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A functional equation arising from multiplication of quantum integers ☆

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Abstract

For the quantum integer $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ there is a natural polynomial multiplication such that $[m]_q \otimes_q [n]_q = [mn]_q$. This multiplication leads to the functional equation $f_m(q) f_n(q^m) = f_{mn}(q)$, defined on a given sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of polynomials. This paper contains various results concerning the construction and classification of polynomial sequences that satisfy the functional equation, as well open problems that arise from the functional equation.

Keywords: Quantum integers; Quantum polynomial; Polynomial functional equation; q-series; Additive bases

1. A polynomial functional equation

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ the set of nonnegative integers. For $n \in \mathbf{N}$, the polynomial

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$

is called the *quantum integer n*. With the usual multiplication of polynomials, we observe that $[m]_q [n]_q \neq [mn]_q$ for all $m \neq 1$ and $n \neq 1$. We would like to define a

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polynomial multiplication such that the product of the quantum integers $[m]_q$ and $[n]_q$ is $[mn]_q$.

Consider polynomials with coefficients in a field. Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials. We define a multiplication operation on the polynomials in $\mathcal{F}$ by

$$f_m(q) \otimes_q f_n(q) = f_m(q) f_n(q^m).$$

We want to determine all sequences $\mathcal{F}$ that satisfy the *functional equation*

$$f_{mn}(q) = f_m(q) \otimes_q f_n(q) = f_m(q) f_n(q^m) \tag{1}$$

for all $m, n \in \mathbf{N}$. If the sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ is a solution of (1), then the operation $\otimes_q$ is commutative on $\mathcal{F}$ since

$$f_m(q) \otimes_q f_n(q) = f_{mn}(q) = f_{nm}(q) = f_n(q) \otimes_q f_m(q).$$

Equivalently,

$$f_m(q) f_n(q^m) = f_n(q) f_m(q^n) \tag{2}$$

for all natural numbers m and n.[1]

Here are three examples of solutions of the functional equation (1). First, the constant sequence defined by $f_n(q) = 1$ for all $n \in \mathbf{N}$ satisfies (1).

Second, let

$$f_n(q) = q^{n-1}$$

for all $n \in \mathbf{N}$. Then

$$f_{mn}(q) = q^{mn-1} = q^{m-1} q^{m(n-1)} = f_m(q) f_n(q^m)$$

and so the polynomial sequence $\{q^{n-1}\}_{n=1}^{\infty}$ also satisfies (1).

Third, let $f_n(q) = [n]_q$ for all $n \in \mathbf{N}$. Then

$$\begin{aligned}
[m]_q \otimes_q [n]_q &= f_m(q) \otimes_q f_n(q) \\
&= f_m(q) f_n(q^m) \\
&= (1 + q + q^2 + \cdots + q^{m-1})(1 + q^m + q^{2m} + \cdots + q^{m(n-1)}) \\
&= 1 + q + \cdots + q^{m-1} + q^m + q^{m+1} \cdots + q^{mn-1} \\
&= [mn]_q
\end{aligned}$$

[1] Note that (1) implies (2), but not conversely, since the sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ with $f_n(q) = 2$ for all $n \in \mathbf{N}$ satisfies (2) but not (1).

and so the polynomial sequence $\{[n]_q\}_{n=1}^{\infty}$ of quantum integers satisfies the functional equation (1).

The identity

$$[m]_q \otimes_q [n]_q = [mn]_q$$

is the q-series expression of the following additive number theoretic identity for sumsets

$$\{0, 1, 2, \ldots, mn-1\} = \{0, 1, \ldots, m-1\} + \{0, m, 2m, \ldots, (n-1)m\}.$$

This paper investigates the following problem.

Problem 1. *Determine all polynomial sequences* $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ *that satisfy the functional equation* (1).

2. Prime semigroups

A *multiplicative subsemigroup* of the natural numbers, or, simply, a *semigroup*, is a set $S \subseteq \mathbf{N}$ such that $1 \in S$ and if $m \in S$ and $n \in S$, then $mn \in S$. For example, for any positive integer n_0, the set $\{1\} \cup \{n \geqslant n_0\}$ is a semigroup. If P is a set of prime numbers, then the set $S(P)$ consisting of the positive integers all of whose prime factors belong to P is a multiplicative subsemigroup of $\mathbf{N}$. If $P = \emptyset$, then $S(P) = \{1\}$. If $P = \{p\}$ contains only one prime, then $S(P) = \{p^k \colon k \in \mathbf{N}_0\}$. A semigroup of the form $S(P)$, where P is a set of primes, will be called a *prime semigroup*.

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials that satisfies the functional equation (1). Since

$$f_1(q) = f_1(q) \otimes_q f_1(q) = f_1(q) f_1(q),$$

it follows that $f_1(q) = 1$ or $f_1(q) = 0$. If $f_1(q) = 0$, then

$$f_n(q) = f_1(q) \otimes_q f_n(q) = f_1(q) f_n(q) = 0$$

for all $n \in \mathbf{N}$, and $\mathcal{F}$ is the sequence of zero polynomials. If $f_n(q) \neq 0$ for some n, then $f_1(q) = 1$.

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be any sequence of functions. The *support* of $\mathcal{F}$ is the set

$$\mathrm{supp}(\mathcal{F}) = \{n \in \mathbf{N} : f_n(q) \neq 0\}.$$

The sequence $\mathcal{F}$ is called *nonzero* if $f_n(q) \neq 0$ for some $n \in \mathbf{N}$, or, equivalently, if $\mathrm{supp}(\mathcal{F}) \neq \emptyset$. If $\mathcal{F}$ satisfies the functional equation (1), then $\mathcal{F}$ is nonzero if and only if $f_1(q) = 1$.

For every positive integer n, let $\Omega(n)$ denote the number of not necessarily distinct prime factors of n. If $n = p_1^{r_1} \cdots p_k^{r_k}$, then $\Omega(n) = r_1 + \cdots + r_k$.

Theorem 1. *Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nonzero sequence of polynomials that satisfies the functional equation* (1). *The support of $\mathscr{F}$ is a prime semigroup. If*

$$\mathrm{supp}(\mathscr{F}) = S(P),$$

where P is a set of prime numbers, then the sequence $\mathscr{F}$ is completely determined by the set of polynomials $\mathscr{F}_P = \{f_p(q)\}_{p \in P}$.

Proof. Since $\mathscr{F}$ is nonzero, we have $f_1(q) = 1$ and so $1 \in \mathrm{supp}(\mathscr{F})$. If $m \in \mathrm{supp}(\mathscr{F})$ and $n \in \mathrm{supp}(\mathscr{F})$, then $f_m(q) \neq 0$ and $f_n(q) \neq 0$, hence

$$f_{mn}(q) = f_m(q) f_n(q^m) \neq 0$$

and $mn \in \mathrm{supp}(\mathscr{F})$. Therefore, $\mathrm{supp}(\mathscr{F})$ is a semigroup.

Let P be the set of prime numbers contained in $\mathrm{supp}(\mathscr{F})$. Then $S(P) \subseteq \mathrm{supp}(\mathscr{F})$. If $n \in \mathrm{supp}(\mathscr{F})$ and the prime number p divides n, then $n = pm$ for some positive integer m. Since $\mathscr{F}$ satisfies the functional equation (1), we have

$$f_n(q) = f_{pm}(q) = f_p(q) f_m(q^p) \neq 0,$$

and so $f_p(q) \neq 0$, hence $p \in \mathrm{supp}(\mathscr{F})$ and $p \in P$. Since every prime divisor of n belongs to $\mathrm{supp}(\mathscr{F})$, it follows that $n \in S(P)$, and so $\mathrm{supp}(\mathscr{F}) \subseteq S(P)$. Therefore, $\mathrm{supp}(\mathscr{F}) = S(P)$ is a prime semigroup.

We use induction on $\Omega(n)$ for $n \in \mathrm{supp}(\mathscr{F})$ to show that the sequence $\mathscr{F}_P = \{f_p(q)\}_{p \in P}$ determines $\mathscr{F}$. If $\Omega(n) = 1$, then $n = p \in P$ and $f_p(q) \in \mathscr{F}_P$. Suppose that $\mathscr{F}_P$ determines $f_m(q)$ for all $m \in \mathrm{supp}(\mathscr{F})$ with $\Omega(m) \leqslant k$. If $n \in \mathrm{supp}(\mathscr{F})$ and $\Omega(n) = k+1$, then $n = pm$, where $p \in P$, $m \in \mathrm{supp}(\mathscr{F})$, and $\Omega(m) = k$. It follows that the polynomial $f_n(q) = f_p(q) f_m(q^p)$ is determined by $\mathscr{F}_P$. $\square$

Let P be a set of prime numbers, and let $S(P)$ be the semigroup generated by P. Define the sequence $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ by

$$f_n(q) = \begin{cases} [n]_q & \text{if } n \in S(P), \\ 0 & \text{if } n \notin S(P). \end{cases}$$

Then $\mathscr{F}$ satisfies (1) and $\mathrm{supp}(\mathscr{F}) = S(P)$. Thus, every semigroup of the form $S(P)$ is the support of some sequence of polynomials satisfying the functional equation (1).

The following theorem provides a general method to construct solutions of the functional equation (1) with support $S(P)$ for any set P of prime numbers.

Theorem 2. *Let P be a set of prime numbers. For each $p \in P$, let $h_p(q)$ be a nonzero polynomial such that*

$$h_{p_1}(q) h_{p_2}(q^{p_1}) = h_{p_2}(q) h_{p_1}(q^{p_2}) \quad \textit{for all } p_1, p_2 \in P. \tag{3}$$

Then there exists a unique sequence $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ *with* $\mathrm{supp}(\mathscr{F}) = S(P)$ *such that* $\mathscr{F}$ *satisfies the functional equation* (1) *and* $f_p(q) = h_p(q)$ *for all* $p \in P$.

The proof uses three lemmas.

Lemma 1. *Let* p *be a prime number and* $h_p(q)$ *a nonzero polynomial. There exists a unique sequence of polynomials* $\{f_{p^k}(q)\}_{k=0}^{\infty}$ *such that* $f_p(q) = h_p(q)$ *and*

$$f_{p^k}(q) = f_{p^i}(q) f_{p^j}(q^{p^i}) \tag{4}$$

for all nonnegative integers i, j *and* k *such that* $i + j = k$.

Proof. We define $f_1(q) = 1$, $f_p(q) = h_p(q)$, and, by induction on k,

$$f_{p^k}(q) = f_p(q) f_{p^{k-1}}(q^p) \tag{5}$$

for $k \geqslant 2$. The proof of (4) is by induction on k. Identity (4) holds for $k = 0, 1$, and 2, and also for $i = 0$ and all j. Assume that (4) is true for some $k \geqslant 1$, and let $k + 1 = i + j$, where $i \geqslant 1$. From the construction of the sequence $\{f_{p^k}(q)\}_{k=0}^{\infty}$ and the induction hypothesis we have

$$\begin{aligned} f_{p^{k+1}}(q) &= f_p(q) f_{p^k}(q^p) \\ &= f_p(q) f_{p^{(i-1)+j}}(q^p) \\ &= f_p(q) f_{p^{i-1}}(q^p) f_{p^j}((q^p)^{p^{i-1}}) \\ &= f_{p^i}(q) f_{p^j}(q^{p^i}). \end{aligned}$$

Conversely, if the sequence $\{f_{p^k}(q)\}_{k=0}^{\infty}$ satisfies (4), then, setting $i = 1$, we obtain (5), and so the sequence $\{f_{p^k}(q)\}_{k=0}^{\infty}$ is unique. □

Lemma 2. *Let* $P = \{p_1, p_2\}$, *where* p_1 *and* p_2 *are distinct prime numbers, and let* $S(P)$ *be the semigroup generated by* P. *Let* $h_{p_1}(q)$ *and* $h_{p_2}(q)$ *be nonzero polynomials such that*

$$h_{p_1}(q) h_{p_2}(q^{p_1}) = h_{p_2}(q) h_{p_1}(q^{p_2}). \tag{6}$$

There exists a unique sequence of polynomials $\{f_n(q)\}_{n \in S(P)}$ *such that* $f_{p_1}(q) = h_{p_1}(q)$, $f_{p_2}(q) = h_{p_2}(q)$, *and*

$$f_{mn}(q) = f_m(q) f_n(q^m) \quad \text{for all } m, n \in S(P). \tag{7}$$

Proof. Every integer $n \in S(P)$ can be written uniquely in the form $n = p_1^i p_2^j$ for some nonnegative integers i and j. We apply Lemma 1 to construct the sets of polynomials

$\{f_{p_1^i}(q)\}_{i=0}^{\infty}$ and $\{f_{p_2^j}(q)\}_{j=0}^{\infty}$. If $n = p_1^i p_2^j$ for positive integers i and j, then we define

$$f_n(q) = f_{p_1^i}(q) f_{p_2^j}(q^{p^i}). \tag{8}$$

This determines the set $\{f_n(q)\}_{n \in S(P)}$.

We shall show that

$$f_{p_1^i}(q) f_{p_2^j}(q^{p_1^i}) = f_{p_2^j}(q) f_{p_1^i}(q^{p_2^j}) \tag{9}$$

for all nonnegative integers i and j. This is true if $i = 0$ or $j = 0$, so we can assume that $i \geqslant 1$ and $j \geqslant 1$.

The proof is by induction on $k = i + j$. If $k = 2$, then $i = j = 1$ and the result follows from (6). Let $k \geqslant 2$, and assume that Eq. (9) is true for all positive integers i and j such that $i + j \leqslant k$. Let $i + j + 1 = k + 1$. By Lemma 1 and the induction assumption,

$$\begin{aligned}
f_{p_1^i}(q) f_{p_2^{j+1}}(q^{p_1^i}) &= f_{p_1^i}(q) f_{p_2^j}(q^{p_1^i}) f_{p_2}(q^{p_1^i p_2^j}) \\
&= f_{p_2^j}(q) f_{p_1^i}(q^{p_2^j}) f_{p_2}(q^{p_2^j p_1^i}) \\
&= f_{p_2^j}(q) f_{p_2}(q^{p_2^j}) f_{p_1^i}(q^{p_2^{j+1}}) \\
&= f_{p_2^{j+1}}(q) f_{p_1^i}(q^{p_2^{j+1}}).
\end{aligned}$$

Similarly,

$$f_{p_1^{i+1}}(q) f_{p_2^j}(q^{p_1^{i+1}}) = f_{p_2^j}(q) f_{p_1^{i+1}}(q^{p_2^j}).$$

This proves (9).

Let $m, n \in S(P)$. There exist nonnegative integers i, j, k, and ℓ such that

$$m = p_1^i p_2^j \quad \text{and} \quad n = p_1^k p_2^\ell.$$

Then

$$\begin{aligned}
f_m(q) f_n(q^m) &= f_{p_1^i}(q) f_{p_2^j}(q^{p_1^i}) f_{p_1^k}(q^{p_1^i p_2^j}) f_{p_2^\ell}(q^{p_1^{i+k} p_2^j}) \\
&= f_{p_1^i}(q) f_{p_1^k}(q^{p_1^i}) f_{p_2^j}(q^{p_1^{i+k}}) f_{p_2^\ell}(q^{p_1^{i+k} p_2^j}) \\
&= f_{p_1^k}(q) f_{p_1^i}(q^{p_1^k}) f_{p_2^\ell}(q^{p_1^{i+k}}) f_{p_2^j}(q^{p_1^{i+k} p_2^\ell}) \\
&= f_{p_1^k}(q) f_{p_2^\ell}(q^{p_1^k}) f_{p_1^i}(q^{p_1^k p_2^\ell}) f_{p_2^j}(q^{p_1^{i+k} p_2^\ell}) \\
&= f_n(q) f_m(q^n).
\end{aligned}$$

Setting $m = p_1^i$ and $n = p_2^j$ in (7) gives (8), and so the sequence of polynomials $\{f_n(q)\}_{n\in S(P)}$ is unique. □

Lemma 3. *Let $P = \{p_1, \dots, p_r\}$ be a set consisting of r prime numbers, and let $S(P)$ be the semigroup generated by P. Let $h_{p_1}(q), \dots, h_{p_r}(q)$ be nonzero polynomials such that*

$$h_{p_i}(q)h_{p_j}(q^{p_i}) = h_{p_j}(q)h_{p_i}(q^{p_j}) \tag{10}$$

for $i,j = 1, \dots, r$. There exists a unique sequence of polynomials $\{f_n(q)\}_{n\in S(P)}$ such that $f_{p_i}(q) = h_{p_i}(q)$ for $i = 1, \dots, r$, and

$$f_{mn}(q) = f_m(q)f_n(q^m) \quad \textit{for all } m,n \in S(P). \tag{11}$$

Proof. The proof is by induction on r. The result holds for $r = 1$ by Lemma 1 and for $r = 2$ by Lemma 2. Let $r \geqslant 3$, and assume that the Lemma holds for every set of $r-1$ primes. Let $P' = P\backslash\{p_r\} = \{p_1, \dots, p_{r-1}\}$. By the induction hypothesis, there exists a unique sequence of polynomials $\{f_n(q)\}_{n\in S(P')}$ such that $f_{p_i}(q) = h_{p_i}(q)$ for $i = 1, \dots, r-1$, and

$$f_{m'n'}(q) = f_{m'}(q)f_{n'}(q^{m'}) \quad \textit{for all } m', n' \in S(P').$$

Every $n \in S(P)\backslash S(P')$ can be written uniquely in the form $n = n'p_r^{a_r}$, where $n' \in S(P')$ and a_r is a positive integer. We define $f_{p_r^{a_r}}(q)$ by Lemma 1 and

$$f_{n'p_r^{a_r}}(q) = f_{n'}(q)f_{p_r^{a_r}}(q^{n'}). \tag{12}$$

We begin by proving that

$$f_{n'}(q)f_{p_r^{a_r}}(q^{n'}) = f_{p_r^{a_r}}(q)f_{n'}(q^{p_r^{a_r}}) \tag{13}$$

for all $n' \in S(P')$ and $a_r \in \mathbf{N}$.

By Lemma 2, Eq. (13) is true if $n' = p_s^{a_s}$ for some prime $p_s \in P'$. Let $n' = n''p_s^{a_s}$, where $n'' \in S(P\backslash\{p_s, p_r\})$. By the induction assumption,

$$f_{n''}(q)f_{p_r^{a_r}}(q^{n''}) = f_{p_r^{a_r}}(q)f_{n''}(q^{p_r^{a_r}})$$

and so

$$\begin{aligned} f_{n'}(q)f_{p_r^{a_r}}(q^{n'}) &= f_{n''}(q)f_{p_s^{a_s}}(q^{n''})f_{p_r^{a_r}}(q^{n''p_s^{a_s}}) \\ &= f_{n''}(q)f_{p_r^{a_r}}(q^{n''})f_{p_s^{a_s}}(q^{n''p_r^{a_r}}) \\ &= f_{p_r^{a_r}}(q)f_{n''}(q^{p_r^{a_r}})f_{p_s^{a_s}}(q^{n''p_r^{a_r}}) \\ &= f_{p_r^{a_r}}(q)f_{n'}(q^{p_r^{a_r}}). \end{aligned}$$

This proves (13).

Let $m, n \in S(P)$. We write $n = n' p_r^{a_r}$ and $m = m' p_r^{b_r}$, where $m', n' \in S(P')$ and a_r, b_r are nonnegative integers. Applying (13) and the induction assumption, we obtain

$$\begin{aligned} f_m(q) f_n(q^m) &= f_{m'}(q) f_{p_r^{b_r}}(q^{m'}) f_{n'}(q^{m' p_r^{b_r}}) f_{p_r^{a_r}}(q^{m' n' p_r^{b_r}}) \\ &= f_{m'}(q) f_{n'}(q^{m'}) f_{p_r^{b_r}}(q^{m' n'}) f_{p_r^{a_r}}(q^{m' n' p_r^{b_r}}) \\ &= f_{n'}(q) f_{m'}(q^{n'}) f_{p_r^{a_r}}(q^{m' n'}) f_{p_r^{b_r}}(q^{m' n' p_r^{a_r}}) \\ &= f_{n'}(q) f_{p_r^{a_r}}(q^{n'}) f_{m'}(q^{n' p_r^{a_r}}) f_{p_r^{b_r}}(q^{m' n' p_r^{a_r}}) \\ &= f_n(q) f_m(q^n). \end{aligned}$$

This proves (11).

Applying (11) with $m = n'$ and $n = p_r^{a_r}$, we obtain (12). This shows that the sequence $\{f_n(q)\}_{n \in S(P)}$ is unique, and completes the proof of the lemma. □

We can now prove Theorem 2.

Proof of Theorem 2. If P is a finite set of prime numbers, then we construct the set of polynomials $\{f_n(q)\}_{n \in S(P)}$ by Lemma 3, and we define $f_n(q) = 0$ for $n \notin S(P)$. This determines the sequence $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ uniquely.

If P is infinite, we write $P = \{p_i\}_{i=1}^{\infty}$. For every positive integer r, let $P_r = \{p_i\}_{i=1}^{r}$ and apply Lemma 3 to construct the set of polynomials $\{f_n(q)\}_{n \in S(P_r)}$. Since

$$P_1 \subseteq \cdots \subseteq P_r \subseteq P_{r+1} \subseteq \cdots \subseteq P$$

and

$$S(P_1) \subseteq \cdots \subseteq S(P_r) \subseteq S(P_{r+1}) \subseteq \cdots \subseteq S(P),$$

we have

$$\{f_n(q)\}_{n \in S(P_1)} \subseteq \cdots \subseteq \{f_n(q)\}_{n \in S(P_r)} \subseteq \{f_n(q)\}_{n \in S(P_{r+1})} \subseteq \cdots .$$

Define

$$\{f_n(q)\}_{n \in S(P)} = \bigcup_{r=1}^{\infty} \{f_n(q)\}_{n \in S(P_r)}.$$

Setting $f_n(q) = 0$ for all $n \notin S(P)$ uniquely determines a sequence $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ that satisfies the functional equation (1) and $f_p(q) = h_p(q)$ for all $p \in P$. This completes the proof. □

For example, for the set $P = \{2, 5, 7\}$, the reciprocal polynomials

$$h_2(q) = 1 - q + q^2,$$

$$h_5(q) = 1 - q + q^3 - q^4 + q^5 - q^7 + q^8,$$

$$h_7(q) = 1 - q + q^3 - q^4 + q^6 - q^8 + q^9 - q^{11} + q^{12}$$

satisfy the commutativity condition (3). There is a unique sequence of polynomials $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ constructed from $\{h_2(q), h_5(q), h_7(q)\}$ by Theorem 2. Since

$$f_p(q) = h_p(q) = \frac{[p]_{q^3}}{[p]_q} \qquad \text{for } p \in P = \{2, 5, 7\},$$

it follows that

$$f_n(q) = \frac{[n]_{q^3}}{[n]_q} \qquad \text{for all } n \in S(P).$$

We have $\deg(f_n) = 2(n-1)$ for all $n \in S(P)$.
We can refine Problem 1 as follows.

Problem 2. *Let P be a set of prime numbers. Determine all polynomial sequences $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ with support $S(P)$ that satisfy the functional equation* (1).

Problem 3. *Let P and P' be sets of prime numbers with $P \subseteq P'$, and let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials with support $S(P)$ that satisfies the functional equation* (1). *Under what conditions does there exist a sequence $\mathcal{F}' = \{f'_n(q)\}_{n=1}^{\infty}$ with support $S(P')$ such that $\mathcal{F}'$ satisfies* (1) *and $f'_p(q) = f_p(q)$ for all primes $p \in P$?*

Problem 4. *Let S be a multiplicative subsemigroup of the positive integers. Determine all sequences $\{f_n(q)\}_{n \in S}$ of polynomials such that*

$$f_{mn}(q) = f_m(q) f_n(q^m) \quad \textit{for all } m, n \in S.$$

This formulation of the problem of classifying solutions of the functional equation does not assume that S is a semigroup of the form $S = S(P)$ for some set P of prime numbers.

3. An arithmetic functional equation

An *arithmetic function* is a function whose domain is the set $\mathbf{N}$ of natural numbers. The *support* of the arithmetic function δ is

$$\mathrm{supp}(\delta) = \{n \in \mathbf{N} : \delta(n) \neq 0\}.$$

Lemma 4. *Let S be a semigroup of the natural numbers, and $\delta(n)$ a complex-valued arithmetic function that satisfies the functional equation*

$$\delta(mn) = \delta(m) + m\delta(n) \quad \textit{for all } m, n \in S. \tag{14}$$

Then there exists a complex number t such that

$$\delta(n) = t(n-1) \quad \textit{for all } n \in S.$$

Proof. Let $\delta(n)$ be a solution of the functional equation (14) on S. Setting $m = n = 1$ in (14), we obtain $\delta(1) = 0$. For all $m, n \in S \backslash \{1\}$ we have

$$\delta(m) + m\delta(n) = \delta(mn) = \delta(nm) = \delta(n) + n\delta(m)$$

and so

$$\frac{\delta(m)}{m-1} = \frac{\delta(n)}{n-1}.$$

It follows that there exists a number t such that $\delta(n) = t(n-1)$ for all $n \in S$. This completes the proof. □

Note that if $\delta(n) = 0$ for some $n \in S \backslash \{1\}$, then $\delta(n) = 0$ for all $n \in S$.
Let $\deg(f)$ denote the degree of the polynomial $f(q)$.

Lemma 5. *Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nonzero sequence of polynomials that satisfies the functional equation* (1). *There exists a nonnegative rational number t such that*

$$\deg(f_n) = t(n-1) \quad \textit{for all } n \in \mathrm{supp}(\mathscr{F}). \tag{15}$$

Proof. Let $S = \mathrm{supp}(\mathscr{F})$. The functional equation (1) implies that

$$\deg(f_{mn}) = \deg(f_m) + m\deg(f_n) \quad \text{for all } m, n \in S,$$

and so $\deg(f_n)$ is an arithmetic function on the semigroup S that satisfies the arithmetic functional equation (14). Statement (15) follows immediately from Lemma 4. □

We note that, in Lemma 5, the number t is rational but not necessarily integral. For example, if $\mathrm{supp}(\mathscr{F}) = \{7^k\colon k \in \mathbf{N}_0\}$ and

$$f_{7^k}(q) = q^{2(1+7+7^2+\cdots+7^{k-1})} = q^{(7^k-1)/3},$$

then $t_1 = \frac{1}{3}$.

An arithmetic function $\lambda(n)$ is *completely multiplicative* if $\lambda(mn) = \lambda(m)\lambda(n)$ for all $m, n \in \mathbf{N}$. A function $\lambda(n)$ is *completely multiplicative on a semigroup S* if $\lambda(n)$ is a function defined on S and $\lambda(mn) = \lambda(m)\lambda(n)$ for all $m, n \in S$.

Theorem 3. *Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nonzero sequence of polynomials that satisfies the functional equation*

$$f_{mn}(q) = f_m(q) f_n(q^m).$$

Then there exist a completely multiplicative arithmetic function $\lambda(n)$, a nonnegative rational number t, and a nonzero sequence $\mathscr{G} = \{g_n(q)\}_{n=1}^{\infty}$ of polynomials such that

$$f_n(q) = \lambda(n) q^{t(n-1)} g_n(q) \quad \textit{for all } n \in \mathbf{N},$$

where

(i) *the sequence $\mathscr{G}$ satisfies the functional equation* (1),

(ii)
$$\mathrm{supp}(\mathscr{F}) = \mathrm{supp}(\mathscr{G}) = \mathrm{supp}(\lambda),$$

(iii)
$$g_n(0) = 1 \quad \textit{for all } n \in \mathrm{supp}(\mathscr{G}).$$

The number t, the arithmetic function $\lambda(n)$, and the sequence $\mathscr{G}$ are unique.

Proof. For every $n \in \mathrm{supp}(\mathscr{F})$ there exist a unique nonnegative integer $\delta(n)$ and polynomial $g'_n(q)$ such that $g'_n(0) \neq 0$ and

$$f_n(q) = q^{\delta(n)} g'_n(q).$$

Let $\lambda(n) = g'_n(0)$ be the constant term of $g'_n(q)$. Dividing $g'_n(q)$ by $\lambda(n)$, we can write

$$g'_n(q) = \lambda(n) g_n(q),$$

where $g_n(q)$ is a polynomial with constant term $g_n(0) = 1$. Define $g_n(q) = 0$ and $\lambda(n) = 0$ for every positive integer $n \notin \mathrm{supp}(\mathscr{F})$, and let $\mathscr{G} = \{g_n(q)\}_{n=1}^{\infty}$. Then $\mathrm{supp}(\mathscr{F}) = \mathrm{supp}(\mathscr{G}) = \mathrm{supp}(\lambda)$. Since the sequence

$$\{\lambda(n) q^{\delta(n)} g_n(q)\}_{n=1}^{\infty}$$

satisfies the functional equation, we have, for all $m, n \in \mathrm{supp}(\mathscr{F})$,

$$\lambda(mn)q^{\delta(mn)}g_{mn}(q) = \lambda(m)q^{\delta(m)}g_m(q)\lambda(n)q^{m\delta(n)}g_n(q^m)$$
$$= \lambda(m)\lambda(n)q^{\delta(m)+m\delta(n)}g_m(q)g_n(q^m).$$

The polynomials $g_m(q), g_n(q)$, and $g_{mn}(q)$ have constant term 1, hence for all $m, n \in \mathrm{supp}(\mathscr{F})$ we have

$$q^{\delta(mn)} = q^{\delta(m)+m\delta(n)},$$

$$\lambda(mn) = \lambda(m)\lambda(n)$$

and

$$g_{mn}(q) = g_m(q)g_n(q^m).$$

It follows that $\lambda(n)$ is a completely multiplicative arithmetic function with $\mathrm{supp}(\mathscr{F})$, and the sequence $\{g_n(q)\}_{n=1}^{\infty}$ also satisfies the functional equation (1). Moreover,

$$\delta(mn) = \delta(m) + m\delta(n) \quad \text{for all } m, n \in \mathrm{supp}(\mathscr{F}).$$

By Lemma 4, there exists a nonnegative rational number t such that $\delta(n) = t(n-1)$. This completes the proof. □

4. Classification problems

Theorem 3 reduces the classification of solutions of the functional equation (1) to the classification of sequences of polynomials $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ with constant term $f_n(0) = 1$ for all $n \in \mathrm{supp}(\mathscr{F})$.

Theorem 4. *Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nonzero sequence of polynomials that satisfies the functional equation* (1).

(i) *Let $\psi(q)$ be a polynomial such that $\psi(q)^m = \psi(q^m)$ for every integer $m \in \mathrm{supp}(\mathscr{F})$. Then the sequence $\{f_n(\psi(q))\}_{n=1}^{\infty}$ satisfies* (1).
(ii) *For every positive integer t, the sequence $\{f_n(q^t)\}_{n=1}^{\infty}$ satisfies* (1).
(iii) *The sequence of reciprocal polynomials $\{q^{\deg(f_n)}f_n(q^{-1})\}_{n=1}^{\infty}$ satisfies* (1).

Proof. Suppose that $\psi(q)^m = \psi(q^m)$ for every integer $m \in \mathrm{supp}(\mathscr{F})$. Replacing q by $\psi(q)$ in the polynomial identity (1), we obtain

$$f_{mn}(\psi(q)) = f_m(\psi(q))f_n(\psi(q)^m) = f_m(\psi(q))f_n(\psi(q^m))$$

for all $m, n \in \mathrm{supp}(\mathscr{F})$. This proves (i).

Since $(q^t)^m = (q^m)^t$ for all integers t, we obtain (ii) from (i) by choosing $\psi(q) = q^t$.

The reciprocal polynomial of $f(q)$ is

$$\tilde{f}(q) = q^{\deg(f)} f(q^{-1}).$$

Then

$$\begin{aligned}\tilde{f}_{mn}(q) &= q^{\deg(f_{mn})} f_{mn}(q^{-1}) \\ &= q^{\deg(f_m)+m\deg(f_n)} f_m(q^{-1}) f_n(q^{-m}) \\ &= q^{\deg(f_m)} f_m(q^{-1}) q^{m\deg(f_n)} f_n((q^m)^{-1}) \\ &= \tilde{f}_m(q)\tilde{f}_n(q^m).\end{aligned}$$

This proves (iii). □

For example, setting

$$[n]_{q^t} = 1 + q^t + q^{2t} + \cdots + q^{(n-1)t},$$

we see that $\{[n]_{q^t}\}_{n=1}^{\infty}$ is a solution of (1) with support $\mathbf{N}$.

The quantum integer $[n]_q$ is a self-reciprocal polynomial of q, and $[n]_{q^t}$ is self-reciprocal for all positive integers t. The reciprocal polynomial of the polynomial q^{n-1} is 1.

The polynomials $\psi(q) = q^t$ are not the only polynomials that generate solutions of the functional equation (1). For example, let p be a prime number, and consider polynomials with coefficients in the finite field $\mathbf{Z}/p\mathbf{Z}$ and solutions $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of the functional equation with $\mathrm{supp}(\mathcal{F}) = S(\{p\}) = \{p^k\colon k \in \mathbf{N}_0\}$. Applying the Frobenius automorphism $z \mapsto z^p$, we see that $\psi(q)^m = \psi(q^m)$ for every polynomial $\psi(q)$ and every $m \in \mathrm{supp}(\mathcal{F})$.

Here is another example of solutions of (1) generated by polynomials satisfying $\psi(q)^m = \psi(q^m)$ for $m \in \mathrm{supp}(\mathcal{F})$.

Theorem 5. *Let P be a nonempty set of prime numbers, and $S(P)$ the multiplicative semigroup generated by P. Let d be the greatest common divisor of the set $\{p-1\colon p \in P\}$. For $\zeta \neq 0$, let*

$$f_n(q) = \sum_{i=0}^{n-1} \zeta^i q^i = [n]_{\zeta q} \quad \text{for } n \in S(P),$$

and let $f_n(q) = 0$ for $n \notin S(P)$. If ζ is a dth root of unity, then the sequence of polynomials

$$\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$$

satisfies the functional equation (1). *Conversely, if* $\mathcal{F}$ *satisfies* (1), *then* ζ *is a dth root of unity.*

Proof. Let ζ be a dth root of unity, and $\psi(q) = \zeta q$. Since $p \equiv 1 \pmod{d}$ for all $p \in P$, it follows that $m \equiv 1 \pmod{d}$ for all $m \in S(P)$. Therefore, if $m \in S(P)$, then

$$\psi(q)^m = (\zeta q)^m = \zeta^m q^m = \zeta q^m = \psi(q^m).$$

It follows from Theorem 4 that the sequence of polynomials $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$, where

$$f_n(q) = [n]_{\zeta q} = \sum_{i=0}^{n-1} \zeta^i q^i \quad \text{for } n \in S(P)$$

and $f_n(q) = 0$ for $n \notin S(P)$, satisfies the functional equation (1).

Conversely, suppose that $\mathcal{F}$ satisfies (1). Let $m, n \in S(P) \backslash \{1\}$. Since

$$f_m(q) f_n(q^m) = \left(\sum_{i=0}^{m-1} \zeta^i q^i \right) \left(\sum_{j=0}^{n-1} \zeta^j q^{mj} \right) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \zeta^{i+j} q^{i+mj},$$

$$f_{mn}(q) = \sum_{k=0}^{mn-1} \zeta^k q^k = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \zeta^{i+mj} q^{i+mj}$$

and

$$f_{mn}(q) = f_m(q) f_n(q^m),$$

it follows that

$$\zeta^{i+j} = \zeta^{i+mj}$$

for $0 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant n-1$. Then

$$\zeta^{j(m-1)} = 1$$

and

$$\zeta^{m-1} = 1 \quad \text{for all } m \in S(P).$$

Thus, ζ is a primitive ℓth root of unity for some positive integer ℓ, and ℓ divides $m-1$ for all $m \in S(P)$. Therefore, ℓ divides d, the greatest common divisor of the integers $m-1$, and so ζ is a dth root of unity. This completes the proof. $\square$

Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ and $\mathscr{G} = \{g_n(q)\}_{n=1}^{\infty}$ be sequences of polynomials. Define the product sequence

$$\mathscr{F}\mathscr{G} = \{f_ng_n(q)\}_{n=1}^{\infty}$$

by $f_ng_n(q) = f_n(q)g_n(q)$.

Theorem 6. *Let $\mathscr{F}$ and $\mathscr{G}$ be nonzero sequences of polynomials that satisfy the functional equation* (1). *The product sequence $\mathscr{F}\mathscr{G}$ also satisfies* (1). *Conversely, if* $\text{supp}(\mathscr{F}) = \text{supp}(\mathscr{G})$ *and if $\mathscr{F}$ and $\mathscr{F}\mathscr{G}$ satisfy* (1), *then $\mathscr{G}$ also satisfies* (1). *The set of all solutions of the functional equation* (1) *is an abelian semigroup, and, for every prime semigroup $S(P)$, the set $\Gamma(P)$ of all solutions $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ of* (1) *with* $\text{supp}(\mathscr{F}) = S(P)$ *is an abelian cancellation semigroup.*

Proof. If $\mathscr{F}$ and $\mathscr{G}$ both satisfy (1), then

$$\begin{aligned} f_{mn}g_{mn}(q) &= f_{mn}(q)g_{mn}(q) \\ &= f_m(q)f_n(q^m)g_m(q)g_n(q^m) \\ &= f_mg_m(q)f_ng_n(q^m) \end{aligned}$$

and so $\mathscr{F}\mathscr{G}$ satisfies (1). Conversely, if $m, n \in \text{supp}(\mathscr{F}) = \text{supp}(\mathscr{G})$,

$$f_{mn}(q)g_{mn}(q) = f_m(q)g_m(q)f_n(q^m)g_n(q^m)$$

and

$$f_{mn}(q) = f_m(q)f_n(q^m),$$

then

$$g_{mn}(q) = g_m(q)g_n(q^m).$$

Multiplication of sequences that satisfy (1) is associative and commutative. For every prime semigroup $S(P)$, we define the sequence $\mathscr{I}_P = \{I_n(q)\}_{n=1}^{\infty}$ by $I_n(q) = 1$ for $n \in S(P)$ and $I_n(q) = 0$ for $n \notin S(P)$. Then $\mathscr{I}_P \in \Gamma(P)$ and $\mathscr{I}_P\mathscr{F} = \mathscr{F}$ for every $\mathscr{F} \in \Gamma(P)$. If $\mathscr{F}, \mathscr{G}, \mathscr{H} \in \Gamma(P)$ and $\mathscr{F}\mathscr{G} = \mathscr{F}\mathscr{H}$, then $\mathscr{G} = \mathscr{H}$. Thus, $\Gamma(P)$ is a cancellation semigroup. This completes the proof. □

Let $S(P)$ be a prime semigroup, and let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ and $\mathscr{G} = \{g_n(q)\}_{n=1}^{\infty}$ be sequences of polynomials with support $S(P)$. We define the sequence of rational functions $\mathscr{F}/\mathscr{G}$ by

$$\frac{\mathscr{F}}{\mathscr{G}} = \left\{\frac{f_n}{g_n}(q)\right\}_{n=1}^{\infty},$$

where

$$\frac{f_n}{g_n}(q) = \frac{f_n(q)}{g_n(q)} \quad \text{if } n \in S(P)$$

and

$$\frac{f_n}{g_n}(q) = 0 \quad \text{if } n \notin S(P).$$

Then $\mathscr{F}/\mathscr{G}$ has support $S(P)$. If $\mathscr{F}$ and $\mathscr{G}$ satisfy the functional equation (1), then the sequence $\mathscr{F}/\mathscr{G}$ of rational functions also satisfies (1).

We recall the definition of the Grothendieck group of a semigroup. If Γ is an abelian cancellation semigroup, then there exists an abelian group $K(\Gamma)$ and an injective semigroup homomorphism $j : \Gamma \to K(\Gamma)$ such that if G is any abelian group and α a semigroup homomorphism from Γ into G, then there exists a unique group homomorphism $\tilde{\alpha}$ from $K(\Gamma)$ into G such that $\tilde{\alpha}j = \alpha$. The group $K(\Gamma)$ is called the *Grothendieck group* of the semigroup Γ.

Theorem 7. *Let $S(P)$ be a prime semigroup, and let $\Gamma(P)$ be the cancellation semigroup of polynomial solutions of the functional equation* (1) *with support $S(P)$. The Grothendieck group of $\Gamma(P)$ is the group of all sequences of rational functions $\mathscr{F}/\mathscr{G}$, where $\mathscr{F}$ and $\mathscr{G}$ are in $\Gamma(P)$.*

Proof. The set $K(\Gamma(P))$ of all sequences of rational functions of the form $\mathscr{F}/\mathscr{G}$, where $\mathscr{F}$ and $\mathscr{G}$ are in $\Gamma(P)$, is an abelian group, and $\mathscr{F} \mapsto \mathscr{F}/\mathscr{I}_P$ is an imbedding of $\Gamma(P)$ into $K(\Gamma(P))$. Let $\alpha : \Gamma(P) \to G$ be a homomorphism from $\Gamma(P)$ into a group G. We define $\tilde{\alpha} : K(\Gamma(P)) \to G$ by

$$\tilde{\alpha}\left(\frac{\mathscr{F}}{\mathscr{G}}\right) = \frac{\alpha(\mathscr{F})}{\alpha(\mathscr{G})}.$$

If $\mathscr{F}/\mathscr{G} = \mathscr{F}_1/\mathscr{G}_1$, then $\mathscr{F}\mathscr{G}_1 = \mathscr{F}_1\mathscr{G}$. Since α is a semigroup homomorphism, we have $\alpha(\mathscr{F})\alpha(\mathscr{G}_1) = \alpha(\mathscr{F}_1)\alpha(\mathscr{G})$, and so

$$\frac{\alpha(\mathscr{F})}{\alpha(\mathscr{G})} = \frac{\alpha(\mathscr{F}_1)}{\alpha(\mathscr{G}_1)}.$$

This proves that $\tilde{\alpha} : K(\Gamma(P)) \to G$ is a well-defined group homomorphism, and $\tilde{\alpha}j = \alpha$. $\square$

Problem 5. *Does every sequence of rational functions that satisfies the functional equation* (1) *and has support* $S(P)$ *belong to the group* $K(\Gamma(P))$?

We recall that if $\mathcal{F}$ is a sequence of nonconstant polynomials that satisfies (1), then there exists a positive rational number t such that $\deg(f_n) = t(n-1)$ is a positive integer for all $n \in \operatorname{supp}(\mathcal{F})$. In particular, if $\operatorname{supp}(\mathcal{F}) = \mathbf{N}$, or if $2 \in \operatorname{supp}(\mathcal{F})$, or, more generally, if $\{n-1\colon n \in \operatorname{supp}(\mathcal{F})\}$ is a set of relatively prime integers, then t is a positive integer.

The result below shows that the quantum integers are the unique solution of the functional equation (1) in the following important case.

Theorem 8. *Let* $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ *be a sequence of polynomials that satisfies the functional equation*

$$f_{mn}(q) = f_m(q) f_n(q^m)$$

for all positive integers m and n. If $\deg(f_n) = n-1$ *and* $f_n(0) = 1$ *for all positive integers n, then* $f_n(q) = [n]_q$ *for all n.*

Theorem 8 is a consequence of the following more general result.

Theorem 9. *Let* $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ *be a sequence of polynomials that satisfies the functional equation*

$$f_{mn}(q) = f_m(q) f_n(q^m)$$

for all positive integers m and n. If $\deg(f_n) = n-1$ *and* $f_n(0) = 1$ *for all* $n \in \operatorname{supp}(\mathcal{F})$, *and if* $\operatorname{supp}(\mathcal{F})$ *contains* 2 *and some odd integer greater than* 1, *then* $f_n(q) = [n]_q$ *for all* $n \in \operatorname{supp}(\mathcal{F})$.

Proof. Since $2 \in \operatorname{supp}(\mathcal{F})$, we have $\deg(f_2) = 1$ and $f_2(0) = 1$, hence

$$f_2(q) = 1 + aq$$

for some $a \neq 0$. If $n = 2r+1 \geqslant 3$ is an odd integer in $\operatorname{supp}(\mathcal{F})$, then

$$f_n(q) = 1 + \sum_{j=1}^{n-1} b_j q^j, \quad \text{with } b_{n-1} \neq 0.$$

We have

$$\begin{aligned} f_n(q)f_2(q^n) &= \left(1+\sum_{j=1}^{n-1} b_j q^j\right)(1+aq^n) \\ &= 1+\sum_{j=1}^{n-1} b_j q^j + aq^n + \sum_{j=1}^{n-1} ab_j q^{n+j} \\ &= 1+b_1q+b_2q^2+\cdots+ab_1q^{n+1}+ab_2q^{n+2}+\cdots \end{aligned}$$

and

$$\begin{aligned} f_2(q)f_n(q^2) &= (1+aq)\left(1+\sum_{j=1}^{n-1} b_j q^{2j}\right) \\ &= 1+aq+\sum_{j=1}^{2r} b_j q^{2j} + \sum_{j=1}^{2r} ab_j q^{2j+1} \\ &= 1+aq+b_1q^2+\cdots+b_{r+1}q^{n+1}+ab_{r+1}q^{n+2}+\cdots\ . \end{aligned}$$

The functional equation with $m=2$ gives

$$f_n(q)f_2(q^n) = f_2(q)f_n(q^2). \tag{16}$$

Equating coefficients in these polynomials, we obtain

$$a = b_1 = b_2,$$

$$b_{r+1} = ab_1 = a^2$$

and

$$ab_{r+1} = ab_2 = a^2.$$

Since $a\neq 0$, it follows that

$$a = 1$$

and

$$f_2(q) = 1+q = [2]_q.$$

By the functional equation, if $f_{2^{k-1}}(q) = [2^{k-1}]_q$ for some integer $k\geqslant 2$, then

$$\begin{aligned} f_{2^k}(q) &= f_{2^{k-1}}(q)f_2(q^{2^{k-1}}) \\ &= (1+q+q^2+\cdots+q^{2^{k-1}-1})(1+q^{2^{k-1}}) \end{aligned}$$

$$= 1 + q + q^2 + \cdots + q^{2^k - 1}$$

$$= [2^k]_q.$$

It follows by induction that $f_{2^k}(q) = [2^k]_q$ for all $k \in \mathbf{N}$.

Let $n = 2r + 1$ be an odd integer in $\mathrm{supp}(\mathscr{F})$, $n \geqslant 3$. Eq. (16) implies that

$$1 + b_1 q + \sum_{j=2}^{n-1} b_j q^j + q^n = 1 + q + \sum_{i=1}^{r} b_i (q^{2i} + q^{2i+1})$$

and so $1 = b_1 = b_r = b_{n-1}$ and

$$b_i = b_{2i} = b_{2i+1} \quad \text{for } i = 1, \ldots, r - 1.$$

If $n = 3$, then $b_1 = b_2 = 1$ and $f_3(q) = [3]_q$. If $n = 5$, then $r = 2$ and $b_1 = b_2 = b_3 = b_4 = 1$, hence $f_5(q) = [5]_q$.

For $n \geqslant 7$ we have $r \geqslant 3$. If $1 \leqslant k \leqslant r - 2$ and $b_i = 1$ for $i = 1, \ldots, 2k - 1$, then $k \leqslant 2k - 1$ and so

$$1 = b_k = b_{2k} = b_{2k+1}.$$

It follows by induction on k that $b_i = 1$ for $i = 1, \ldots, n - 1$, and $f_n(q) = [n]_q$ for every odd integer $n \in \mathrm{supp}(\mathscr{F})$.

If $2^k n \in \mathrm{supp}(\mathscr{F})$, where n is odd, then

$$f_{2^k n}(q) = f_{2^k}(q) f_n(q^{2^k}) = [2^k]_q [n]_{q^{2^k}} = [2^k n]_q.$$

This completes the proof. □

Problem 6. *Let* $t \geqslant 2$, *and let* $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ *be a sequence of polynomials satisfying the functional equation* (1) *such that* $f_n(q)$ *has degree* $t(n-1)$ *and* $f_n(0) = 1$ *for all* $n \in \mathbf{N}$. *Is* $\mathscr{F}$ *constructed from the quantum integers*? *More precisely*, *do there exist positive integers* $t_1, \ldots, t_k$ and integers $u_1, \ldots, u_k$ *such that*

$$t = t_1 u_1 + \cdots + t_k u_k$$

and, *for all* $n \in \mathbf{N}$,

$$f_n(q) = \prod_{i=1}^{k} ([n]_{q^{t_i}})^{u_i}?$$

5. Addition of quantum integers

It is natural to consider the analogous problem of addition of quantum integers. With the usual rule for addition of polynomials, $[m]_q + [n]_q \neq [m+n]_q$ for all positive integers m and n. However, we observe that

$$[m]_q + q^m [n]_q = [m+n]_q \quad \text{for all } m, n \in \mathbf{N}.$$

This suggests the following definition. Let $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials. We define

$$f_m(q) \oplus_q f_n(q) = f_m(q) + q^m f_n(q). \tag{17}$$

If $h(q)$ is any polynomial, then the sequence $\mathscr{F} = \{f_n(q)\}_{n=1}^{\infty}$ defined by $f_n(q) = h(q)[n]_q$ is a solution of the additive functional equation (17), and, conversely, every solution of (17) is of this form. This is discussed in [1].

References

[1] M.B. Nathanson, Additive number theory and the ring of quantum integers, www.arXiv.org:-math.NT/0204006.