All hereditary torsion theories are differential

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\begin{abstract}
Let $\alpha$ and $\beta$ be automorphisms on a ring $R$ and $\delta : R \to R$ an $(\alpha, \beta)$-derivation. It is shown that if $\mathfrak{T}$ is a right Gabriel filter on $R$ then $\mathfrak{T}$ is $\delta$-invariant if it is both $\alpha$ and $\beta$-invariant. A consequence of this result is that every hereditary torsion theory on the category of right $R$-modules is differential in the sense of Bland (2006). This answers in the affirmative a question posed by Vaš (2007) and strengthens a result due to Golan (1981) on the extendability of a derivation map from a module to its module of quotients at a hereditary torsion theory.
\end{abstract}

\section{Introduction}

In [4, Corollary 1] Golan proves that if $R$ is a ring endowed with a derivation map $\delta : M \to M$, and $\tau$ a hereditary torsion theory on the category of right $R$-modules such that $d[\tau(M)] \subseteq \tau(M)$, then $d$ extends to a $\delta$-derivation map on the module of quotients $Q_{\mathfrak{T}}(M)$ of $M$ at $\tau$. This result is sharpened by Bland [3] who calls a hereditary torsion theory \textit{differential} if the aforementioned containment $d[\tau(M)] \subseteq \tau(M)$ holds for all $M$ and $\delta$-derivations $d : M \to M$, and then proves that the differential hereditary torsion theories are precisely those hereditary torsion theories $\tau$ for which all $\delta$-derivation maps are extendable in the above sense [3, Proposition 2.3].

In a recent paper Vaš [9] identifies several classes of hereditary torsion theories that are differential and poses the question [9, page 852]: is every hereditary torsion theory differential? In this paper we shall answer this question in the affirmative by proving a slightly more general result on skew-derivations.

\section{Preliminaries}

Throughout this paper $R$ will denote an associative ring with identity and $\text{Mod}-R$ the category of unital right $R$-modules. If $N, M \in \text{Mod}-R$ we write $N \subseteq M$ if $N$ is a submodule of $M$. If $X, Y$ are nonempty subsets of $M$ we define $(X : Y) = \{ r \in R | yr \subseteq X\}$. If $X, Y \subseteq R$, then $(X : Y)$ will be taken as above with $R$ interpreted as a right module over itself.

If $d : R \to R$ is an additive map, we say that a nonempty family $\mathfrak{T}$ of right ideals of $R$ is \textit{d-invariant} if, for any $I \in \mathfrak{T}$, there exists $J \in \mathfrak{T}$ such that $d[J] \subseteq I$.

A \textit{hereditary torsion theory} on $\text{Mod}-R$ is a pair $\tau = (\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ is a class of right $R$-modules that is closed under submodules, homomorphic images, direct sums and module extensions, and $\mathcal{F}$ comprises all $N \in \text{Mod}-R$ such that $\text{Hom}_R(M, E(N)) = 0$ for all $M \in \mathcal{T}$. The modules in $\mathcal{T}$ are called $\tau$-\textit{torsion} and those in $\mathcal{F}$ $\tau$-\textit{torsion-free}. For each $M \in \text{Mod}-R$ there is a largest $\tau$-torsion submodule of $M$ that we shall denote by $\tau(M)$.\textsuperscript{*}

\begin{footnotesize}
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A nonempty family $\mathfrak{F}$ of right ideals of a ring $R$ is called a right Gabriel filter on $R$ if it satisfies the following two conditions:

(G1) if $I \in \mathfrak{F}$ then $(I : r) \in \mathfrak{F}$ for all $r \in R$;

(G2) if $I \in \mathfrak{F}$ and $J \subseteq R_\mathfrak{F}$ is such that $(J : a) \in \mathfrak{F}$ for all $a \in I$, then $J \in \mathfrak{F}$.

If $\tau = (\mathcal{T}, \mathcal{F})$ is an arbitrary hereditary torsion theory on $\text{Mod}-R$, then

$$\mathfrak{T}_\tau := \{l \subseteq R_\mathfrak{F} \mid R/l \in \mathcal{T}\}$$

is a right Gabriel filter on $\text{Mod}-R$. If $\mathfrak{F}$ is an arbitrary right Gabriel filter on $R$, then there is a (unique) hereditary torsion theory, denoted by $\tau_\mathfrak{F}$, whose torsion class $\mathcal{T}$ is given by

$$\mathcal{T} = \{M \in \text{Mod}-R \mid (0 : x) \in \mathfrak{F} \text{ for all } x \in M\}.$$

For every ring $R$ the maps $\tau \mapsto \mathfrak{T}_\tau$ and $\mathfrak{F} \mapsto \tau_\mathfrak{F}$ constitute a pair of mutually inverse maps between the sets of hereditary torsion theories on $\text{Mod}-R$ and right Gabriel filters on $R$ (see [8, Theorem VI.5.1, page 146]).

We refer the reader to [1,5,8] for further background information on torsion theories and Gabriel filters.

2. Differential torsion theories

Let $\alpha$ and $\beta$ be automorphisms on a ring $R$. An additive map $\delta : R \to R$ is called an $(\alpha, \beta)$-derivation on $R$ if

$$\delta(ab) = \delta(a)\alpha(b) + \beta(a)\delta(b) \quad \text{for all } a, b \in R.$$  

If $\alpha$ and $\beta$ coincide with the identity map on $R$, it is customary to omit the prefix $(\alpha, \beta)$ and speak simply of a derivation on $R$.

If $\delta$ is a derivation on $R$ and $M \in \text{Mod}-R$, then an additive map $d : M \to M$ is called a $\delta$-derivation on $M$ if

$$d(xr) = d(x)r + x\delta(r) \quad \text{for all } x \in M \text{ and } r \in R.$$  

The following result is due to Bland [3, Lemma 1.5].

**Theorem 1.** Let $\delta$ be a derivation on a ring $R$. The following conditions are equivalent for a hereditary torsion theory $\tau$ on $\text{Mod}-R$:

(i) for every $M \in \text{Mod}-R$ and $\delta$-derivation $d$ on $M$, $d[I_\tau(M)] \subseteq I_\tau(M)$;

(ii) $\mathfrak{T}_\tau$ is $\delta$-invariant.

A hereditary torsion theory $\tau$ satisfying the equivalent conditions of Theorem 1 is called differential. Differential torsion theories have the important property that every $\delta$-derivation on a module $M$ extends uniquely to a derivation on the module of quotients of $M$ at the given torsion theory, as shown in [4, Corollary 1] and [3, Proposition 2.1].

We refer the reader to [4,3,2,9] as sources of further information on torsion theories in the context of rings endowed with a derivation map.

We now prove our main theorem from which it shall follow that all hereditary torsion theories are differential thus answering in the affirmative a question posed by Vaš [9, page 852].

**Theorem 2.** Let $\alpha$ and $\beta$ be automorphisms on a ring $R$ and $\delta : R \to R$ an $(\alpha, \beta)$-derivation on $R$. If $\mathfrak{F}$ is a right Gabriel filter on $R$ that is both $\alpha$ and $\beta$-invariant, then $\mathfrak{F}$ is $\delta$-invariant.

**Proof.** Let $I \in \mathfrak{F}$. We have to show that there exists $J \in \mathfrak{F}$ with $\delta[J] \subseteq I$. Since $\mathfrak{F}$ is $\alpha$ and $\beta$-invariant, $L = \alpha^{-1}[I] \cap \beta^{-1}[I] \in \mathfrak{F}$.

Let

$$J = \{x \in L \mid \delta(x) \in I\}.$$  

Since $\delta$ is additive, $J$ is an additive subgroup of $R$. Take any $x \in J$ and $r \in R$. Then $\delta(xr) = \delta(x)\alpha(r) + \beta(x)\delta(r) \in I$, because $\beta(x) \in \beta[L] \subseteq I$ and $\delta(x) \in I$ by definition, whence $xr \in J$. We conclude that $J$ is a right ideal.

For each $x \in L$ we claim that

$$(L : \alpha^{-1}(\delta(x))) \subseteq (J : x).$$  

To prove (1) note that

$$y \in (L : \alpha^{-1}(\delta(x))) \iff \alpha^{-1}(\delta(x))y \in L \iff \delta(x)\alpha(y) \in \alpha[L] \subseteq I.$$  

Since $x \in L$ we also have $\beta(x)\delta(y) \in I$, whence

$$\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y) \in I.$$  

Inasmuch as $xy \in L$ and $\delta(xy) \in I$, we have $xy \in J$. This establishes (1).

Since $L \in \mathfrak{F}$, it follows from (G1) that $(L : \alpha^{-1}(\delta(x))) \in \mathfrak{F}$. Hence by (1), $(J : x) \in \mathfrak{F}$ for all $x \in L$. We conclude from (G2) that $J \in \mathfrak{F}$, as required. □
Since every nonempty family of right ideals of \( R \) is trivially invariant with respect to the identity map on \( R \), the following corollary follows immediately from the two previous theorems.

**Corollary 3.** Let \( R \) be any ring endowed with a derivation map \( \delta : R \to R \). Then every hereditary torsion theory on \( \text{Mod}-R \) is differential.

**Remark.** The problem of extending derivations to rings of quotients of algebras over fields is a special case of extending Hopf algebra actions to rings of quotients. Let \( H \) be a Hopf algebra over a field \( k \) acting on a \( k \)-algebra \( A \) and let \( \mathfrak{g} \) be a right Gabriel filter on \( A \) with associated ring of quotients \( Q_\mathfrak{g}(A) \). Denote by \( \lambda_\mathfrak{g}(a) := h \cdot a \) the action of an element \( h \in H \) to \( a \in A \), which is an additive map. A necessary condition for extending the \( H \)-action on \( A \) to \( Q_\mathfrak{g}(A) \) is that \( H \) act \( \mathfrak{g} \)-continuously, i.e., \( \mathfrak{g} \) is \( \lambda_\mathfrak{g} \)-invariant for all \( h \in H \) (see [6]). The terminology is justified if \( A \) is considered a topological ring whose topology is induced by \( \mathfrak{g} \) and interpreting the condition \( \lambda_\mathfrak{g}^{-1}(l) \in \mathfrak{g} \) for any \( l \in \mathfrak{g} \) as continuity.

In [6] it is shown that if the Hopf algebra \( H \) is pointed, i.e., all simple subcoalgebras are one-dimensional, then \( H \) always acts \( \mathfrak{g} \)-continuously on an algebra \( A \). In the case of a derivation \( \delta \) of \( A \) one might consider the enveloping algebra \( H \) of the 1-dimensional Lie algebra which acts as \( \delta \) on \( A \). Here \( H = k[X] \) is a pointed Hopf algebra and hence the action extends to \( Q_\mathfrak{g}(A) \).

A purely coalgebraic version was given by Rumynin in [7]: a coalgebra \( C \) is said to measure an algebra \( A \) if there exists an action \( \cdot : C \otimes A \to A \) such that for all \( c \in C \) and \( a, b \in A \), \( c \cdot (ab) = \sum c(c_1 \cdot a)(c_2 \cdot b) \) and \( c \cdot 1 = \epsilon(c)1 \) where \( \Delta(c) = \sum c(c_1 \otimes c_2) \in C \otimes C \) denotes the comultiplication and \( \epsilon(c) \) the counit of \( c \). Rumynin proved that if every simple subcoalagebra of \( C \) is 1-dimensional and measures \( A \) \( \mathfrak{g} \)-continuously, then \( C \) also measures \( A \) \( \mathfrak{g} \)-continuously.

Let \( \alpha \) and \( \beta \) be automorphisms on \( A \) and \( \delta : A \to A \) an \( (\alpha, \beta) \)-derivation. Let \( C \) be the 4-dimensional vector space over \( k \) with basis \( 1, g, h \) and \( x \) which becomes a coalgebra with comultiplication

\[
\Delta(1) = 1 \otimes 1, \quad \Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes g + h \otimes x
\]

and counit \( \epsilon(1) = \epsilon(g) = \epsilon(h) = 1 \) and \( \epsilon(x) = 0 \). Define the measuring \( \cdot : C \otimes A \to A \) by \( 1 \cdot a = a, g \cdot a = \alpha(a), h \cdot a = \beta(a) \) and \( x \cdot a = \delta(a) \). The simple subcoalgebras of \( C \) are \( k1, k^2 \) and \( k^3h \) which are 1-dimensional. If \( \mathfrak{g} \) is \( \alpha \) and \( \beta \)-invariant, then by [7, Lemma 9], \( C \) acts \( \mathfrak{g} \)-continuously on \( A \), i.e., \( \mathfrak{g} \) is \( \delta \)-invariant. This yields another proof of Theorem 2 for the special case of algebras over fields.

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**References**


