)}_{cont}(x)$. Since $g^{(n+1)}_{cont}(x) = c$, and $f^{(n+1)}_{cont}(x) = f^{(n+1)}(x)$ for $x \in (a,b)$ λ a.e., the theorem is proved. □

Corollary 4.16. *Let $c > 0$, $n \in \mathbb{N}$ and $f : (a,b) \to \mathbb{R}$ be a function. Then f is strongly n-convex with modulus c if and only if f is of the form*

$$f(x) = f_{cont}(x) + R(x) \quad \big(x \in (a,b)\big),$$

where $f_{cont} : (a,b) \to \mathbb{R}$ is an $(n+1)$-times differentiable strongly n-convex function with modulus c, and $R : (a,b) \to \mathbb{R}$ is an n-convex function such that $R^{(n+1)}(x) = 0$ for $x \in (a,b)$ λ a.e.

Corollary 4.17. *Let $c > 0$, $n \in \mathbb{N}$ and let $f : (a,b) \to \mathbb{R}$ be an $(n+1)$-times differentiable function. Then f is strongly n-convex with modulus c if and only if*

$$f^{(n+1)}(x) \geqslant c, \quad x \in (a,b).$$

5. Interpolation of functions by *n*-convex functions

It is well known that every convex function $f : I \to \mathbb{R}$ admits an affine support at every interior point of I (i.e. for any $x_0 \in \mathit{Int}\, I$ there exists an affine function $a : I \to \mathbb{R}$ such that $a(x_0) = f(x_0)$ and $a \leqslant f$ on I). Convex functions of higher orders (precisely of an odd orders) have similar property; they are supported by polynomials of degree no greater than the order of convexity.

The following important property of convex functions of higher order (cf. Kuczma [12], Popoviciu [22], Roberts and Varberg [24]) is well known: a function $f : I \to \mathbb{R}$ is n-convex ($I \subset \mathbb{R}$ is an interval) if and only if for any $x_1, \dots, x_{n+1} \in I$ with $x_1 < \dots < x_{n+1}$ the graph of an interpolating polynomial $p := P(x_1, \dots, x_{n+1}; f)$ passing through the points $(x_i, f(x_i))$, $i = 1, \dots, n+1$, changes successively from one side of the graph of f to another (always $p(x) \leqslant f(x)$ for $x \in I$ such that $x > x_{n+1}$ if such points exist). More precisely, $(-1)^{n+1}(f(x) - p(x)) \geqslant 0$, $x < x_1$, $x \in I$, $(-1)^{n+1-i}(f(x) - p(x)) \geqslant 0$, $x_i < x < x_{i+1}$, $i = 1, \dots, n$, $f(x) - p(x) \geqslant 0$, $x > x_{n+1}$, $x \in I$. It is not difficult to observe that the n-convexity reduces to convexity in the usual sense if $n = 1$.

In the Wąsowicz paper [29] certain attaching method is developed. The method is applied in Theorem 5.1 to obtain a general result, from which the mentioned above support theorem and some related properties of convex functions of higher order are derived.

Theorem 5.1. *Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be an n-convex function. Let us fix $k \in \mathbb{N}$, $k \leqslant n$, and take $x_1, \dots, x_k \in I$ such that $x_1 < \dots < x_k$. To each point x_j ($j = 1, \dots, k$) assign the multiplicity $l_j \in \mathbb{N}$ such that $l_1 + \dots + l_j = n+1$. Additionally assume that if $x_1 = \inf I$, then $l_1 = 1$, and if $x_k = \sup I$, then $l_k = 1$. Denote $I_0 = (-\infty, x_1)$, $I_j = (x_j, x_{j+1})$, $j = 1, \dots, k-1$, and $I_k = (x_k, \infty)$. Under these assumptions there exists a polynomial $p \in \Pi_n$ such that $p(x_j) = f(x_j)$, $j = 1, \dots, k$, and such that $(-1)^{n+1}(f(x) - p(x)) \geqslant 0$ for $x \in I_0 \cap I$, $(-1)^{n+1-(l_1+\dots+l_j)}(f(x) - p(x)) \geqslant 0$ for $x \in I_j$, $j = 1, \dots, k-1$, $f(x) - p(x) \geqslant 0$ for $x \in I_k \cap I$.*

The numbers $l_1, \dots, l_k$ can be interpreted as multiplicities of the points $x_1, \dots, x_k$, respectively. The polynomial $p(x)$ in the above theorem will be called the *support of $(l_1, \dots, l_k)$-type*.

Remark 5.2. This fact is shown by Wąsowicz [30] in a more general setting, i.e. for functions convex with respect to Chebyshev systems (for Chebyshev's polynomial system $(1, x, \dots, x^n)$ such convexity reduces to n-convexity).

Observation 5.3. The polynomial $p(x)$ described in Theorem 5.1 has the following properties:

(i) $p(x) \leqslant f(x)$, $x > x_k$, $x \in I$,
(ii) if l_j (i.e. the multiplicity of x_j) is even, then the graph of $p(x)$ passing through x_j remains on the same side of the graph of f, while it changes the side, if l_j is odd.

We apply Theorem 5.1 to obtain a general result, that for any two n-convex functions f and g, such that f is n-convex with respect to g, the function g is a support of $(l_1, \ldots, l_k)$-type for function f, up to some polynomial $p \in \Pi_n$.

Theorem 5.4. *Let $n \in \mathbb{N}$ and let f and $g : I \to \mathbb{R}$ be two n-convex functions such that f is n-convex with respect to g. Fix $k \in \mathbb{N}$, $k \leqslant n$, and let $x_1, \ldots, x_k \in I$ be such that $x_1 < \cdots < x_k$. Suppose that l_j, I_j satisfy conditions of Theorem* 5.1. *Then there exists a polynomial $p \in \Pi_n$, such that*

$$f(x_j) = g(x_j) + p(x_j), \quad j = 1, \ldots, k, \tag{5.1}$$

and additionally

$$\begin{aligned} &(-1)^{n+1}\big[f(x) - \big(g(x) + p(x)\big)\big] \geqslant 0 \quad \text{for } x \in I_0 \cap I, \\ &(-1)^{n+1-(l_1+\cdots+l_j)}\big[f(x) - \big(g(x) + p(x)\big)\big] \geqslant 0 \quad \text{for } x \in I_j,\ j = 1, \ldots, k-1, \\ &f(x) - \big(g(x) + p(x)\big) \geqslant 0 \quad \text{for } x \in I_k \cap I. \end{aligned} \tag{5.2}$$

The function $g(x) + p(x)$ will be called the support of $(l_1, \ldots, l_k)$-type for the function f.

Proof. Since $f(x)$ is n-convex with respect to g, $f(x) - g(x)$ is n-convex. Applying Theorem 5.1 with the function $f(x) - g(x)$ in place of $f(x)$, we obtain that there exists a polynomial $p \in \Pi_n$ such that

$$\begin{aligned} &f(x_j) - g(x_j) = p(x_j), \quad j = 1, \ldots, k, \\ &(-1)^{n+1}\big[\big(f(x) - g(x)\big) - p(x)\big] \geqslant 0 \quad \text{for } x \in I_0 \cap I, \\ &(-1)^{n+1-(l_1+\cdots+l_j)}\big[\big(f(x) - g(x)\big) - p(x)\big] \geqslant 0 \quad \text{for } x \in I_j,\ j = 1, \ldots, k-1, \\ &\big(f(x) - g(x)\big) - p(x) \geqslant 0 \quad \text{for } x \in I_k \cap I. \end{aligned}$$

Thus (5.1) and (5.2) are satisfied. This completes the proof. □

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