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Representations of Finite Posets and Valuated Groups

D. ARNOLD* AND F. RICHMAN*

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

AND

C. VINSONHALER[†]

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269

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Valuated p-groups [RW] arose in the study of torsion abelian groups. If A is a subgroup of a finite p-group B, then A becomes a valuated group by assigning to each element of A its height as an element of B. Classifying such pairs (A, B) is the same as classifying finite valuated p-groups with values in the positive integers. Finite valuated p-groups give rise to finite-dimensional representations of partially ordered sets over the p-element field via the notion of *v-height*, and any such representation can be so realized [A, Theorem 3.2].

A *v-height* is an equivalence class of valuated trees under the natural quasi-ordering (reflexive and transitive) on valuated trees given in [HRW1] and in Section 2 below. The v-heights form a distributive lattice under this ordering. Each nonzero element x of a valuated p-group has a v-height $vh(x)$ given by the valuated tree $\{y \mid p^n y = x \text{ for some } n\}$. If P is a set of v-heights, and G is a valuated p-group, then we get a representation of P^{op} over the p-element field by considering $G[p]$ together with the subspaces $G(\tau)[p] = \{x \in G[p] \mid vh(x) \geq \tau\}$ for each $\tau \in P$. We will be more concerned with the subrepresentation $H_P(G)$ whose underlying space is $\sum_{\tau \in P} G(\tau)[p]$ rather than $G[p]$.

Let $\mathcal{V}$ be the category of finite valuated p-groups, and, for Q a finite *poset* (partially ordered set), let Rep(Q) be the category of finite-dimensional representations of Q over the p-element field. The representations of the form $H_P(G)$, for G in $\mathcal{V}$, are not arbitrary, even in $\text{Rep}_0(P) = \{U \in \text{Rep}(P) \mid U = \sum_{s \in P} U(s)\}$, because $G(\tau_1 \vee \tau_2)[p] = G(\tau_1)[p] \cap$

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$G(\tau_2)[p]$. A necessary condition for the realization of all representations in $\mathrm{Rep}_0(P)$ is that P be *join-irreducible*, that is, no element of P is the join of a finite number of strictly smaller elements in the lattice generated by P.

For each positive integer n there is a join-irreducible poset P_n consisting of n poles (trees without branches), no two comparable, and each with n nodes; this construction is a special case of Proposition 2.3, and is illustrated in Example 2.4(a). If Q is a poset of cardinality n, then any one-to-one map from $P_n = P_n^{\mathrm{op}}$ onto Q induces a functorial embedding of $\mathrm{Rep}(Q)$ as a full subcategory of $\mathrm{Rep}(P_n^{\mathrm{op}})$. In [A, Theorem 3.2] a map I_n of $\mathrm{Rep}(P_n^{\mathrm{op}})$ into $\mathscr{V}$ is constructed, preserving isomorphism and indecomposability, so that $U \cong H_{P_n} I_n(U)$ for each $U \in \mathrm{Rep}(P_n^{\mathrm{op}})$. Thus classifying finite valuated p-groups is at least as difficult as classifying finite-dimensional representations over the p-element field. The image of I_n is fairly small (see Section 3) and I_n is not a functor.

In this paper we initiate a study of $\mathscr{V}$ in terms of the functor $H_P: \mathscr{V} \to \mathrm{Rep}(P^{\mathrm{op}})$. The goal is to introduce ideas and techniques arising from the extensive literature on representations of finite posets, a survey of which is given in [A], to the subject of finite valuated p-groups. The theory of $\mathrm{Rep}(P^{\mathrm{op}})$ could be applied directly to $\mathscr{V}$ if there were a full additive functorial embedding of $\mathrm{Rep}(P^{\mathrm{op}})$ in $\mathscr{V}$. But the image of any additive functor from $\mathrm{Rep}(P^{\mathrm{op}})$ to $\mathscr{V}$ consists of p-bounded finite valuated groups, which are known to be direct sums of cyclics [HRW2]. As most finite posets have wild representation type (see [A]), additive functorial embeddings of $\mathrm{Rep}(P^{\mathrm{op}})$ in $\mathscr{V}$ do not exist in general.

The problem of functoriality can be partially resolved by passing to quotient categories. Let $\mathscr{V}/\mathscr{A}$ be the category whose objects are the objects of $\mathscr{V}$, and whose maps are the maps in $\mathscr{V}$ modulo those maps that annihilate the p-socle of their domain. Isomorphism in $\mathscr{V}$ coincides with isomorphism in $\mathscr{V}/\mathscr{A}$, and a valuated group is indecomposable in $\mathscr{V}$ if and only if it is indecomposable in $\mathscr{V}/\mathscr{A}$ (Proposition 2.1). Note that H_P is naturally a functor on $\mathscr{V}/\mathscr{A}$.

Let $\mathrm{Rep}_0(Q)$ denote the full subcategory of $\mathrm{Rep}(Q)$ consisting of those representations with no nonzero trivial summands; a representation U in $\mathrm{Rep}(Q)$ is in $\mathrm{Rep}_0(Q)$ if and only if $U = \sum_{s \in Q} U(s)$. The functor taking U to $\sum_{s \in Q} U(s)$ is a retraction of $\mathrm{Rep}(Q)$ onto $\mathrm{Rep}_0(Q)$. Clearly H_P takes $\mathscr{V}/\mathscr{A}$ to $\mathrm{Rep}_0(P^{\mathrm{op}})$.

We generalize and improve upon the splitting map I_n in [A, Theorem 3.2] for the functor $H_{P_n}: \mathscr{V}/\mathscr{A} \to \mathrm{Rep}_0(P_n^{\mathrm{op}})$ by constructing a map F_P which is an additive functor when the trees in P are poles, and which splits H_P when P is a join-irreducible weak antichain (defined just before Theorem II). The posets P_n are join-irreducible weak antichains. The construction of F_P depends on a choice of representatives of the v-heights, so we consider posets of valuated trees rather than posets of v-heights.

THEOREM I. *Let p be a prime and P a finite poset of finite valuated trees.*

(a) *There is a correspondence* $F_P\colon \mathrm{Rep}_0(P^{\mathrm{op}}) \to \mathcal{V}$ *preserving isomorphism and direct sums, and sending rank-1 representations to simply presented valuated groups.*

(b) *If each tree in P is a pole, then* F_P *is an additive functor to* $\mathcal{V}/\mathcal{A}$.

The next theorem provides a necessary and sufficient condition for F_P to split H_P. If T_1 and T_2 are finite valuated trees, then write $T_1 \ll T_2$ if there is a map of valuated trees $T_1 \to T_2$ that does not take root T_1 to root T_2 [HRW1]. We say that a set P of trees us a *weak antichain* if $T_1 \ll T_2$ does not hold for any T_1, T_2 in P.

THEOREM II. *Let P be a join-irreducible poset of finite valuated trees. Then* $H_P F_P(U) \cong U$ *for each U in* $\mathrm{Rep}_0(P^{\mathrm{op}})$ *if and only if P is a weak antichain.*

If P is a weak antichain, then the groups in $\mathcal{V}$ of the form $F_P(U)$, for U in $\mathrm{Rep}_0(P^{\mathrm{op}})$, are classified in terms of the v-height structure of their p-socles. If U is indecomposable, then $F_P(U) = K \oplus L$ for some indecomposable K with $H_P F_P(U) = H_P(K)$. Moreover, if U has rank 1, then K is indecomposable, simply presented, and isomorphic to $S(T)$ for T the unique irretractable tree equivalent to the join of $\{T' \in P \mid U(T') \neq 0\}$ (Proposition 3.1).

The correspondence F_P need not be split by H_P, even when P consists of poles (Example2.5). However, for each finite poset Q, there is a poset of valuated poles anti-isomorphic to Q that is a join-irreducible weak antichain (Proposition 2.3). This together with Theorem I demonstrates the complexity of $\mathcal{V}$:

COROLLARY III. *Let Q be a finite poset. Then* $\mathrm{Rep}_0(Q)$ *is an additive retract of* $\mathcal{V}/\mathcal{A}$.

Valuated p-groups of the form $F_P(U)$, for U a representation of rank greater than 1, need not be simply presented, so they constitute a class of finite valuated p-groups that have not been studied in depth. In view of Corollary III, constructions and classifications in $\mathrm{Rep}(Q)$ can be carried over to $\mathcal{V}$ via $\mathcal{V}/\mathcal{A}$. In particular, duality, Coxeter correspondences, classification of preinjectives and preprojectives, and classification of indecomposable representations of the form $G(X_1, ..., X_n)$ and $G[X_1, ..., X_n]$, for an n-tuple $(X_1, ..., X_n)$ of rank-1 representations (as described in [AV] have direct analogs in $\mathcal{V}/\mathcal{A}$ and $\mathcal{V}$.

This leads to the problem, addressed in Corollary 3.2, of finding a group-theoretic description of those groups in $\mathcal{V}$ that are in the image of F_P for some finite P. Such a group is an extension of a p-bounded valuated group by a simply presented valuated group (Corollary 3.2). We include an unpublished example, due to L. Hughes, of a group in $\mathscr{V}$ that is not in the image of F_P for any finite P (Example 3.3).

Given a finite poset Q, there are complete sets of invariants for finite direct sums of rank-1 representations (Corollary 1.6(b)). We call these invariants "Baer invariants" because of their close relationship to the classical invariants of R. Baer for direct sums of rank-1 torsion-free abelian groups.

The final corollary demonstrates a correspondence between Baer invariants for representations and Ulm invariants for valuated p-groups. Given a valuated group G and a valuated tree T, the T^{th} *Ulm invariant of* G is the dimension of

$$\frac{G(T)[p]}{G(T)[p] \cap G(T^*)[p]},$$

where $G(T^*) = \sum \{G(T') \mid \neg(T' \leqslant T)\}$. Clearly the T^{th} Ulm invariant depends only on the equivalence class of T. For a poset P of valuated trees, the T^{th} *Ulm invariant of* G *relative to* P is as defined above, replacing $G(T^*)$ by $\sum \{G(T') \mid T'$ is a join of trees in P, and $\neg(T' \leqslant T)\}$.

COROLLARY IV. *Let P be a finite join-irreducible poset of finite valuated trees that is a weak antichain. There is a correspondence F'_P*: $\text{Rep}_0(P^{\text{op}}) \to \mathscr{V}$ *such that*:

(a) *F'_P preserves isomorphism, indecomposables, and direct sums.*

(b) *If X is a rank-1 representation, then $F'_P(X) = S(T)$ is simply presented and indecomposable for T the unique irretractable tree equivalent to the join of $\{T' \in P \mid X(T') \neq 0\}$.*

(c) *$H_P F_P(U) \cong U$ for each U in* $\text{Rep}(P^{\text{op}})$.

(d) *If U is in* $\text{Rep}_0(P^{\text{op}})$, *$X$ is a rank-1 representation of type τ, and $F'_P(X) = S(T)$, then the τ^{th} Baer invariant of U is equal to the T^{th} Ulm invariant of $F'_P(U)$ relative to P.*

The T^{th} Ulm invariant relative to a poset is at least as big as the T^{th} Ulm invariant. Let G be a finite valuated p-group, let L be the lattice generated by the v-heights of nonzero elements of G, let P' be the join-irreducible elements of L, and let P consist of one representative from each v-height in P'. Then P is a finite join-irreducible poset of finite valuated trees and the Ulm invariants of G coincide with the Ulm invariants of G relative to P.

The existence of a nonzero Baer invariant for a representation guarantees the existence of a rank-1 summand determined by that invariant (Corollary 1.6(a)). It follows from Corollary IV that valuated groups in the image of F_P, for P a join-irreducible weak antichain, have a simply presented summand if there is a corresponding nonzero Ulm invariant. This extends [HRW1, Theorem 3.4], wherein it is shown that if a valuated p-group has a nonzero Ulm invariant determined by a pole T, then the group has a cyclic summand isomorphic to $S(T)$. L. Hughes has shown that the group in Example 3.3 has a nonzero T^{th} Ulm invariant but does not have a summand isomorphic to $S(T)$. This provides an alternate proof of the fact that this group cannot be in the image of F_P for any join-irreducible finite poset P which is a weak antichain.

Theorem I(b) and Corollary III, for which F_P is an additive functor, suggest that valuated cyclics are the analogs of rank-1 representations. However, as demonstrated in Corollary IV, if one does not insist on a functor, indecomposable simply presented groups in $\mathscr{V}$ can be thought of as analogous to rank-1 representations.

There are other choices for embeddings. For P join-irreducible [ARV, Theorem 5.1 and 4.1] provides the construction of a functor λ: $\text{Rep}_0(P^{\text{op}}) \to \mathscr{V}$ that splits H_P. As noted above, λ cannot be additive; it takes values in $\mathscr{V}$ instead of $\mathscr{V}/\mathscr{A}$, and it sends rank-1 representations to simply presented groups. The functor λ has yet to be examined carefully for the special case of finite-dimensional representations and finite valuated groups.

In this paper, attention is restricted to finite valuated p-groups and finite-dimensional representations, even though some of the definitions and results hold in greater generality. This restriction is predicated on the assumptions that finite-dimensional representations of finite posets are better understood, and that finite valuated p-groups need to be considered in detail before attempting the more general case.

1. Representations of Posets

Let k be a field and Q a finite poset. The objects in the *category* $\text{Rep}(k, Q)$ *of representations of* Q, are finite-dimensional k-vector spaces U, together with subspaces $U(s)$ for each $s \in Q$, such that $U(s) \subseteq U(t)$ whenever $s \leqslant t$. The maps in $\text{Rep}(k, Q)$ are k-linear transformations $f: U \to V$ such that $f(U(s)) \subseteq V(s)$ for each $s \in Q$. Finite direct sums in $\text{Rep}(k, Q)$ are given by $U \oplus V$ with $(U \oplus V)(s) = U(s) \oplus V(s)$.

The category $\text{Rep}(k, Q)$ is a *Krull–Schmidt category*: each object has a finite direct sum decomposition into indecomposables unique up to order and isomorphism. This is a consequence of the Krull–Schmidt theorem for additive categories [B, Theorem 3.6], since the endomorphism ring of an indecomposable representation is a finite-dimensional k-algebra with no nontrivial idempotents, hence local, and idempotents split in $\text{Rep}(k, Q)$.

The category $\operatorname{Rep}(k, Q)$ has kernels and cokernels. If $f: U \to V$, then the kernel of f is $K = f^{-1}(0)$, with $K(s) = K \cap U(s)$ for $s \in Q$, and the cokernel of f is $C = V/f(U)$, with $C(s) = (V(s) + f(U))/f(U)$ for $s \in Q$. Consequently, $0 \to U \to V \to W \to 0$ is exact in $\operatorname{Rep}(k, Q)$ if and only if it is an exact sequence of vector spaces, and $0 \to U(s) \to V(s) \to W(s) \to 0$ is exact for each $s \in Q$.

The *rank of a representation* U is defined to be $\dim_k U$. Note that rank $U = 0$ if and only if U is the zero object in $\operatorname{Rep}(k, Q)$. A *trivial representation* is a representation U with $U(s) = 0$ for each $s \in Q$.

Define $\operatorname{Rep}_0(k, Q)$ to be the full subcategory of $\operatorname{Rep}(k, Q)$ consisting of those objects U such that $\sum_{s \in Q} U(s) = U$. We can also think of $\operatorname{Rep}_0(k, Q)$ as a quotient category of $\operatorname{Rep}(k, Q)$ constructed by dividing out by the subspaces $\Lambda(U, V) = \{f \in \operatorname{Hom}(U, V) \mid f(\sum_{s \in Q} U(s)) = 0\}$.

PROPOSITION 1.1. *The following are equivalent for U and V in* $\operatorname{Rep}(k, Q)$.

(a) $U \cong V$ *modulo* Λ,

(b) $\sum_{s \in Q} U(s) \cong \sum_{s \in Q} V(s)$ *as representations,*

(c) $U \oplus X \cong V \oplus Y$ *for some trivial representations X and Y.*

Proof. If $f: U \to V$ represents an isomorphism modulo Λ, or $f: U \oplus X \to V \oplus Y$ is an isomorphism with X and Y trivial representations, then f restricts to an isomorphism from $\sum_{s \in Q} U(s)$ to $\sum_{s \in Q} V(s)$; thus (a) or (c) implies (b). Let U_0 and V_0 be complementary vector space summands of $\sum_{s \in Q} U(s)$ and $\sum_{s \in Q} V(s)$ in U and V; then these are complementary trivial summands in $\operatorname{Rep}(k, Q)$. If (b) holds, let f be an isomorphism from $\sum_{s \in Q} U(s)$ to $\sum_{s \in Q} V(s)$. Then $U \oplus V_0 \cong V \oplus U_0$, whence (c) holds, and extending f to U by setting $f(U_0) = 0$ gives an isomorphism from U to V in $\operatorname{Rep}_0(k, Q)$, whence (a) holds. ∎

A representation U is a *subrepresentation* of V if U is a subspace of V and $U(s) = U \cap V(s)$ for each $s \in Q$. If U is a subrepresentation of V, then $0 \to U \to V \to V/U \to 0$ is exact in $\operatorname{Rep}(k, Q)$, where $(V/U)(s) = (V(s) + U)/U$ for each $s \in Q$. Any subspace U of V becomes a subrepresentation by setting $U(s) = U \cap V(s)$ for each $s \in Q$.

A *type* τ in Q is a subset τ (possibly empty) of Q such that $s \in \tau$ and $s \leqslant t$ in Q imply $t \in \tau$. The set of types in Q forms a distributive lattice under inclusion. The *type of a rank*-1 *representation* X is type $X = \{s \in Q \mid X(s) \neq 0\}$; note that $X(s) = X$ or 0 for each s if rank $X = 1$. The type function gives a 1–1 correspondence between isomorphism classes of rank-1 representations and types.

Let U be a representation of Q and τ a type in Q. Define the subrepresentations $U(\tau) = \bigcap_{s \in \tau} U(s)$ and $U[\tau] = \sum_{s \notin \tau} U(s)$.

LEMMA 1.2. *Let U, V be in* Rep(k, Q), *and τ a type in Q.*

(a) $U(\tau) = \sum \{Y \subseteq U \mid \text{rank } Y = 1 \text{ and } \tau \leqslant \text{type } Y\}$.

(b) $U[\tau] = \sum \{Y \subseteq U \mid \text{rank } Y = 1 \text{ and } \neg(\tau \geqslant \text{type } Y)\}$.

(c) $(U \oplus V)(\tau) = U(\tau) \oplus V(\tau)$.

(d) $(U \oplus V)[\tau] = U[\tau] \oplus V[\tau]$.

Proof. This is a routine consequence of the definitions. ▮

A representation U is *τ-homogeneous* for a type τ if $U(s) = U$ for $s \in \tau$, and $U(s) = 0$ for $s \notin \tau$.

PROPOSITION 1.3. *Suppose that U is a τ-homogeneous subrepresentation of V.*

(a) *If $V = U \oplus W$ is a vector space decomposition with $V[\tau] \subseteq W$, then $V = U \oplus W$ as representations.*

(b) *U is a summand of V if and only if $U \cap V[\tau] = 0$.*

Proof. (a) If $s \in \tau$, then $U = U(s) \subseteq V(s)$. If $s \notin \tau$, then $V(s) \subseteq V[\tau] \subseteq W$. In either case, $V(s) = U(s) \oplus (W \cap V(s)) = U(s) \oplus W(s)$, as desired.

(b) If $V = U \oplus W$ as representations, then $V[\tau] = U[\tau] \oplus W[\tau] = W[\tau] \subseteq W$, because U is τ-homogeneous. The converse follows from (a) by choosing a vector space decomposition $V = U \oplus W$ with $V[\tau] \subseteq W$. ▮

COROLLARY 1.4. *Suppose $V \in$* Rep(k, Q) *and τ is a type in Q.*

(a) *If V is τ-homogeneous, then V is isomorphic to a direct sum of rank-1 representations of type τ.*

(b) *$V = U \oplus W$ in* Rep(k, Q), *where $U \cong V(\tau)/(V(\tau) \cap V[\tau]$ is τ-homogeneous and W has no τ-homogeneous summands.*

Proof. (a) As V is τ-homogeneous, $V[\tau] = 0$, and each subrepresentation of V is τ-homogeneous. Therefore, by Proposition 1.3(b), each subrepresentation of V is a summand.

(b) Write $V(\tau) = U \oplus (V(\tau) \cap V[\tau])$ as vector spaces. Then U is a τ-homogeneous representation, since if $s \in \tau$, then $U(s) = U$ because $U \subseteq V(\tau)$, and if $s \notin \tau$, then $U(s) = 0$ because $U \cap V[\tau] = 0$. Let W be a complementary vector-space summand of U containing $V[\tau]$. That $V = U \oplus W$ in Rep(k, Q) follows from Proposition 1.3(b). As $V(\tau) = U \oplus W(\tau) = U \oplus (V(\tau) \cap V[\tau])$, and $V[\tau] \subseteq W[\tau]$, it follows that $W(\tau) = V(\tau) \cap V[\tau] \subseteq W[\tau]$, so W has no τ-homogeneous summands by Proposition 1.3(b). ▮

COROLLARY 1.5. *Let U and X be in* Rep(k, Q) *with X rank* 1 *of type τ. Then*

$$U[\tau] = \bigcap \{\text{Kernel } f \mid f\colon U \to X\}.$$

Proof. Note that $U[\tau]$ is in the kernel of each $f\colon U \to X$, as $f(U(s)) \subseteq X(s) = 0$ for each $s \notin \tau$. Conversely, suppose $y \in U \setminus U[\tau]$. Let f be a linear transformation from U to X such that $f(U[\tau]) = 0$ and $f(y) \neq 0$. It suffices to show that f is a map in Rep(k, Q). If $s \in \tau$, then $f(U(s)) \subseteq X = X(s)$. If $s \notin \tau$, then $U(s) \subseteq U[\tau]$, so $f(U(s)) = 0 \subseteq X(s)$. ∎

Given U in Rep(k, Q), and a type τ, define the τ^{th} *Baer invariant* $B(U, \tau)$ to be $\dim_k (U(\tau)/(U(\tau) \cap U[\tau]))$. Note that if U is a direct sum of rank-1 representations, then $B(U, \tau)$ is the number of rank-1 summands of type τ. The following corollary summarizes the preceding discussion.

COROLLARY 1.6. *Let U be in* Rep(k, Q) *and τ a type in Q.*

(a) *There is a decomposition $U = V \oplus W$, where V is τ-homogeneous, $\dim_k V = B(U, \tau)$, and W has no τ-homogeneous summands.*

(b) *If U and V are direct sums of rank-*1 *representations, then U and V are isomorphic if and only if $B(U, \tau) = B(V, \tau)$ for each type τ in Q.* ∎

2. FINITE VALUATED p-GROUPS

We summarize some of the definitions and properties of finite valuated p-groups, as found in [RW, HRW1, HRW2]. A valuated p-group, for a prime p, is an abelian p-group together with a function $v\colon G \to$ ordinals $\cup \{\infty\}$ such that

(i) $v(px) > v(x)$ if $v(x) < \infty$,

(ii) $v(x+y) \geqslant \min(v(x), v(y))$, and

(iii) $v(nx) = v(x)$ if n is an integer relatively prime to p.

Condition (iii) can be shown to be redundant. Define $\mathscr{V}$ to be the category with objects finite valuated p-groups, and morphisms

$$\mathscr{V}(G, H) = \{f \in \text{Hom}_{\mathbb{Z}}(G, H) \mid v(f(x)) \geqslant v(x) \text{ for each } x \in G\}.$$

A subgroup H of a valuated group G is a valuated group under the *induced valuation*: the value of each element x of H is its value in G. A *direct sum* of valuated groups becomes a valuated group upon assigning to each element the minimum of the values of its coordinates; this is the categorical coproduct.

The *p-socle* $G[p]$ of a group G is the subgroup $\{x \in G \mid px = 0\}$. Define the quotient category $\mathcal{V}/\mathcal{A}$ by letting the objects of $\mathcal{V}/\mathcal{A}$ be the objects of $\mathcal{V}$, and letting the set of maps from G to H be $\mathcal{V}(G, H)/\mathcal{A}(G, H)$, where $\mathcal{A}(G, H) = \{f \in \mathcal{V}(G, H) \mid f(G[p]) = 0\}$.

PROPOSITION 2.1. (a) *$\mathcal{V}$ and $\mathcal{V}/\mathcal{A}$ are additive Krull–Schmidt categories,*

(b) *Objects in $\mathcal{V}$ are isomorphic if and only if they are isomorphic in in $\mathcal{V}/\mathcal{A}$,*

(c) *$A \cong A_1 \oplus A_2$ in $\mathcal{V}$ if and only if $A \cong A_1 \oplus A_2$ in $\mathcal{V}/\mathcal{A}$.*

Proof. Each group in either category, being finite, is a finite direct sum of indecomposables. The categories are Krull–Schmidt because the endomorphism ring of any indecomposable is finite with no nontrivial idempotents, hence local.

Taking $A_2 = 0$ in (c), we see that (b) follows from (c). Half of (c) is trivial, so suppose $A \cong A_1 \oplus A_2$ in $\mathcal{V}/\mathcal{A}$. Let ι_1, ι_2, π_1, and π_2 be representatives of the injections and projections of the direct sum. First note that if $G \in \mathcal{V}$, then $\mathcal{A}(G, G)$ is a nilpotent ideal in the endomorphism ring of G because endomorphisms in $\mathcal{A}(G, G)$ strictly decrease the order of nonzero elements of G, and G is bounded. So $\pi_1\tau_1 - 1 \in \mathcal{A}(A_1, A_1)$ is nilpotent, whence there is an automorphism α_1 of A_1 such that $\pi_1\iota_1\alpha_1 = 1$. As $\pi_2(1 - \iota_1\alpha_1\pi_1)\iota_2 \in 1 - \mathcal{A}(A_2, A_2)$, there is an automorphism α_2 of A_2 such that $\pi_2(1 - \iota_1\alpha_1\pi_1)\iota_2\alpha_2 = 1$. Let

$$\bar{\iota}_1 = \iota_1\alpha_1, \qquad \bar{\iota}_2 = \iota_2\alpha_2, \qquad \bar{\pi}_2 = \pi_2(1 - \bar{\iota}_1\pi_1), \qquad \bar{\pi}_1 = \pi_1(1 - \bar{\iota}_2\bar{\pi}_2).$$

It is readily checked that this exhibits A as the direct sum of A_1 and A_2 in $\mathcal{V}$. ∎

A *rooted tree* is a set T of nodes, a partial function $\rho: T \to T$, and a distinguished node r called the *root* of T, such that the domain of definition of ρ is $T \setminus \{r\}$, and for each $x \in T \setminus \{r\}$ there is a positive integer n such that $\rho^n x = r$. The node ρx is the parent of x. A tree T is partially ordered by setting $y \leqslant x$ if $y = \rho^n x$ for some $n \geqslant 0$. A *valuation* on T is a function v: $T \to$ ordinals $\cup \{\infty\}$ with $v(\rho x) > v(x)$ whenever ρx is defined. If G is a valuated p-group, and $x \in G$ is nonzero, then the *tree on* x is $T(x) = \{y \in G \mid p^n y = x$ for some $n \geqslant 0\}$ with each node given its value in G (the partial function ρ is multiplication by p); we let $T(0)$ be the infinite rooted pole with each node valued by ∞. A *map of valuated trees* is a function f: $T_1 \to T_2$ such that $f(\rho(x)) = \rho f(x)$ whenever $\rho(x)$ is defined and $v(f(x)) \geqslant v(x)$ for each $x \in T_1$.

Associated with each valuated rooted tree is a valuated p-group $S(T) = F/R$, where F is the free group on the nodes of T, and R is generated by the relations $px = \rho x$ if ρx is defined, and $px = 0$ if ρx is not defined. The valuation on $S(T)$ is given as follows [HRW2]: Each element x of $S(T)$ has a unique representative $\sum u_i x_i$ in F with $x_i \in T$ and $u_i \in \{1, ..., p-1\}$; define $v(x) = \min\{v(x_i)\}$. Note that if T is a finite pole (linearly ordered) with n elements, then $S(T)$ is a cyclic valuated group with p^n elements. The valuated group $S(T)$ is indecomposable in $\mathscr{V}$ if and only if T is *irretractable* (any idempotent map from T to T is the identity) [HRW2, Theorem 7]. A group G in $\mathscr{V}$ is called *simply presented* if $G \cong \bigoplus_{i=1}^n S(T_i)$, where each T_i is a finite valuated tree.

Given two valuated trees T_1 and T_2, define $T_1 \leqslant T_2$ if there is a map of valuated trees $T_1 \to T_2$. This quasi-orders the collection of valuated trees. Two valuated trees T_1 and T_2 are *equivalent* if $T_1 \leqslant T_2$ and $T_2 \leqslant T_1$; the equivalence classes are called *v-heights*, the equivalence class of T is denoted by $[T]$. Each finite valuated tree is equivalent to a unique irretractable tree. The v-heights form a complete distributive lattice as follows. Let $\{T_i\}_{i \in I}$ be a family of valuated trees. Then $\sup_i T_i$ is the disjoint union of the trees T_i with the roots of the T_i identified to form the root of $\sup_i T_i$, and the value of this root set equal to the supremum of the values of the roots of the T_i, the values of all other nodes being left unchanged. The infimum, $\inf_i T_i$ is constructed by forming the product $\prod_i T_i$, valuating each element by the minimum of the values of its coordinates, and passing to the subtree consisting of those $(t_i) \in \prod_i T_i$ for which there is a nonnegative integer n such that $p^n t_i = \text{root } T_i$ for all i. Finally, define $T_1 \ll T_2$ if there is a map $T_1 \to T_2$ of valuated trees that sends root T_1 into $T_2 \setminus \{\text{root } T_2\}$.

If x is an element of a valuated p-group G, then the *v-height* of x in G, written $vh(x)$, is the equivalence class of $T(x)$, the tree on x. Now $vh(px) > vh(x)$ if $x \neq 0$, and $vh(x+y) \geqslant vh(x) \wedge vh(y)$ for each x, y in G [HRW1, Theorem 3.2]. Consequently, if τ is a v-height, then $G(\tau) = \{x \in G \mid vh(x) \geqslant \tau\}$ is a subgroup of G. If $T \in \tau$, let $G(T) = G(\tau) = \{x \in G \mid T(x) \geqslant T\}$. A subgroup H of a group G in $\mathscr{V}$ is *v-nice* if $\{vh(x+y) \mid y \in H\}$ contains a greatest element for each $x \in G$ [HRW1].

PROOF OF THEOREM I. Let U be in $\text{Rep}_0(P^{\text{op}})$. There is an exact sequence

$$0 \to K \to \bigoplus_{T \in P} U(T) \to U \to 0$$

of vector spaces induced by inclusion of the $U(T)$'s in U. For $T \in P$, let C^T be the direct sum of dim $U(T)$ copies of the indecomposable valuated group $S(T)$. Then $U(T)$ may be identified with the subgroup of $C^T[p]$ generated by the roots of the T's. Under this identification, the vector space K is a p-bounded subgroup of $C = \bigoplus_{T \in P} C^T$. Define $F_P(U) = C/K$. Note that $U = (\bigoplus_{T \in P} U(T))/K$ is contained in $F_P(U)[p]$. As any automorphism of $U(T)$ extends to an automorphism of C^T, isomorphism and direct sums are preserved by F_P.

If each T is a pole, then $C[p] = \bigoplus_{T \in P} U(T)$; we can define F_P on maps in this case. If $f\colon U_1 \to U_2$ is a map of representation, then f induces a map $f'\colon \bigoplus_{T \in P} U_1(T) \to \bigoplus_{T \in P} U_2(T)$, resulting in a commutative diagram of vector spaces with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & C_1[p] = \bigoplus_{T \in P} U_1(T) & \longrightarrow & U_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f' & & \downarrow f & & \\
0 & \longrightarrow & K_2 & \longrightarrow & C_2[p] = \bigoplus_{T \in P} U_2(T) & \longrightarrow & U_2 & \longrightarrow & 0
\end{array}$$

Extend f' to a homomorphism $h\colon C_1 \to C_2$, by extending each map $U_1(T) \to U_2(T)$ to a map $C_1^T \to C_2^T$. Such extensions exist because C_1^T is a direct sum of copies of the cyclic valuated group $S(T)$, and $C_1^T[p] = U_1(T)$. The extension h is unique modulo $\mathscr{A}(C_1, C_2)$ because the difference of any two extensions of f' annihilates $C_1[p]$. As T is a pole $\mathscr{A}(C_1^T, C_2^T) = p\mathscr{V}(C_1^T, C_2^T)$, so $h + \mathscr{A}(C_1, C_2)$ induces a unique map from C_1/K_1 to C_2/K_2 in $\mathscr{V}/\mathscr{A}$, which we define to be $F_P(f)$. It follows that $F_P\colon \mathrm{Rep}(P^{\mathrm{op}}) \to \mathscr{V}/\mathscr{A}$ is a well-defined additive functor.

Returning to the general case, let X be a rank-1 representation, so that each $X(T)$ is either X or 0. Then $F_P(X) = (\bigoplus_{T \in \text{type } X} S(T))/K \cong S(\sup_{T \in \text{type } X} T)$, noting that K is generated by elements of the form $(0, ..., 0, v_T, 0, ..., 0, -v_{T'}, 0, ..., 0)$ with $v = v_T = v_{T'}$ a basis element of $X = X(T) = X(T')$ for $T, T' \in \text{type } X$. ∎

Given a poset P of finite valuated trees, the functor $H_P\colon \mathscr{V}/\mathscr{A} \to \mathrm{Rep}_0(P^{\mathrm{op}})$ is defined by setting $H_P(G) = (\sum_{T \in P} G(T)[p], G(T)[p] \mid T \in P)$, and letting $H_P(f)$ be f restricted to $\sum_{T \in P} G(T)[p]$ for $f \in \mathscr{V}(G, H)$.

The proof of Theorem II uses the following theorem which we state and prove for arbitrary valuated p-groups.

THEOREM 2.2. *Let C be a valuated p-group, K' a v-nice subgroup of C, and K a subgroup of K' such that K'/K is p-bounded. If $x \in C/K$, then*

$$vh(x) = \sup\{vh(c) \mid c + K = x\}.$$

Proof. Let $\pi\colon C \to C/K$ be the natural map and $\varphi(x) = \sup\{vh(c) \mid \pi(c) = x\}$. In order to show that $\varphi(x) = vh(x)$, it is sufficient, by [HRW1, Lemma 7.1], to verify:

(i) $v(\text{root } \varphi(x)) = v(x)$,

(ii) $\varphi(x) \ll \varphi(px)$,

(iii) If T is a valuated tree with $T \ll \varphi(x)$, then there is $y \in C/K$ with $py = x$ and $T \leqslant vh(y)$.

Given $c \in C$ with $\pi(c) = x$, there is $k' \in K'$ with $vh(c+k') = \sup_{k \in K'} vh(c+k)$, since K' is v-nice in C. In particular, $K \subseteq K'$ implies $\varphi(x) \leqslant vh(c+k')$. Now K'/K is p-bounded so that $vh(pc+pk') \leqslant \varphi(px)$. Therefore $\varphi(x) \leqslant vh(c+k') \ll vh(pc+pk') \leqslant \varphi(px)$ proves (ii).

To show (i), it suffices to show that $\psi(x) = v(\text{root } \varphi(x))$ is a valuation on C/K, in which case $\psi = v$ is the valuation on C/K induced by the valuation of C [RW]. In view of the definitions of φ and ψ, the only problem is verifying that $\psi(px) > \psi(x)$. But this is a consequence of (ii), since $p^i(\text{root } \varphi(x)) = \text{root } \varphi(px)$ for some $i \geqslant 1$.

Finally, suppose $T \ll \varphi(x)$. As K' is v-nice in C, and K'/K is p-bounded, there is $c \in C$ with $\pi(c) = x$ and $T \ll vh(c)$. Now (iii) follows from the definition of $\ll$. ∎

PROOF OF THEOREM II. Let U be in $\text{Rep}_0(P^{\text{op}})$. Write $F_P(U) = C/K$, with $C = \bigoplus_{T \in P} S(T)^{\dim U(T)}$ and K the kernel of $\bigoplus_{T \in P} U(T) \to U$, as in the proof of Theorem I. Then K is a subgroup of $K' = \bigoplus_{T \in P} U(T)$ and K' is generated by the roots of the various T's. Thus, K' is a v-nice subgroup of C by [HRW2], Theorem 7.3]. As K' is p-bounded, Theorem 2.2 applies.

($\rightarrow$) Suppose that $T_1 \ll T_2$ for $T_1, T_2 \in P$. Define a rank-1 U in $\text{Rep}_0(P^{\text{op}})$ by $U = \mathbb{Z}/p\mathbb{Z}$; $U(T) = U$ if $T \leqslant T_2$, and $U(T) = 0$ otherwise. Then $\dim U(T_1) = 1$ but $\dim F_P(U)(T_1)[p] \geqslant 2$, since $T_1 \ll T$ implies that $((S(T_1) \oplus S(T_2))/K)[p]$ has at least two independent elements (one from root T_1 and one from some power of p times the image of root T_1 in T_2) and is contained in $F_P(U)(T_1)[p]$. This contradicts the assumption that $H_P F_P(U)$ is isomorphic to U in $\text{Rep}_0(P^{\text{op}})$.

($\leftarrow$) As $U \cong K'/K$, it suffices to show that $(K'/K)(T) = F_P(U)(T)[p]$ for each $T \in P$. Accordingly, let $x \in F_P(U)(T)[p]$, so $vh(x) \geqslant \tau = [T]$. As P is a weak antichain, $C(T) = C(T)[p] = \bigoplus_{T' \geqslant T} U(T') \subseteq K'$. In particular, $C(T)/K \subseteq (K'/K)(T) \subseteq F_P(U)[p]$. By Theorem 2.2,

$$vh(x) = \sup\{vh(c) \mid c + K = x\}.$$

Since v-heights form a distributive lattice, it follows that

$$\tau = \sup\{\tau \wedge vh(c) \mid c + K = x\}.$$

Each $\tau \wedge vh(c)$ is in the lattice of v-heights generated by P. To see this, observe that K' v-nice in C implies that $(C/K')(T) = (C(T) + K')/K' = 0$, since $C(T) \subseteq K'$. In particular, c is in K' so that $vh(c)$ is the meet of the v-heights of elements of P corresponding to nonzero coordinates of c in K'. But P is join-irreducible, so $\tau \wedge vh(c) = \tau$ for some c with $c + K = x$. Thus $vh(c) \geqslant \tau$, and $x \in C(T)/K \subseteq (K'/K)(T)$, as desired. ∎

PROPOSITION 2.3. *Let Q be a finite poset. There is a poset of valuated poles anti-isomorphic to Q that is a join-irreducible weak antichain.*

Proof. Let Q be $\{q_0, q_1, ..., q_n\}$. For $i \in \{0, 1, ..., n\}$, define a pole $T_i = \{t_i^0, ..., t_i^n\}$ with $\rho(t_i^j) = t_i^{j+1}$ for $0 \leqslant j < n$, and valuate T_i by $v(t_i^j) = 2j + 1$ if $q_i \leqslant q_j$, and $v(t_i^j) = 2j$ otherwise. If $q_i \leqslant q_k$, then $v(t_i^j) = 2j + 1$ whenever $v(t_k^j) = 2j + 1$, so $T_k \leqslant T_i$. Conversely, if $T_k \leqslant T_i$, then $2i + 1 = v(t_i^i) \leqslant v(t_k^i)$ so $q_i \leqslant q_k$. It follows that $P = \{T_i \mid q_i \in Q\}$ is a poset anti-isomorphic to Q.

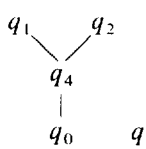
Clearly P is a weak antichain; it remains to show that P is join-irreducible. Suppose $T_i \leqslant \sup_{k \in I} T_k$ for some $I \subseteq \{0, 1, ..., n\}$. Then, by definition of supremum, there exists $k \in I$ such that $v(t_i^j) \leqslant v(t_k^j)$ for each $j < n$. If $i < n$, then $2i + 1 = v(t_i^i) \leqslant v(t_k^i)$, so $q_k \leqslant q_i$ whence $T_i \leqslant T_k$. If $i = n$ then either $q_k \leqslant q_n$ whence $T_n \leqslant T_k$, or $v(t_k^n) = 2n$ so $T_n \leqslant T_k$ by the choice of k. ∎

We give examples of the construction of the poles in Proposition 2.3.

EXAMPLE 2.4. (a) Suppose that Q is a poset of 5 pairwise incomparable elements. Then

1	0	0	0	0
2	3	2	2	2
$T_0 = 4$	$T_1 = 4$	$T_2 = 5$	$T_3 = 4$	$T_4 = 4.$
6	6	6	7	6
8	8	8	8	9

(b) Suppose that Q is the poset with Hasse diagram



Then

1	0	0	0	0
3	3	2	2	3
$T_0 = 5$	$T_1 = 4$	$T_2 = 5$	$T_4 = 4$	$T_4 = 5.$
6	6	6	7	6
9	8	8	8	9

The following example shows that $H_P F_P(U)$ need not be isomorphic to U in $\mathrm{Rep}_0(P^{\mathrm{op}})$, even for valuated poles. These poles form a weak antichain but are not join-irreducible.

EXAMPLE 2.5. Let $Q = \{1, 2, 3\}$ be a poset of three pairwise incomparable elements and

$$U = (kx \oplus ky, U(1) = kx, U(2) = ky, U(3) = k(x+y)) \in \operatorname{Rep}(Q).$$

Define a poset P of valuated poles $\{T_1, T_2, T_3\}$ anti-isomorphic to Q by

$$T_1 = \begin{matrix} 2 \\ 6 \\ 7 \end{matrix} \qquad T_2 = \begin{matrix} 2 \\ 5 \\ 8 \end{matrix} \qquad T_3 = \begin{matrix} 2 \\ 4 \\ 9 \end{matrix}.$$

Let $C = S(T_1) \oplus S(T_2) \oplus S(T_3)$ and $G = F_P(U) = C/K$, where K is the kernel of the map $U(1) \oplus U(2) \oplus U(3) \to U$ and $S(T_i)[p] = U(i)$. Choose a generator a_i for each cyclic group $S(T_i)$. A routine computation of v-heights shows that the v-heights of the respective elements are

$$p^2a_1 + K = \begin{matrix} 2 \\ 6 \\ 8 \end{matrix} \qquad p^2a_2 + K = \begin{matrix} 2 \\ 5 \\ 8 \end{matrix} \qquad p^2a_3 + K = \begin{matrix} 2 & & 2 \\ 4 & & 5 \\ & 9 & \end{matrix} = \begin{matrix} 2 \\ 5 \\ 9 \end{matrix}$$

observing that K is the subgroup of C generated by $(p^2a_1, p^2a_2, -p^2a_3)$. Thus, $G(T_2)[p]$ has dimension greater than 1 while $C(T_2)[p] = U(2)$ has dimension 1.

3. EXAMPLES AND APPLICATIONS

We give a simplified and corrected version of the construction of $G = I_n(U)$ in [A]. Construct $P_n = \{T_1, ..., T_n\}$ by applying Proposition 2.3 to a poset $\mathscr{Q}$ of n mutually incomparable elements (see Example 2.4(a)). Let $U \in \operatorname{Rep}(P_n)$ be of dimension k. Set $G = (\mathbb{Z}/p^n\mathbb{Z})^k$ with $U = G[p]$. If $x \in G$ is nonzero of height m, let $v(x) = 1 + 2m$ if $\mathbb{Z}x \cap U(T_m) \neq 0$, and $v(x) = 2m$ otherwise. If $U \in \operatorname{Rep}_0(P_n)$, then $H_{P_n}I_n(U) = U$. As $U = G[p]$, if U is indecomposable then so is G.

Let P be a finite join-irreducible poset of valuated trees that is a weak antichain. Proposition 2.3 shows that we can realize any abstract finite poset by such a P. Then $H_PF_P(U) \cong U$ for each U in $\operatorname{Rep}_0(P^{\mathrm{op}})$. For each $T \in P$, the valuated group $S(T)$ is simply presented, and $F_P(U) = C/K$, where $C = \bigoplus_{T \in P} S(T)^{\dim U(T)}$ and K is the kernel of the map $\bigoplus_{T \in P} U(T) \to U$. In general, $G = F_P(U)$ is decomposable for an indecomposable U, and U is properly contained in $G[p]$. However, as a special case of the following proposition, $G = F_P(U)$ has an indecomposable summand A with $H_P(A) \cong H_P(G)$ modulo A.

Proposition 3.1. *Let P be a join-irreducible poset of finite valuated trees such that P is a weak antichain, and let $U=\sum_{T\in P} U(T)$ be indecomposable in* $\mathrm{Rep}(P^{\mathrm{op}})$.

(a) *There is an indecomposable summand of $F_P(U)$ in $\mathscr{V}$, unique up to isomorphism, containing $U=\sum_{T\in P} F_P(U)(T)[p]$.*

(b) *If U has rank 1, then this summand is isomorphic to $S(T_0)$, for T_0 the unique irretractable tree equivalent to* $\sup\{T \mid U(T)\neq 0\}$.

Proof. (a) By Theorem II we can identify U with $\sum_{T\in P} F_P(U)(T)[p]$. Write $F_P(U)$ as a direct sum of indecomposables in $\mathscr{V}$. As U is indecomposable, and fully invariant in $F_P(U)$, one of the summands contains U. Because $\mathscr{V}$ is a Krull–Schmidt category, this summand is unique up to isomorphism.

(b) Let $T'=\sup\{T\in P \mid U(T)\neq 0\}$. By construction, $F(U)=S(T')$. As T_0 is a retract of T', and is irretractable, $S(T_0)$ is a summand of $S(T')$ and is indecomposable [HRW2]. Clearly $S(T_0)$ contains U. ∎

The following is a description of the groups in $\mathscr{V}$ that are in the image of F_P for some finite poset P.

Corollary 3.2. *Let G be in $\mathscr{V}$, and P a finite poset of finite valuated trees. Then there is U in $\mathrm{Rep}(P^{\mathrm{op}})$ with $F_P(U)\cong G$ if and only if $G\cong C/K$, where $C=\bigoplus_{T\in P} S(T)^{u_i}$, and $K\subseteq K'=\bigoplus_{T\in P} W_T$ where W_T is the subgroup of C generated by the copies of the root of T in $S(T)^{u_i}$.*

In this case there is an exact sequence $0\to A\to G\to B\to 0$ in $\mathscr{V}$ with A p-bounded and B simply presented.

Proof. ($\to$) This is just the construction of F.

($\leftarrow$) If $K\cap W_T\neq 0$, then it is a valuated summand of both of the p-bounded valuated groups K and W_T. Thus, we may assume that each $K\cap W_T=0$, since $S(T)$ modulo the subgroup generated by the root of T is again simply presented.

Define U in $\mathrm{Rep}(P^{\mathrm{op}})$ by letting $U(T)$ be the image of W_T in G, and $U=\sum_{T\in P} U(T)$. The assumption that $K\cap W_T=0$ guarantees that $W_T\cong U(T)$, whence $F_P(U)\cong G$ by the construction of F.

For the last statement of the corollary, let $A=K'/K$, a p-bounded group and $B=C/K'$, a simply presented group. ∎

Example 3.3 (L. Hughes). Let $G=(\mathbb{Z}/p^4\mathbb{Z})a\oplus(\mathbb{Z}/p^3\mathbb{Z})c$ and valuate G by setting

$$\text{values of } p^i a=\begin{matrix}0\\3\\4\\6\end{matrix}\qquad \text{values of } p^i c=\begin{matrix}0\\2\\5\end{matrix}\qquad \text{values of } p^i b=\begin{matrix}1\\2\\4\\5\end{matrix},$$

where $b = a - c$. Then G is not in the image of F for any finite poset P of finite valuated trees.

Proof. The group G can be represented by a *hang diagram*,

$$\begin{array}{ccc} a & & b \\ 0 & & 1 \\ 3 & & 2 \\ 4 & — 5 — & 4 \\ & \diagdown \; \diagup & \\ & 6 & \end{array}$$

where $p^2c = p^2a - p^2b$ has value 5 while $v(p^2a) = v(p^2b) = 4$. Moreover, G is indecomposable and not simply presented. Note that $G[p] = \langle p^3a \rangle \oplus \langle p^2c \rangle$ with

$$T_1 = T(p^3a, G) = T(p^3b, G) = \begin{array}{cc} 0 & 1 \\ 3 & 2 \\ 4 & 4 \\ \multicolumn{2}{c}{6} \end{array} \quad \text{and} \quad T_2 = T(p^2c, G) = \begin{array}{c} 0 \\ 2 \\ 5 \end{array}.$$

Since $T_1 > T_2$, $[T_1]$ and $[\mathrm{T}_2]$ are the only v-heights of elements in $G[p]$.

It now follows that G cannot be of the form C/K for $C = \bigoplus_i S(T_i)^{u_i}$ and K a subgroup of the subgroup K' generated by the roots of the various T_i's, as required by Corollary 3.2. The difficulty is that while $p^2a - p^2b$ has order p, it has value greater than $v(p^2a) = v(p^2b)$ but is not in the subgroup of G generated by the roots of T_1 and T_2. ∎

PROOF OF COROLLARY IV. Parts (a), (b), and (c) follow directly from Proposition 3.1 and Theorem II. If U is indecomposable, then define $F'_P(U)$ to be an indecomposable summand of $F_P(U)$ with $H_P F'_P(U) \cong H_P F_P(U) \cong U$ in $\mathrm{Rep}_0(P)$ as given by Proposition 3.1. Using the fact that $\mathrm{Rep}_0(P)$ is a Krull–Schmidt category guarantees that F'_P may be extended to a well-defined correspondence satisfying (a), (b), and (c).

(d) Let $G = F'_P(U)$. Then $U(T') = G(T')[p]$ for each T' in P by (c). Since $[\mathrm{T}] = [\sup\{T' \in P \mid T' \in \tau\}]$, we have $\bigcap_{T' \in \tau} G(T')[p] = G(T)[p]$. Thus, the identity map induces a vector space epimorphism

$$U(\tau)/(U(\tau) \cap U[\tau]) \to G(T)[p]/(G(T)[p] \cap G(T^*)[p]),$$

noting that $U(\tau) = \bigcap_{T' \in \tau} U(T') = G(T)[p]$ and $U[\tau] = \sum_{\neg(\tau' \leqslant \tau)} U(\tau')$ by Lemma 1.2. This proves that $B(U, \tau) \geqslant U_T(G)$.

Conversely, note that the image of $U(\tau) \cap U[\tau]$ in $G(T)[p]$ is

$G(T)[p] \cap G_P(T^*)$, where $G_P(T^*) = \sum \{G(T')[p] \mid T'$ is a join of trees in P, and $\neg(T' \leqslant T)\}$. On the other hand,

$$G_P(T^*) = \sum \{G(T') \mid T' \text{ is a join of trees in } P, \text{ and } \neg(T' \leqslant T)\}$$

$$= \sum \{G(T'') \mid T'' \in P \text{ and } \neg(T'' \leqslant T)\},$$

since if T' is a join of trees in P, and $\neg(T' \leqslant T)$, then $G(T') \subseteq G(T'')$ for some T'' in P such that $\neg(T'' \leqslant T)$. Consequently, $G_P(T^*)$ is p-bounded, as in the proof of Theorem II. Thus

$$\text{image } U(\tau) \cap U[\tau] = G(T)[p] \cap G_P(T^*)[p],$$

$$U(\tau)/(U(\tau) \cap U[\tau]) \cong G(T)[p]/G(T)[p] \cap G_P(T^*)[p],$$

as desired.

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