ELSEVIER

Journal of Computational and Applied Mathematics 83 (1997) 257–268

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Symmetrizability of differential equations having orthogonal polynomial solutions

K.H. Kwon *, G.J. Yoon

Department of Mathematics, KAIST, 373-1 Kusong-dong, Yusong-ku, Taejon 305-701, South Korea

Received 25 June 1996

Dedicated to our beloved late Professor Jongsik Kim

Abstract

We show that if a linear differential equation of spectral type with polynomial coefficients

$$L_N[y](x) = \sum_{i=0}^{N} \ell_i(x) y^{(i)}(x) = \lambda_n y(x)$$

has an orthogonal polynomial system of solutions, then the differential operator $L_N[\cdot]$ must be symmetrizable. We also give a few applications of this result.

Keywords: Differential equations; Symmetrizability; Orthogonal polynomials

AMS classifications: 33C45; 34L05

1. Introduction

In this work, we are interested in differential equations of the form

$$L_N[y](x) = \sum_{i=0}^{N} \ell_i(x) y^{(i)}(x) = \sum_{i=0}^{N} \sum_{j=0}^{i} \ell_{ij} x^j y^{(i)}(x) = \lambda_n y(x) \tag{1.1}$$

* Corresponding author. E-mail: khkwon@jacobi.kaist.ac.kr.

PII S0377-0427(97)00110-6

having orthogonal polynomials of solutions, where ℓ_{ij} are real constants and the eigenvalue parameter λ_n is given by

$$\lambda_n = \ell_{00} + \ell_{11}n + \cdots + \ell_{NN}n(n-1)\cdots(n-N+1), \quad n = 0, 1, \ldots \tag{1.2}$$

We always assume $\ell_N(x) \not\equiv 0$ and $\ell_{11}^2 + \ell_{22}^2 + \cdots + \ell_{NN}^2 \neq 0$.

In 1929, Bochner [1] showed that there are essentially five polynomial systems (namely, four classical orthogonal polynomials of Jacobi, Bessel, Laguerre, and Hermite, and $\{x^n\}_{n=0}^{\infty}$) that satisfy the differential equation (1.1) with $N=2$. In 1938, Krall [13] found necessary and sufficient conditions in order for an orthogonal polynomial system (OPS) to satisfy the differential equation (1.1) and then [14] classified all fourth-order differential equations having an OPS of solutions (see also [10]). For $N>4$, the complete classification of such differential equations remain open but several examples have been found in [7, 8, 11, 19, 21].

Interests in such differential equations lie partly in the fact that they provide excellent examples to illustrate the general Titchmarsh–Weyl theory [4] of singular boundary value problems. We refer the reader to Everitt and Littlejohn [5] (and references therein) for the up to date survey on the known results linking spectral theory of differential operators and orthogonal polynomials.

In order to develop the spectral theory of such differential operators, which seeks self-adjoint differential operators in some Hilbert or Krein space, we first need to put differential operators into symmetric forms. In other words, we should first know that differential operators involved are symmetrizable. The differential operator $L_N[\cdot]$ in (1.1) is always symmetrizable for $N=2$ but need not be so in general for $N>2$. Necessary and sufficient conditions for any linear differential operator (more general than (1.1)) to be symmetrizable were found by Littlejohn [20] and Littlejohn and Race [22].

The main result of this work (see Theorem 3.4) is to show that if the differential equation (1.1) has an OPS of solutions, then the differential operator $L_N[\cdot]$ must be symmetrizable. It answers a question in [5]. Sufficient conditions for symmetrizability of such $L_N[\cdot]$ were previously given in [12, 17]. As a consequence, we can also refine results in [17] on the structure of distributional orthogonalizing weights for OPS's satisfying the differential equation (1.1).

2. Preliminaries

All polynomials in this work are assumed to be real polynomials in one variable and we let $\mathscr{P}$ be the space of all real polynomials. We denote the degree of a polynomial $\pi(x)$ by $\deg(\pi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n)=n$, $n \geqslant 0$. Note that a PS forms a basis of $\mathscr{P}$.

We call any linear functional σ on $\mathscr{P}$ a moment functional and denote its action on a polynomial $\pi(x)$ by $\langle \sigma, \pi \rangle$. For a moment functional σ, we call

$$\sigma_n := \langle \sigma, x^n \rangle, \quad n = 0, 1, \ldots$$

the moments of σ. We say that a moment functional σ is quasi-definite [2] if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j}^{n} \neq 0 \tag{2.1}$$

for every $n \geqslant 0$. Any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a nonzero constant multiple), called a canonical moment functional of $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \quad n \geqslant 1. \tag{2.2}$$

Definition 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial system (OPS) if there is a moment functional σ such that

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn}, \quad m \text{ and } n \geqslant 0, \tag{2.3}$$

where K_n are nonzero real constants and δ_{mn} is the Kronecker delta function. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

It is immediate from (2.3) that for any OPS $\{P_n(x)\}_{n=0}^{\infty}$, its orthogonalizing moment functional σ must be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. It is well known (see [2, Chapter 1]) that a moment functional σ is quasi-definite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ and that each $P_n(x)$ is unique up to a nonzero multiplicative constant.

For a moment functional σ and a polynomial $\pi(x)$, we let σ' (the derivative of σ) and $\pi\sigma$ (the left multiplication of σ by $\pi(x)$) be the moment functionals defined by

$$\langle \sigma', \phi \rangle = -\langle \sigma, \phi' \rangle \tag{2.4}$$

and

$$\langle \pi\sigma, \phi \rangle = \langle \sigma, \pi\phi \rangle \tag{2.5}$$

for every polynomial $\phi(x)$. Then it is easy to obtain the following.

Lemma 2.2 (Kwon et al. [18]). *For a moment functional σ and a polynomial $\pi(x)$, we have*

(i) *Leibniz' rule*: $(\pi\sigma)' = \pi'\sigma + \pi\sigma'$;

(ii) $\sigma' = 0$ *if and only if* $\sigma = 0$.

If σ is quasi-definite, then

(iii) $\pi\sigma = 0$ *if and only if* $\pi(x) = 0$.

3. Symmetrizability

Consider a linear differential operator of order $N \geqslant 1$ of the form

$$L[\cdot] = \sum_{i=0}^{N} a_i(x) \mathrm{D}^i, \tag{3.1}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}x$, $a_i(x)$ are real-valued functions in $\mathscr{C}^i(I)$, $a_N(x) \neq 0$ in I, and I is an open interval on the real line $\mathbb{R}$. The formal adjoint of $L[\cdot]$ is the differential operator $L^*[\cdot]$ defined by

$$L^*[y] := \sum_{i=0}^{N} (-1)^i (a_i(x) y)^{(i)}. \tag{3.2}$$

The differential operator $L[\cdot]$ is called symmetric (or formally self-adjoint) on I if $L[\cdot]=L^*[\cdot]$. It is called symmetrizable on I if there is a real-valued function $s(x)\not\equiv 0$ in $\mathscr{C}^N(I)$ such that $sL[\cdot]$ is symmetric on I. Then we call $s(x)$ a symmetry factor of $L[\cdot]$.

It is well known [3] that any symmetric differential operator must be of even order and the most general symmetric differential operator of order $N=2r$ must be of the form

$$L[y]=\sum_{i=0}^{r}(-1)^i(f_i y^{(i)})^{(i)}. \tag{3.3}$$

Necessary and sufficient conditions for a function $s(x)$ to be a symmetry factor of a linear differential operator were found by Littlejohn [20] and Littlejohn and Race [22].

Proposition 3.1. *Let $L[\cdot]$ be a differential operator as in* (3.1). *Then for any real-valued function $s(x)\not\equiv 0$ in $\mathscr{C}^N(I)$, the following are all equivalent*:

(i) *$s(x)$ is a symmetry factor for $L[\cdot]$, that is, $sL=(sL)^*$*;

(ii) *$s(x)$, $x\in I$, satisfies the $r:=[\frac{N+1}{2}]$ equations*:

$$R_k(s):=\sum_{i=2k+1}^{N}(-1)^i\binom{i-k-1}{k}(a_is)^{(i-2k-1)}=0,\quad k=0,1,\ldots,r-1; \tag{3.4}$$

(iii) *for any real-valued functions $y(x)$ and $z(x)$ in $\mathscr{C}^N(I)$, one of which has compact support in I, we have*

$$\langle sL[y],z\rangle := \int_I z(x)(sL[y])(x)\,\mathrm{d}x=\int_I y(x)(sL[z])(x)\,\mathrm{d}x:=\langle y,sL[z]\rangle. \tag{3.5}$$

Furthermore, if any one of the above equivalent conditions holds, then $N=2r$ must be even.

Proof. See [22, Theorem 5.3; 20, Theorem 4]. We call the r overdetermined system of equations in (3.4) the symmetry equations for $L[\cdot]$. Note that each $R_k[\cdot]$, $0\leqslant k\leqslant r-1$, is a differential operator of order $2r-2k-1$.

In particular, for $k=r-1$, we have

$$R_{r-1}(s)=r(a_{2r}s)'-a_{2r-1}s=0, \tag{3.6}$$

which has only one linearly independent classical solution given by

$$s(x)=\frac{1}{a_{2r}(x)}\exp\left[\frac{1}{r}\int\frac{a_{2r-1}(x)}{a_{2r}(x)}\,\mathrm{d}x\right]. \tag{3.7}$$

Hence, we may restate Proposition 3.1 as: the differential operator $L[\cdot]$ in (3.1) is symmetrizable if and only if $s(x)$ in (3.7) satisfies the remaining $r-1$ symmetry equations $R_k(s)=0$, $0\leqslant k\leqslant r-2$. As a consequence, we have that a symmetry factor $s(x)$ of $L[\cdot]$ is unique up to a nonzero multiplicative constant and $s(x)\neq 0$ in I if it exists.

The fact that $s(x)$ in (3.7) is the only one candidate for a symmetry factor of $L[\cdot]$ motivates a question: Are the r symmetry equations in (3.4) really independent of each other? We show that all $R_k[\cdot]$, $0\leqslant k\leqslant r-2$, are, in fact, derivable from $R_{r-1}[\cdot]$ if the differential operator $L[\cdot]$ is symmetrizable.

Theorem 3.2. *The differential operator $L[\cdot]$ in* (3.1) *with $N=2r$ is symmetrizable on I if and only if*

$$a_{2r}^{2r-2k-2}R_k[\cdot]=\sum_{i=0}^{2r-2k-2} b_{ik}(x)\mathrm{D}^iR_{r-1}[\cdot],\quad k=0,1,\ldots,r-1, \tag{3.8}$$

where $b_{ik}(x)$ are real-valued continuous functions on I.

Proof. Assume that differential operator $L[\cdot]$ is symmetrizable and let $s(x)$ be a symmetry factor of $L[\cdot]$. For $k=r-1$, (3.8) holds with $b_{0,r-1}(x)=1$. For any fixed k with $0\leqslant k\leqslant r-2$, we have

$$\begin{aligned}R_k[\cdot]&=\binom{2r-k-1}{k}a_{2r}(x)\mathrm{D}^{2r-2k-1}+\mathrm{DO}(2r-2k-2)\\&=\frac{1}{r}\binom{2r-k-1}{k}\mathrm{D}^{2r-2k-2}R_{r-1}[\cdot]+\mathrm{DO}(2r-2k-2),\end{aligned} \tag{3.9}$$

where $\mathrm{DO}(k)$ denotes a differential operator of order $\leqslant k$ with continuous functions as coefficients. Multiplying (3.9) by $a_{2r}(x)$ gives

$$\begin{aligned}a_{2r}(x)R_k[\cdot]=&\frac{1}{r}\binom{2r-k-1}{k}a_{2r}(x)\mathrm{D}^{2r-2k-2}R_{r-1}[\cdot]\\&+c(x)\mathrm{D}^{2r-2k-3}R_{r-1}[\cdot]+\mathrm{DO}(2r-2k-3),\end{aligned}$$

where $c(x)$ is a continuous function. Continuing the same process, we obtain

$$a_{2r}^{2r-2k-2}R_k[\cdot]=\sum_{i=-1}^{2r-2k-2} b_{ik}(x)\mathrm{D}^iR_{r-1}[\cdot], \tag{3.10}$$

where $b_{ik}(x)$ are continuous functions and $\mathrm{D}^{-1}R_{r-1}[\cdot]=1$ (identity operator). If we apply (3.10) to $s(x)$, then we obtain $b_{-1,k}(x)s(x)=0$ and so $b_{-1,k}(x)=0$ in I since $R_k(s)=0$, $0\leqslant k\leqslant r-1$. Conversely, we have (3.6) with $s(x)$ in (3.7). Then, (3.8) implies that $R_k(s)=0$ on I, $0\leqslant k\leqslant r-1$. Hence, $s(x)$ is a symmetry factor of $L[\cdot]$ on I. □

Returning now to the differential equation (1.1), we have:

Proposition 3.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS and σ a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then the following are all equivalent.*

(i) *for each $n\geqslant 0$, $P_n(x)$ satisfies the differential equation* (1.1).

(ii) *The moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ satisfy $r:=[\frac{N+1}{2}]$ recurrence relations*:

$$S_k(m):=\sum_{i=2k+1}^{N}\sum_{j=0}^{i}\binom{i-k-1}{k}P(m-2k-1,i-2k-1)\ell_{i,i-j}\sigma_{m-j}=0 \tag{3.11}$$

for $k=0,1,\ldots,r-1$ and $m=2k+1,2k+2,\ldots$, where $P(n,k)=n(n-1)(n-2)\cdots(n-k+1)$.

(iii) σ *satisfies* $r := [\frac{N+1}{2}]$ *functional equations*:

$$R_k(\sigma) := 0, \quad k = 0, 1, \ldots, r-1, \tag{3.12}$$

where $R_k[\cdot]$ *are differential operators in* (3.4) *with* $\ell_i(x)$ *instead of* $a_i(x)$.

(iv) $\sigma L_N[\cdot]$ *is symmetric on polynomials in the sense that*

$$\langle L_N(\phi)\sigma, \psi\rangle = \langle L_N(\psi)\sigma, \phi\rangle, \tag{3.13}$$

for all polynomials $\phi(x)$ *and* $\psi(x)$.

Furthermore, if any one of the above equivalent conditions holds, then $N = 2r$ *must be even.*

Proof. See [18, Theorem 2.4].

The equivalence of the statements (i) and (ii) was first proved by Krall [13]. We call the r functional equations in (3.12) the moment equations for the differential equation (1.1).

Now we are ready to give our main result. In the following, we always assume $N = 2r$ for some integer $r \geqslant 1$.

Theorem 3.4. *If the differential equation* (1.1) *has an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then the differential operator* $L_N[\cdot]$ *is symmetrizable on any open interval not containing any root of* $\ell_N(x)$ *with*

$$s(x) = \frac{1}{\ell_N(x)} \exp\left[\frac{1}{r}\int \frac{\ell_{N-1}(x)}{\ell_N(x)}\mathrm{d}x\right] \tag{3.14}$$

as a symmetry factor.

Proof. Let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then σ is quasi-definite and satisfies (3.12) by Proposition 3.3. As in the proof of Theorem 3.2, we have (3.10) with $\ell_i(x)$ instead of $a_i(x)$, where $b_{ik}(x)$ are polynomials. If we apply (3.10) to σ, then we obtain $b_{-1,k}(x)\sigma = 0$ so that $b_{-1,k}(x) = 0$ by Lemma 2.2. Hence, we also have (3.8). Then with $s(x)$ in (3.14), we have $R_k(s) = 0$, $0 \leqslant k \leqslant r-1$, on any open interval not containing any root of $\ell_N(x)$. That is, $s(x)$ is a symmetry factor of $L_N[\cdot]$ on any such interval. □

Remark 3.5. In Theorem 3.4, the differential operator $L_N[\cdot] - \lambda_n$ is also symmetrizable with $s(x)$ in (3.14) as a symmetry factor since the sum of any two symmetric differential operators is also symmetric.

Next example shows that the converse of Theorem 3.4 does not hold in general even though the differential equation (1.1) has a PS of solutions.

Example 3.6. Consider the following fourth-order differential equation:

$$L_4[y](x) = y^{(4)}(x) + y^{(3)}(x) + (2x - \tfrac{15}{4})y''(x) + xy'(x) = ny(x). \tag{3.15}$$

It is easy to see that the differential equation has a unique monic PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions. The corresponding symmetry equations

$$R_1(s) = 2s'(x) - s(x) = 0,$$

$$R_0(s) = s^{(3)}(x) - s''(x) + ((2x - \tfrac{15}{4})s(x))' - xs(x) = 0$$

have a common solution $s(x) = \exp(x/2)$ so that the differential operator $L_4[\cdot]$ in (3.15) is symmetrizable. However, the corresponding moment equations have only the trivial solution. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ cannot be an OPS.

4. Applications and examples

In the following, we let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS that satisfies the differential equation (1.1). In [17], we have proved the following.

Theorem 4.1. *If $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to a distribution $w(x)$, then*

$$R_k(w) = g_k, \quad k = 0, 1, \ldots, r-1, \tag{4.1}$$

where $g_k(x)$, $k = 0, 1, \ldots, r-1$, are distributions with zero moments, that is,

$$\langle g_k, x^n \rangle = 0, \quad k = 0, 1, \ldots, r-1 \quad \text{and} \quad n = 0, 1, \ldots.$$

Conversely, if a distribution $w(x)$ is such that
(i) *$w(x)$ decays very rapidly as $|x|$ tends to ∞ so that $w(x)$ can act on polynomials;*
(ii) *$\langle w, x^n \rangle \neq 0$ for some $n \geqslant 0$ (i.e., $w(x)$ is nontrivial as a moment functional);*
(iii) *$w(x)$ satisfies (4.1),*
then $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to $w(x)$.

We call the r equation in (4.1) the nonhomogeneous weight equations for the differential equation (1.1). In general, we cannot take $g_k(x) \equiv 0$, $k = 0, 1, \ldots, r-1$, unless the distribution $w(x)$ has a compact support (see Example 4.6 below). However, the proof of Theorem 3.4 implies that the distributions $g_k(x)$, $k = 0, 1, \ldots, r-1$, satisfy the following relations (see (3.8)):

$$\ell_{2r}^{2r-2k-2}(x) g_k(x) = \sum_{i=0}^{2r-2k-2} b_{ik}(x) \mathrm{D}^i g_{r-1}(x), \quad k = 0, 1, \ldots, r-1, \tag{4.2}$$

where $b_{ik}(x)$ are polynomials. Hence, we obtain:

Theorem 4.2. *Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to a distribution $w(x)$. If $R_{r-1}(w) = 0$, then*
(i) *$w(x)$ satisfies r homogeneous weight equations*

$$R_k(w) = 0, \quad k = 0, 1, \ldots, r-1 \tag{4.3}$$

and

(ii) *the restriction of* $w(x)$ *onto* $U := \mathrm{Int}(\mathrm{supp}(w))\backslash\{x \in R \mid \ell_N(x) = 0\}$ *is a symmetry factor of* $L_N[\cdot]$, *where* $\mathrm{Int}(A)$ *denotes the interior of a set* A *in* $\mathbb{R}$.

Conversely, if supp(w) *has finitely many connected components and the statements* (ii) *holds, then* $w(x)$ *satisfies* (4.3).

Proof. Assume $R_{r-1}(w) = 0$. Then, by Theorem 4.1 and (4.2), we have for any fixed $k = 0, 1, \ldots, r-2$,

$$\ell_{2r}^{2r-2k-2}(x) g_k(x) = 0.$$

Hence, $g_k(x) = 0$ on $\mathbb{R}\backslash\{x \in \mathbb{R} \mid \ell_N(x) = 0\}$ so that supp(g_k) must be finite. Then $g_k(x) \equiv 0$ on $\mathbb{R}$ since $g_k(x)$ has zero moments. From $R_{r-1}(w) = r(\ell_N w)' - \ell_{N-1} w = 0$, we have $w(x) = cf(x)$ on any open interval (a, b) in which $\ell_N(x) \neq 0$, where c is a constant and

$$f(x) = \frac{1}{\ell_N(x)} \exp\left[\frac{1}{r} \int \frac{\ell_{N-1}(x)}{\ell_N(x)}\,\mathrm{d}x\right].$$

Moreover, $f(x)(\neq 0)$ is (real-) analytic in (a, b). If (a, b) is contained in U, then $c \neq 0$ and so $R_k(f) = 0$, $0 \leqslant k \leqslant r-1$, on (a, b). Hence, $f(x)$ is a symmetry factor of $L_N[\cdot]$ on (a, b). Conversely, assume that supp(w) has finitely many connected components and the statement (ii) holds. Then $\mathrm{supp}(R_{r-1}(w)) = \mathrm{supp}(g_{r-1})$ must be finite so that $g_{r-1}(x) \equiv 0$ on $\mathbb{R}$. Hence, by the first part of theorem, we have (4.3). □

Remark 4.3. In fact, we can show: if $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to a distribution $w(x)$ satisfying $R_{r-1}(w) = 0$, then supp(w) can have at most three connected components, among which at most one is compact (see [17, Section 4] for details). The conclusion (ii) in the first part of Theorem 4.2 is also proved in [17].

Now, we give examples illustrating above theorems. In 1984, Koornwinder [9] found a new class of OPS $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, called the generalized Jacobi polynomials, which is orthogonal relative to the weight distribution

$$\begin{aligned} w^{\alpha,\beta,M,N}(x) := {} & \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} H(1-x^2)(1-x)^{\alpha}(1+x)^{\beta} \\ & + M\delta(x+1) + N\delta(x-1), \end{aligned} \tag{4.4}$$

where $\alpha, \beta > -1$ and $M, N \geqslant 0$ and $H(x)$ is the Heaviside step function. As a limit case, he also found another OPS $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$, called the generalized Laguerre polynomials, which is orthogonal relative to the weight distribution

$$w^{\alpha,M}(x) := \frac{1}{\Gamma(\alpha+1)} H(x) x^{\alpha} \mathrm{e}^{-x} + M\delta(x), \tag{4.5}$$

where $\alpha > -1$ and $M \geqslant 0$.

For $M = N = 0$, $\{P_n^{\alpha,\beta,0,0}(x)\}_{n=0}^{\infty}$ and $\{L_n^{\alpha,0}(x)\}_{n=0}^{\infty}$ are classical Jacobi and Laguerre polynomials, which satisfy second-order differential equations. For $M > 0$, differential equation of order four, six, and eight satisfied by $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$ for $\alpha=0,1$, and 2 were found by Krall [14], Littlejohn [19], and Littlejohn [21], respectively. Recently, Koekoek and Koekoek [7] proved

Theorem 4.4. *For $M>0$, the generalized Laguerre polynomials $\{L_m^{\alpha,M}(x)\}_{n=0}^{\infty}$ satisfy the differential equation*:

$$M\sum_{i=0}^{\infty}\ell_i(x)y^{(i)}(x)+xy''(x)+(\alpha+1-x)y'(x)+ny(x)=0, \tag{4.6}$$

where

$$\ell_0(x)=\binom{n+\alpha+1}{n-1}$$

and

$$\ell_i(x)=\frac{1}{i!}\sum_{j=1}^{i}(-1)^{i+j+1}\binom{\alpha+1}{j-1}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}x^j,\quad i=1,2,\ldots$$

(here, $(a)_k=a(a+1)\cdots(a+k-1)$). Moreover, if α is a nonnegative integer, then $\ell_i(x)=0$ for $i>2\alpha+4$ so that Eq. (4.6) *reduces to*

$$L_{2\alpha+4}^{M}[y](x)=M\sum_{i=0}^{2\alpha+4}\ell_i(x)y^{(i)}(x)+xy''(x)+(\alpha+1-x)y'(x)+ny(x)=0. \tag{4.7}$$

Now, Theorem 3.4 (see also Remark 3.5) implies that the differential operator $L_{2\alpha+4}^{M}[\cdot]$ in (4.7) is symmetrizable and its symmetry factor $s(x)$ satisfies

$$(\alpha+2)(x^{\alpha+2}s(x))'-((\alpha+1)(2\alpha+4)x^{\alpha+1}-(\alpha+2)x^{\alpha+2})s(x)=0.$$

Hence we have

$$s(x)=x^{\alpha}e^{-x}. \tag{4.8}$$

Since $\mathrm{supp}(w^{\alpha,M})=[0,\infty)$ is connected, Theorem 4.2 implies that $w^{\alpha,M}(x)$ satisfies $\alpha+2$ homogeneous weight equations corresponding to (4.7). Symmetrizability of $L_{2\alpha+4}^{M}[\cdot]$ was first shown in Wellman [23] (see also [6]). In order to show this, he applied Green's formula to the operator $s(x)L_{2\alpha+4}^{M}[\cdot]$.

Krall [14] found a fourth-order differential equation satisfied by $\{P_n^{0,0,M,M}(x)\}_{n=0}^{\infty}$ $(M>0)$, which is called the Legendre-type polynomials in [10]. Recently, Koekoek [8] proved

Theorem 4.5. *For $M>0$, the generalized ultraspherical polynomials $\{P_n^{\alpha,\alpha,M,M}(x)\}_{n=0}^{\infty}$ satisfy the differential equation*:

$$M\sum_{i=0}^{\infty}\ell_i(x)y^{(i)}(x)+(1-x^2)y''(x)-2(\alpha+1)xy'(x)+n(n+2\alpha+1)y(x)=0, \tag{4.9}$$

where

$$\ell_0(x) = 4(2\alpha+3)\binom{n+2\alpha+2}{n-2}$$

$$\ell_i(x) = (2\alpha+3)(1-x^2)\ell_i^*(x), \quad i=1,2,\ldots,$$

where

$$\ell_1^*(x)=0$$

$$\ell_i^*(x)=\frac{2^i}{i!}\sum_{k=0}^{i-2}\binom{\alpha+1}{i-k-2}\binom{i-2\alpha-5}{k}\left(\frac{1-x}{2}\right)^k, \quad i=2,3,\ldots.$$

Moreover, if α is a nonnegative integer, then $\ell_i(x)=0$ for $i>2\alpha+4$ so that Eq. (4.9) *reduces to*

$$L_{2\alpha+4}^{M,M}[y](x)=M\sum_{i=0}^{2\alpha+4}\ell_i(x)y^{(i)}(x)$$
$$+(1-x^2)y''(x)-2(\alpha+1)xy'(x)+n(n+2\alpha+1)y(x)=0. \tag{4.10}$$

Now, Theorem 3.4 implies that the differential operator $L_{2\alpha+4}^{M,M}[\cdot]$ in (4.10) is symmetrizable and its symmetry factor $s(x)$ satisfies

$$((1-x^2)^{\alpha+2}s(x))'+4(\alpha+1)(1-x^2)^{\alpha+1}xs(x)=0.$$

Hence we have

$$s(x)=(1-x^2)^{\alpha}. \tag{4.11}$$

Since $\operatorname{supp}(w^{\alpha,\alpha,M,M})=[-1,1]$ is connected, Theorem 4.2 implies that $w^{\alpha,\alpha,M,M}(x)$ satisfies $\alpha+2$ homogeneous weight equations corresponding to (4.10).

Finally, we give an example showing that we may not take $g_k(x)=0$, $0\leqslant k\leqslant r-1$, in (4.1) and that $w(x)$ may not, in general, be a symmetry factor of $L_N[\cdot]$ on any interval.

Example 4.6. Consider the following second-order differential equation:

$$L_2[y](x)=x^2y''(x)+(2x+2)y'(x)=n(n+1)y(x). \tag{4.12}$$

It is well known that the differential equation (4.12) has on OPS $\{B_n(x)\}_{n=0}^{\infty}$, called the Bessel polynomials [15], of solutions. The corresponding homogeneous weight equation is

$$x^2w'(x)-2w(x)=0. \tag{4.13}$$

Hence, the differential operator $L_2[\cdot]$ in (4.12) is symmetrizable with symmetry factor $s(x)=\exp(-2/x)$. On the other hand, (4.13) has only one linearly independent distributional solution with support in $[0,\infty)$:

$$w_+(x)=\begin{cases}0, & x\leqslant 0,\\ \exp(-2/x), & x>0,\end{cases}$$

which cannot even define a moment functional since $\lim_{x\to\infty} w_+(x) = 1$. However, if we consider the nonhomogeneous weight equation

$$x^2 w'(x) - 2w(x) = g(x), \tag{4.14}$$

where

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp(-x^{1/4}) \sin x^{1/4}, & x > 0, \end{cases}$$

then Eq. (4.14) has

$$w(x) = \begin{cases} 0, & x \leqslant 0, \\ -\exp(-2/x) \int_x^\infty t^{-2} \exp(-t^{1/4} + \frac{2}{t}) \sin t^{1/4}\, dt, & x > 0, \end{cases}$$

as a distributional orthogonalizing weight for $\{B_n(x)\}_{n=0}^\infty$ (see [16] for details). Even though $w(x)$ is analytic for $x \neq 0$, $w(x)$ cannot be a symmetry factor of $L_2[\cdot]$ on any interval.

Acknowledgements

This work is partially supported by GARC at Seoul National University and KOSEF(95-0701-02-01-3).

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