The solution of the bipartite analogue of the Oberwolfach problem

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In memory of my teacher and friend Egmont Köhler.

Abstract


Let $F$ be a 2-regular graph. We prove that the complete bipartite graph $K_{n,n}$ can be decomposed into pairwise edge-disjoint factors (spanning subgraphs) which are isomorphic to $F$ if and only if the trivial necessary conditions are fulfilled and $F$ is not a vertex-disjoint union of two cycles of length 6. This solves the bipartite analogue of the well-known Oberwolfach problem.

In order to prove this in the case $n = 2 \mod 4$ (the case $n = 0 \mod 4$ is completely known and easy to prove (Häggkvist (1985)); for $n = 2 \mod 4$ only decompositions in cycles whose lengths are divisible by 4 are known to exist (Alspach and Häggkvist (1985)) we introduce the definition of 'pathlike factorisations' and answer the question of the existence of regular pathlike factorisations completely.

Pathlike factorisations with some modifications are applicable to many decomposition problems; as an example we state that the complete graph $K_n$ can be decomposed into edge-disjoint copies of a 2-regular factor if $n = 2t + 3 \mod (4t + 4)$ with $t \geq 2$ holds and the factor contains no cycle having a length less than $(2t + 1)(4t + 5) + 1$.

0. Notations

Sets 0.1. (i) Throughout this paper $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{N}_n$ the subset $\{i \in \mathbb{N} \mid 0 \leq i < n\}$ for $n \in \mathbb{N}$.

(ii) Families of elements are often considered as being a representative of a list of elements (multiset). Two lists (represented by) $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are said to be equal if and only if there is a bijection $\sigma : I \rightarrow J$ with $a_i = b_{\sigma(i)}$ for all $i \in I$ and the analogue definition applies if a list is given by a $n$-tuple.
Graphs 0.2. (i) Let $G$ be a graph. Then $V(G)$ and $E(G)$ denote the set of its vertices and the set of its edges, respectively. We always assume that $E(G)$ is a subset of $\{(x, y) \mid x \neq y, x, y \in V(G)\}$, i.e. all our graphs are *simple* graphs.

(ii) Path $(x_0, \ldots, x_e)$ denotes the path of the length $e$ with the edges $\{x_i, x_{i+1}\}$ ($i = 0, \ldots, e - 1$) and $e + 1$ pairwise distinct vertices $x_0, \ldots, x_e$; we say that this path joins $x_0$ with $x_e$ ($e = 0$ is allowed). Similarly, cycle $(x_0, \ldots, x_{k-1})$ denotes the cycle with the $k$ edges $\{x_i, x_{i+1}\}$ ($i = 0, \ldots, k - 2$, $x_{k-1}, x_0$) ($k$ must be $\geq 3$ and $x_i \neq x_j$ whenever $i \neq j$).

(iii) For a graph $G$ and a non-empty set $M$ let $G(M)$ denote the graph with $V(G(M)) = V(G) \times M$, $E(G(M)) = \{(a, x), (b, y) \mid (a, b) \in E(G), x, y \in M\}$; for $n \geq 1$ we use also the notation $G(n)$ for any graph $G(M)$ with $|M| = n$.

(iv) We use the standard notation $K_n$ for a complete graph on $n$ vertices, $K_{n,n}$ for a complete bipartite graph on two independent sets of each $n$ vertices, and $P_e$ and $C_e$ for the path, respectively cycle, with $e$ edges. $C_{a_1, a_2, \ldots, a_k}$ denotes each graph which is a vertex-disjoint union of $k$ cycles $C_{a_1}, C_{a_2}, \ldots, C_{a_k}$.

Factorisations 0.3. (i) If $G$, $F_i$ ($i \in I$) are graphs then the notation $G = \sum_{i \in I} F_i$ always means that $G$ is the union of the graphs $F_i$ and these $F_i$ are pairwise edge-disjoint. If $G = \sum_{i \in I} F_i$ holds and each $F_i$ is a factor of $G$ (i.e. $V(F_i) = V(G)$), we say also that $(F_i)_{i \in I}$ is a factorisation of $G$.

(ii) If $F, G$ are graphs such that a family $\Phi = (\sigma_i)_{i \in I}$ of the bijections $\sigma_i : V(F) \to V(G)$ exists with $G = \sum_{i \in I} \sigma_i(F)$ we write $F \parallel G$ and call $(\Phi, F)$ a factorisation of $G$ into isomorphic parts. If such a family $\Phi$ does not exist we write $FMG$.

1. Introduction and known results

The well-known Oberwolfach problem [3] can be formulated as follows.

**Conjecture.** Let $F$ be a 2-regular factor of a $K_n$. Then $F \parallel K_n$ if and only if $n$ is an odd integer and $F$ is not isomorphic to a $C_{4,5}$ or a $C_{3,3,5}$.

Of course, the 'problem' is to decide whether the above conjecture is true or false. (The first 'exceptional case', namely $C_{3,3,5} \parallel K_6$, has been proven by several authors; in an unpublished paper [6] the author proved $C_{3,3,5} \parallel K_{11}$ with the aid of a computer.)

In this paper we investigate the (much easier) bipartite variant of the Oberwolfach problem; we prove the following.

**Theorem 1.1.** Let $F$ be a 2-regular graph, i.e. $F = C_{a_1, a_2, \ldots, a_s}$ with integers $s \geq 1$, $a_i \geq 3$. Then $F \parallel K_{n,n}$ if and only if the following conditions are satisfied:

(i) $n$ and all $a_i$ are even integers with $\sum a_i = 2n$,

(ii) $F$ is not isomorphic to a $C_{6,6}$.
In the case \( n = 0 \) (4) this theorem is known and seems to have been first published by Häggkvist [4]. For \( n = 2 \) (4) the fact \( C_{6,6} \parallel K_{6,6} \) is given without proof in [5], and in [1] the authors proved \( C_{a_1, \ldots, a_k} \parallel K_{n,n} \) for \( n = 2 \) (4) under the additional restriction \( a_i \equiv 0 \) (4) for all \( i \)—and, of course, \( n = 2 \) (4), \( \sum a_i = 2n \), \( a_i \geq 4 \). For reasons of completeness and as the proofs for the case \( n = 0 \) (4) and of \( C_{6,6} \parallel K_{6,6} \) are short and easy we will give them below in this section. The proof of the constructive part of Theorem 1.1 for \( n = 2 \) (4) requires new ideas (we do not use any result of [1]) and is more complicated. This proof will be furnished at the end of Section 4.

Section 5 contains further investigation of the structures used in the proof of Theorem 1.1, and finally in Section 6 we give an application to the original Oberwolfach problem.

As already indicated we intend here to prove the 'easy' parts of Theorem 1.1. First note that the necessary conditions stated in the theorem are trivial with the following exception.

**Lemma 1.2.** \( C_{6,6} \parallel K_{6,6} \).

**Proof.** Otherwise, a factorisation \( F_1 + F_2 + F_3 = C_4(3) = K_{6,6} \) with \( F_1 = C_{6,6} \) exists. The first factor \( F_1 = C_{6,6} \) of \( C_4(3) \) can be chosen arbitrarily (There is an—up to the naming of the vertices—unique graph '\( K_{6,6} - C_{6,6} \)'). Now compare with Fig. 1 where the graph \( F_1 \) we choose is given. (The vertex in the line 'i' and column 'j' gets the name \((i, j)\), vertices with the same name must be identified; one only has to check \( F_1 = C_{6,6} \) and \( F_1 + (F_2 + F_3) = \text{cycle (0, 1, 2, 3)(N,)} = C_4(3) \)). Let \( A := \text{path (1, 2)(N,)} \) and \( B := \text{path (3, 0)(N,)} \). Then \( A \) and \( B \) are subgraphs of \( F_2 + F_3 \). One of the two factors of \( F_2 + F_3 \), say \( F_2 \), must contain at least five of the nine edges of \( A = K_{3,3} \). Since any subgraph of the \( K_{3,3} \) with at least five edges is a connected graph or contains a \( C_4 \), we conclude that \( A \cap F_2 \) is a connected graph. It follows \( A \cap F_2 \approx C_6 \) and then \( B \cap F_2 \approx C_6 \). Thus \( F_2 \) is a subgraph of \( A \cup B \) and \((A \cup B) - F_2 \) consists of the edges \( \{(1, x), (2, \sigma(x))\}, \{(3, x), (0, \mu(x))\} \) \( x \in N, \) \( \) \( Fig. 1. \)}
where $\sigma, \mu$ are certain permutations on $\mathbb{N}_3$. Hence $E(F_3) = E(F_2 + F_3) - E(F_2)$ is the union of

\[\text{path}((0, x), (1, x), (2, \sigma(x)), (3, \sigma(x)), (0, \mu \sigma(x))) \quad (x \in \mathbb{N}_3)\]

(where in the case $x = \mu \sigma(x)$ 'path' must be replaced by 'cycle'). But this means that $F_3$ must consist of cycles of length 4, 8, or 12, in contradiction to $F_3 = C_{6,6}$. □

The following lemma is the fundamental observation in [4].

**Lemma 1.3.** Let $(a_1, a_2, \ldots, a_s)$ be a list of integers and $s \geq 1$. Then

$$C_{a_1, a_2, \ldots, a_s} \| C_n(2)$$

holds with even integers $\geq 4$.

**Proof.** If $F$ is a 2-regular factor of a $C_n(2)$ and $F$ contains a cycle of odd length then one sees easily that $F$ must be isomorphic to a $C_{n,n}$ with odd $n$. But in this case $F - C_{n,n}$ is isomorphic to a $C_{2n}$. Therefore $C_{a_1, \ldots, a_s} \| C_n(2)$ implies that all $a_i$ are even. The other conditions on the $a_i$ given in the lemma are trivial consequences of the definition of 'cycle' and 'factor'. Now it remains to show the constructive part of the lemma. Whenever $a_1, a_2, \ldots, a_s$ are given with $\sum a_i = 2n$ and even $a_i \geq 4$ then one can easily check that a factor $F = C_{a_1, \ldots, a_s}$ of $G = \text{cycle}(0, \ldots, n - 1)(\mathbb{N}_2) = C_n(2)$ exists such that $F$ contains the four edges $\{(0, 0), (1, x)\}, \{(n - 1, x), (0, 1)\}$ ($x = 0, 1$). Such a factor $F$ has the property that $G - F$ is isomorphic to $F$. (The reader may make a drawing in order to check the last two assertions). Thus we have proved the lemma. □

**Lemma 1.4.** If $n \equiv 0 \mod 4$ then $C_n \| C_{4}(n/4)$.

**Proof.** The assertion is a special case of a lemma which we will prove later (Corollary 5.7). But it can also be proven easily by an explicit construction which we leave to the reader as an exercise. □

Now let $\sum_{i=1}^s a_i = 2n$, all $a_i$ even and $\geq 4$, $s \geq 1$, $n \equiv 0 \mod 4$. Then $C_{a_1, \ldots, a_s} \| C_n(2)$ by Lemma 1.3. Lemma 1.4 gives $C_n(2) \| C_{4}(n/4)$ (generally, for all integers $k, l \equiv 1$ and graphs $A, B$ one has obviously $(A(k))(1) = A(kl)$ and $A \| B$ implies $A(k) \| B(k)$). Since ‘$\|$’ is transitive, $C_{a_1, \ldots, a_s} \| C_{4}(n/2)$. Because of $C_{4}(n/2) = K_{n,n}$ the proof of the case $n \equiv 0 \mod 4$ of Theorem 1.1 is now completed.

2. Pathlike and arithmetical factors

In this section we consider 'pathlike factorisations' of a $P_n(n)$ and show how 'arithmetical pathlike factors' can be used to construct pathlike factorisations
The bipartite analogue of the Oberwolfach problem.

In the next section we will modify these pathlike factorisations of a $P_e(n)$ to factorisations of a $C_e(n)$ into isomorphic 2-regular graphs and our aim, the proof of Theorem 1.1 in the case $n = 2 (4)$, will be reached finally in Section 4 (note $C_4(n/2) = K_{n, n}$ for $n = 0 (2)$).

**Definition 2.1.** Let $n, e > 0$. A factor $F$ of $P = \text{path}(0, \ldots, e)(\mathbb{N}_n)$ is called a pathlike factor of $P$ if the following conditions are satisfied:

(i) The maximal degree of a vertex in $F$ is less than or equal to 2;

(ii) $|E(F) \cap E(\text{path}(i, i + 1)(\mathbb{N}_n))| = n$ for all $i \in \{0, \ldots, e - 1\}$;

(iii) $F$ contains no cycle.

(Under systematic aspects 2.1 (iii) is not essential but of course it justifies the name 'pathlike'—and we need no 'pathlike factors with cycles' in this paper.) For pathlike factors we must introduce some further technical notations. (Examples can be taken from Fig. 2.)

**Definition 2.2.** Let $F$ be a pathlike factor of $P = \text{path}(0, \ldots, e)(\mathbb{N}_n)$ with $e, n > 0$.

(i) The maximal connected subgraphs of $F$ are called its paths.

(ii) The vertices of degree $n$ in $P$ are called the endvertices of $P$. The endvertices $(0, x)$ are called the top-vertices and the (remaining) endvertices $(e, x)$ the bottom-vertices ($x \in \mathbb{N}_n$).

(iii) We call a path of $F$ a top-to-bottom path (briefly TB-path) if it joins a top-vertex with a bottom-vertex; we call it a top-to-top path (TT-path) if it joins top-vertices and finally a bottom-to-bottom path (BB-path) if it joins bottom-vertices.

![Fig. 2.](image-url)
(iv) Let \( L_{TB} := (|E(P)|)_{P \in U_{TB}} \) where \( U_{TB} \) denotes the set of the TB-paths of \( F \) and let \( L_{TT}, L_{BB} \) the analogously defined lists of the pathlengths of the TT- and BB-paths, respectively. Then the triple \((L_{TB}, L_{TT}, L_{BB})\) is called the type of \( F \).

Some remarks should be given. First of all we should emphasise that paths of length 0 are allowed. Now let \( F \) be a pathlike factor of \( P = \text{path}(0, \ldots, e)(\mathbb{N}_n) \). From 2.1 (i) and (ii) we deduce easily that each vertex which is no endvertex of \( P \) must have the degree 2 in \( F \). Therefore each path of \( F \) is a path in the usual sense (with a length \( \geq 0 \)) and it is either a TB-, a TT-, or a BB-path, and a path of \( F \) with the length 0 is necessarily either a TT- or a BB-path. (It can be seen easily too that the number of TT-paths always equals the number of BB-paths, but we do not need this fact in the sequel.)

**Definition 2.3.** A factorisation \( \Phi = ((\sigma_i)_{i \in I}, F) \) of \( Q = \text{path}(0, \ldots, e)(\mathbb{N}_n) \) with \( e, n > 0 \) is called a pathlike factorisation if each factor \( \sigma_i(F) \) is a pathlike factor and \( \sigma_i^{-1}(v) \) is independent of \( i \in I \) for each endvertex \( v \) of \( Q \); the type of \( \Phi \) is the type of any factor \( \sigma_i(F) \) (which is necessarily independent of \( i \in I \)).

Note, if \( \Phi = ((\sigma_i)_{i \in I}, F) \) is a pathlike factorisation of \( \text{path}(0, \ldots, e)(\mathbb{N}_n) \) and if \( u, w \) are (end-)vertices which are joined by a path \( U \) of a factor \( \sigma_i(F) \) then there is a path of the same length in each factor \( \sigma_i(F) \) which joins \( u \) to \( w \) (namely \( \sigma_i \sigma_i^{-1}(U) \)), especially, all factors \( \sigma_i(F) \) have the same type.

An endvertex \( v \) has necessarily in all factors the same degree (namely the degree of \( \sigma_i^{-1}(v) \) in \( F \), independent of \( i \in I \)); since the degree of an endvertex in a \( P_s(n) \) is \( n \) and since there are \( n \) factors, this degree must be 1. (Especially there are no factors with paths of length 0 in a pathlike factorisation).

In an obvious sense the above definitions will be used also in arbitrary labelled graphs \( G = P_s(n) \) (with \( e, n > 0 \)); of course ‘topvertex’, ‘type of a pathlike factor’ etc. depends in general on the chosen isomorphism \( G \rightarrow \text{path}(0, \ldots, e)(\mathbb{N}_n) \).

Now we give the general definitions which will be used later to construct many pathlike factorisations.

**Definition 2.4.** Let \( F \) be a factor of \( G(\mathbb{N}_n) \) where \( G \) is a graph and \( n > 0 \) and let \( D_{a,b} \) be the list of the differences \( x - y \) with \( \{(a, x), (b, y)\} \in E(G) \) for all \( (a, b) \in E(G) \) (i.e. \( D_{a,b} \) contains the integer \( d \) exactly \(|\{(a, x), (b, x + d)\} \in E(F)\}| \) times).

(i) \( F \) is called a rotational factor if for all \( (a, b) \) with \( \{(a, x), (b, y)\} \in E(G) \) the list \( D_{a,b} \) is a list of \( n \) pairwise incongruent integers modulo \( n \).

(ii) \( F \) is called an arithmetical factor if all lists \( D_{a,b} \) with \( \{(a, x), (b, y)\} \in E(G) \) are equal to the list \( D_n := (- (n - 1), -(n - 3), \ldots, (n - 3), (n - 1)) \).

Note, that \( D_{a,b} \) is defined as a list of integers (not as a list of classes of residues modulo \( n \)). If \( n \) is odd then the list \( D_n \) consists of \( n \) pairwise incongruent integers mod \( n \): An arithmetical factor of a \( G(\mathbb{N}_n) \) with odd \( n \) is a special case of a rotational factor.
Lemma 2.5. Let $G$ be a graph and let $F$ be a rotational factor of $G(N_n)$. Then \(((\sigma_i)_{i \in \mathbb{N}_n}, F)\) with $\sigma_i(v, x) = (v, x + i)$ for all $(v, x) \in V(G)$ and $i \in \mathbb{N}_n$ (where $x + i$ is calculated modulo $n$) is a factorisation of $G(N_n)$.

**Proof.** This lemma follows by detailed verifications. \(\square\)

Lemma 2.6. Let $G$ be a graph containing no cycle of odd length and let $F_1, F_2$ be arithmetical factors of $G(N_{n_1})$ and $G(N_{n_2})$, respectively. Then an arithmetical factor $F = F_1 \cup F_2$ with $V(F) = V(F_1) \cap V(F_2) = \emptyset$ of $G(N_{n_1+n_2})$ exists.

**Proof.** As $G$ has no odd cycles it is known (and easy to show) that a partition $V(G) = V_1 + V_2$ can be chosen such that each edge of $G$ joins a vertex of $V_1$ with a vertex of $V_2$. (In other words, we use the fact that the chromatic number of such a graph is at most 2.)

Consider the mappings $\tau_1 : V(F_i) \rightarrow V(G(N_{n_1+n_2}))$ for $i = 1, 2$ defined by

$$
\tau_1(a, x) := (a, x), \quad \tau_1(b, x) := (b, n_2 + x),
$$

$$
\tau_2(a, y) := (a, n_1 + y), \quad \tau_2(b, y) := (b, y)
$$

for all $a \in V_1$, $b \in V_2$, $0 \leq x < n_1$, $0 \leq y < n_2$. Then it can be easily verified that $F := \tau_1(F_1) \cup \tau_2(F_2)$ is an arithmetical factor of $G(N_{n_1+n_2})$, and $\tau_1$, $\tau_2$ are injections with $V(\tau_1(F_1)) \cap V(\tau_2(F_2)) = \emptyset$. \(\square\)

Mainly as a special case of Lemma 2.6 we get a method to construct arithmetical pathlike factors recursively.

Lemma 2.7. The existence of arithmetical pathlike factors $F, F'$ with the types \(((a_1, \ldots, a_r), (b_1, \ldots, b_s), (c_1, \ldots, c_t))\) in a $P_e(n)$ and \(((a'_1, \ldots, a'_w), (b'_1, \ldots, b'_v), (c'_1, \ldots, c'_w))\) in a $P_e(n')$, respectively, implies the existence of an arithmetical pathlike factor with the type

\(((a_1, \ldots, a_r, a'_1, \ldots, a'_w), (b_1, \ldots, b_s, b'_1, \ldots, b'_v), (c_1, \ldots, c_t, c'_1, \ldots, c'_w))\)

in a $P_e(n + n')$.

**Proof.** Use the construction of Lemma 2.6. \(\square\)

The following slight modification of Lemma 2.5 shows how rotational pathlike factors of a $P_e(n)$ (especially arithmetical pathlike factors if $n$ is odd) can be used to construct pathlike factorisations of a $P_{e+2}(n)$.

Lemma 2.8. Let $F$ be a rotational pathlike factor of a $P_e(n)$ with the type \(((a_1, \ldots, a_r), (b_1, \ldots, b_s), (c_1, \ldots, c_t))\). Then a pathlike factorisation of a $P_{e+2}(n)$ with the type \(((a_1 + 2, \ldots, a_r + 2), (b_1 + 2, \ldots, b_s + 2), (c_1 + 2, \ldots, c_t + 2))\) exists.
Proof. We may assume that $F$ is a factor in $\text{path}(0, 1, \ldots, e)(\mathbb{N}_n)$ with the type which is given above. Let $g_H(v)$ denote the degree of the vertex $v$ in a graph $H$. As the set $E(F) \cap E(\text{path}(0, 1)(\mathbb{N}_n))$ contains $n$ edges and since this set is equal to the set of edges of $F$ which are incident with one vertex in $\{0\} \times \mathbb{N}_n$ we have $\sum_{x \in \mathbb{N}_n} (2 - g_F(0, x)) = n$. Since $2 - g_F(0, x) \geq 0$ for all $x \in \mathbb{N}_n$ clearly a factor $F_0$ of $\text{path}(-1, 0)(\mathbb{N}_n)$ exists with $g_{F_0}(-1, x) = 1, g_{F_0}(0, x) = 2 - g_F(0, x)$ for all $x \in \mathbb{N}_n$. By reasons of symmetry there is also a factor $F_e$ of $\text{path}(e, e + 1)(\mathbb{N}_n)$ with $g_{F_e}(e, x) = 2 - g_F(e, x), g_{F_e}(e + 1, x) = 1$ for all $x \in \mathbb{N}_n$.

Now consider $F := F_0 \cup F \cup F_e$. One sees that $F$ is a pathlike factor of $\text{path}(-1, 0, \ldots, e, e + 1)(\mathbb{N}_n)$ and has the desired type (with the vertices $(-1, x)$ and $(e + 1, x)$ as new ‘bottom-vertices’ and new ‘top-vertices’, respectively). Furthermore, $((\sigma_i)_{i \in \mathbb{N}_n}, F)$ with $\sigma_i(v) = v$ for $v \in \{-1, e + 1\} \times \mathbb{N}_n$ and $\sigma_i(j, x) = (j, i + x)$—the second component reduced mod $n$—for all other vertices is a factorisation such that $\sigma_i^{-1}(v)$ for $v \in \{-1, e + 1\} \times \mathbb{N}_n$ is independent of $i \in \mathbb{N}_n$. 

3. Identifications and constant vertices of a factorisation

We study the relation between ‘identifications of vertices’ and ‘factorisations’ of a graph and apply the results to pathlike factorisations.

Definition 3.1. (1) An equivalence relation $\sim$ on the set $V(G)$ of a graph $G$ shall be called an identification for $G$ if it has the following properties:

(i) $x \sim y$ implies $(x, y) \notin E(G)$,

(ii) $(x, y), (x', y') \in E(G)$ with $x \sim x', y \sim y'$ implies $x = x'$ and $y = y'$.

(2) If $\sim$ is an identification for a graph $G$ then $G_\sim$ shall denote the graph with $V(G_\sim) = \{v_\sim \mid v \in V(G)\}$ and $E(G_\sim) = \{\{v_\sim, y_\sim\} \mid \{x, y\} \in E(G)\}$ (where we use the standard notation $v_\sim$ for the class which contains $v$).

Note, that if identifying of vertices in a graph $G$ means simply ‘glue equivalent vertices together, but do not identify or remove edges’, then the resulting graph $\tilde{G}$ has no loops iff 3.1 (i) holds, and it has no multiple edge iff 3.1 (ii) holds; i.e. if this type of identification yields a simple graph $\tilde{G}$ then this graph will be just $G_\sim$.

The following are the basic properties of an identification for a graph $G$.

Lemma 3.2. Let $\sim$ be an identification for a graph $G$, and let $F, F_1, \ldots$ be factors of $G$. Then the following is satisfied:

(i) $\sim$ is an identification for $F$; $F_\sim$ is a factor of $G_\sim$.

(ii) $(F_1 \cap F_2)_\sim = (F_1)_\sim \cap (F_2)_\sim$.

(iii) $(F_1 \cup F_2)_\sim = (F_1)_\sim \cup (F_2)_\sim$.

(iv) $G = \Sigma_i F_i$ implies $G_\sim = \Sigma_i (F_i)_\sim$. 


Proof. Each part of Lemma 3.2 can be immediately checked; as an example we give the formal proof of $(F_1)_- \cap (F_2)_- \subseteq (F_1 \cap F_2)_-$. If $e$ is an edge of $(F_1)_- \cap (F_2)_-$ then $\{x, y\} \in E(F_1)$ and $\{x', y'\} \in E(F_2)$ must exist with $x \sim x'$, $y \sim y'$, $e = \{x-, y-\}$. By Definition 3.1 (ii), we have $x = x'$ and $y = y'$. Hence $\{x, y\} \in E(F_1) \cap E(F_2) = E(F_1 \cap F_2)$ and $e = \{x-, y-\}$ is an edge of $F_1 \cap F_2$. (All other parts of Lemma 3.2 can be seen even easier.) □

Definition 3.3. Let $((\sigma_i)_{i \in I}, F)$ be a factorisation of a graph $G$ into isomorphic parts. Then a vertex $v \in V(G)$ is called a constant vertex of the factorisation if $\sigma_i^{-1}(v)$ is independent of $i \in I$. (In other words, $v$ is a constant vertex of $\Phi$ iff $v$ is a fixed point of $\sigma_i \sigma^{-1}_i$ for all $i, j \in I$).

Of course the property 'constant vertex' of a factorisation $((\sigma_i)_{i \in I}, F)$ into isomorphic parts is mainly a property of the family of mappings $(\sigma_i)_{i \in I}$ and not of the associated family of the factors $((\sigma_i(F))_{i \in I}$. Note that in a pathlike factorisation of a $P_e(n)$ at least all top- and bottom-vertices are constant vertices by definition.

Lemma 3.4. Let $\Phi = ((\sigma_i)_{i \in I}, F)$ be a factorisation of a graph $G$ into isomorphic parts, and let $\sim$ be an identification for $G$ such that with the notation $A := \{v \in V(G) \mid |v-| > 1\}$ the following conditions in $G$ hold:

(i) there is no edge between vertices $x, y \in A$,
(ii) each $v \in A$ is a constant vertex of $\Phi$,
(iii) equivalent vertices are never joined by a path of length 2. Then there is a factorisation $\tilde{\Phi} = ((\tilde{\sigma}_i)_{i \in I}, F)$ of $G_-$ into isomorphic parts with $\tilde{F} = (\sigma_i(F))_-$ for $i \in I$.

Proof. Let $C$ denote the set of constant vertices of $\Phi$. Definition 3.1 (i) holds for $\sim$ on $G$; $\{x, y\} \in E(G)$ with $x \sim y$ is impossible (otherwise $x \neq y$, $x, y \in A$ and Lemma 3.4 (i) gives the contradiction). Now let $\{x, y\}, \{x', y'\} \in E(G)$ be given with $x \sim x'$, $y \sim y'$. By Lemma 3.4 (i) again, one of two vertices $x, y$, say $x$, is not in $A$. Hence $x = x'$ by the definition of $A$. Because of Lemma 3.4 (iii) we get $y = y'$ and we have checked Definition 3.1 (ii) too. Hence $\sim$ is an identification for $G$ and, by Lemma 3.2, the assumption $G = \sum_i \sigma_i(F)$ implies $G_- = \sum_i (\sigma_i(F))_-$. It remains only to show $(\sigma_i(F))_- = (\sigma_j(F))_- (i, j \in I)$. The bijection $\mu_{ij} := \sigma_i \sigma^{-1}_j$ is the identity on $C$ (by definition of $C$), therefore (using Lemma 3.4 (iii)) each class $x_-$ with $|x_-| > 1$ is (even pointwise) fixed by $\mu_{ij}$. Hence $\mu_{ij}$ can be considered as a bijection on $V(G_-)$ (with $\mu_{ij}(x_-) = \mu_{ij}(x_-)$) and a straightforward check shows that this mapping is an isomorphism $\sigma_i(F)_- \rightarrow \sigma_j(F)_-$.

(Let us remark that the condition (i) in Lemma 3.4. can be removed if $\Phi = ((\sigma_i)_{i \in I}, F)$ contains $|I| > 1$ factors—and these are the interesting cases; Proof: Let $e = \{x, y\} \in E(G)$ be given with $x, y \in A$. Then exactly one $i \in I$ with $e \in (\sigma_i(F)$ must exist, and, because $x$ and $y$ are constant vertices (Lemma 3.4 (ii)), we have $e = (\sigma_i(F)^{-1}e) \in (\sigma_j(F)$ for all $j \in I$, and therefore $|I| = 1$.)
From now on we will forget about our rather formal definition of identification of vertices in a graph and speak more freely of vertices which shall be identified without explicitly using the underlying equivalence relation.

If a pathlike factorisation of a $P_e(n)$ with $e > 2$ is given and we identify each topvertex with a bottomvertex $\tau(x)$, where $\tau$ is injective, we can apply the last lemma. (Lemma 3.4 (i) is satisfied: no edge between endvertices exists in a $P_e(n)$; endvertices are constant vertices by definition of pathlike factorisation (hence Lemma 3.4 (ii) holds); and finally Lemma 3.4 (iii) follows because of $e > 2$).

The result of the identification is obviously a factorisation of a $C_e(n)$ into isomorphic 2-regular factors (in each factor $\sigma_i(F)$ of a pathlike factorisation the endvertices in the $P_e(n)$ have degree 1 and the other vertices degree 2). The resulting 2-regular factor depends only on the type of the factorisation and on the chosen identification (i.e. the mapping $\tau$).

For a further investigation it is necessary to give a technical definition.

**Definition 3.5.** Let $M = (m_1, m_2, \ldots, m_n)$ be a list of integers. Then $L = (a_1, a_2, \ldots, a_k)$ is called a contraction of $M$ if a partition $(L_i)_{i \in \{1, \ldots, k\}}$ of $\{1, 2, \ldots, n\}$ with $a_i = \sum_{j \in L_i} m_j$ exists.

**Lemma 3.6.** Let $\Phi$ be a pathlike factorisation of a $P_e(n)$, $e > 2$, with the type

$$T = ((a_1, a_2, \ldots, a_r), (b_1, b_2, \ldots, b_s), (c_1, c_2, \ldots, c_t))$$

and let $L = (d_1, d_2, \ldots, d_k)$ be a contraction of the list $(a_1, a_2, \ldots, a_r, b_1 + c_1, b_2 + c_2, \ldots, b_s + c_t)$. Then $C_{d_1, d_2, \ldots, d_k} \parallel C_e(n)$.

**Proof.** Let $\Phi = ((\sigma_i, s_i, F)$ be a factorisation of a $P_e(n)$ as assumed and let $F_0 \in \{\sigma_i(F) | i \in I\}$ be arbitrarily chosen. Then a partition of the paths of $F_0$ into sets $U_i$ ($i = 1, \ldots, k$) can be chosen such that the sum of the lengths of the paths in $U_i$ is $d_i$, and the number of top-to-top paths equals the number of bottom-to-bottom paths in $U_i$. (Choose the partition which corresponds to the partition on $T$ which is induced by the contraction. Especially, for each top-to-top path with length $b_i$ there will be a bottom-to-bottom path with length $c_j$ in $U_i$.)

A suitable identification of the top vertices with the bottom vertices in $U_i$ yields a cycle of length $d_i$. If these identifications are made for all $i \in \{1, \ldots, k\}$ the graph $P_e(n)$ becomes a $C_e(n)$, and $F_0$ (as each other factor $\sigma_i(F)$ of the factorisation—the arguments have already been given some lines before Definition 3.5) modifies to a factor $C_{d_1, \ldots, d_k}$ of the $C_e(n)$. \(\square\)

4. Final step in the proof of Theorem 1.1

First we construct five arithmetical pathlike factors. Within a list we will use here and later the notation \('n \ast x'\) which stands for \('x, x, \ldots, x'\) where $x$ is repeated $n$ times ($n \in \mathbb{N}$).
Lemma 4.1. There are arithmetical pathlike factors with the types:

- (T1) \(((1 \cdot 2), 0, 0)\) in \(P_2(1)\),
- (T2) \(((3 \cdot 4), (2 \cdot 0, 1 \cdot 2), (1 \cdot 0, 2 \cdot 2))\) in \(P_2(9)\),
- (T3) \(((2 \cdot 4), (1 \cdot 0), (1 \cdot 0))\) in \(P_2(4)\),
- (T4) \(((2 \cdot 4), (2 \cdot 0), (2 \cdot 2))\) in \(P_2(6)\).

Proof. Such factors are given in Fig. 2. The vertices of each of these factors are arranged in the form of a matrix. Number the rows and columns from top to bottom and from left to right, respectively, with 0, 1, 2... and call the vertex in row \(i\), column \(j\) the vertex \((i, j)\). □

Repeated application of Lemma 2.7 with \(2x + 1\) factors (T1), \(y\) factors (T3) and \(z\) factors (T4) yields an arithmetical pathlike factor in \(P_2(n)\) with \(n = 2x + 1 + 4y + 6z\) and with the type

\((TA)\) \(((2x + 1) \cdot 2, (2y + 2z) \cdot 4), ((y + 2z) \cdot 0), (y \cdot 0, 2z \cdot 2))\).

If one takes \(2x\) factors of the type (T1), one factor of type (T2) and \(y\) factors of type (T4) then one gets an arithmetical pathlike factor in \(P_2(n)\) with \(n = 2x + 9 + 6y\) and with the type

\((TB)\) \(((2x \cdot 2, (2y + 3) \cdot 4), ((2y + 2) \cdot 0, 1 \cdot 2), ((2y + 2) \cdot 2, 1 \cdot 0))\).

Lemma 2.8 yields the existence of corresponding pathlike factorisations of \(P_2(n)\) (add simply 2 to each element of the lists).

Finally, Lemma 3.6 yields the following.

4.2. \(C_{a_1, \ldots, a_k} \parallel K_{2n, 2n}\) with \(4n = \sum a_i\) for all \((a_i, \ldots, a_k)\) which are a contraction of a list

- \((LA)\) \(L = ((2x + 1) \cdot 4, (2y + 2z) \cdot 6, y \cdot (2 + 2), (2z) \cdot (2 + 4)) = ((2x + y + 1) \cdot 4, (2y + 4z) \cdot 6)\)

with \(n = 2x + 1 + 4y + 6z\) or of a list

- \((LB)\) \(L = ((2x) \cdot 4, (2y + 3) \cdot 6, (2y + 2) \cdot (2 + 4), 1 \cdot (4 + 2)) = ((2x) \cdot 4, (4y + 6) \cdot 6)\)

with \(n = 2x + 9 + 6y\) and arbitrary parameters \(x, y, z \geq 0\).

Now we are able to complete the proof of Theorem 1.1; we have to prove the following.

Lemma 4.3. Let \(L = (a_1, \ldots, a_k)\) be a list of integers with

- (i) \(k \geq 1\),
- (ii) \(a_i = 0 (2), a_i \geq 4\) for all \(i\),
- (iii) \(4n = \sum a_i - 4 (8)\),
- (iv) \(L \neq (6, 6)\).

Then \(C_{a_1, \ldots, a_k} \parallel K_{2n, 2n}\).
Proof. Let \( L \) be as assumed. Because of 4.2 it suffices to show that \( L \) is a contraction of a list of the form (LA) or (LB) given in 4.2 with suitable integers \( x, y, z \geq 0 \). Because of (i), (ii) we can write \( L \) as

\[
L = (4b_1, \ldots, 4b_r, 6 + 4c_1, \ldots, 6 + 4c_s)
\]

with \( b_i \geq 1, c_i \geq 0, r, s \geq 0, r + s > 0 \).

Let \( B := \sum_i b_i, C := \sum_i c_i \). Then (iii) and (iv) can be written as

\[
4n = 4B + 6s + 4C = 4 \quad (8)
\]

and

\[
B + C \neq 0 \quad \text{if } s = 2. \quad (* *)
\]

By (8), \( s \) is even. Hence there are two possibilities modulo 4.

Case \( s = 0 \) (4).

By (8), \( B + C \) must be odd. It is now plain that \( L \) is a contraction of a list which has the form (LA) with \( x := (B + C - 1)/2, y := 0 \) and \( z := s/4 \).

Again by (*), \( B + C \) must be even. If \( s = 2 \) then \( B + C \neq 0 \) by (**), and \( L \) is a contraction of a list (LA) with \( x := (B + C - 2)/2, y := 1 \) and \( z := 0 \). If \( s > 2 \) then \( L \) is a contraction of a list (LB) with \( x := (B + C)/2, y := (s - 6)/4 \).

5. Regular pathlike factorisations

In this section we present our second theorem. It concerns without doubt the most important class of pathlike factorisations which we call RPFs (the definition will be given some lines later). These RPFs are closely related to latin squares and pairs of orthogonal latin squares. (We assume the reader is familiar with the relevant definitions.) Thus it is not surprising that the famous result [2] of Bose, Shrikhande and Parker on the existence of pairs of orthogonal latin squares can be translated into a statement on RPFs. But on the other hand there is Tarry's result [7] on the non-existence of orthogonal latin squares of order 6 and this fact makes the things a bit more complicated.

Definition 5.1. A pathlike factorisation of a \( P_e(n) \) with the type \( (n * e, 0, 0) \) will be called an RPF\((e, n)\) (a regular pathlike factorisation of a \( P_e(n) \)).

In other words an RPF\((e, n)\) is a factorisation of a \( P_e(n) \) in factors which are vertex-disjoint unions of top-to-bottom paths of length \( e \) such that if \( x \) is joined to \( y \) in any factor then \( x \) and \( y \) are joined in all factors.

Theorem 5.2. Let \( e, n > 1 \). An RPF\((e, n)\) does not exist in the cases:

(i) \( e = 1 \mod 2 \) and \( n = 2 \),
(ii) \( e = 1 \) and \( n > 1 \),
(iii) \( e = 3 \) and \( n = 6 \),

but in all other cases an RPF\((e, n)\) exists.
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Proof. First we will formulate in Observations 5.3–5.5 three basic (and obvious) observations. In Observations 5.3 and 5.4 a LQ(n), a latin square of order n, is always assumed to have its row and column indices as well as its entries from the set \( \mathbb{N}_n \). Furthermore, we require in Observations 5.3 and 5.4 that an RPF is a factorisation of \( \text{path}(0, \ldots, e)(\mathbb{N}_n) \) and it is considered as ordered \( n \)-tuple \( (F_0, F_1, \ldots, F_{n-1}) \) of its factors and finally the paths of its factors always join vertices \((0, x)\) with vertices \((e, x)\). (Of course these are no real restrictions.)

**Observation 5.3.** There is a bijection between a \( \text{LQ}(n) \) and an \( \text{RPF}(2, n) \).

**Proof.** If \((a_{ij})\) is the \( \text{LQ}(n) \) then take simply path \(((0, i), (1, a_{ij}), (2, i))\) as \( i \)th path of the \( j \)th factor in \( \text{path}(0, 1, 2)(\mathbb{N}_n) \) and conversely. \( \Box \)

**Observation 5.4.** There is a bijection between an ordered pair of orthogonal \( \text{LQ}(n)s \) and an \( \text{RPF}(3, n) \).

**Proof.** If \(((a_{ij}), (b_{ij}))\) is the pair of orthogonal squares then take path \(((0, i), (1, a_{ij}), (2, b_{ij}), (3, i))\) as \( i \)th path of the \( j \)th factor in \( \text{path}(0, 1, 2, 3)(\mathbb{N}_n) \) and conversely. \( \Box \)

**Observation 5.5.** If an \( \text{RPF}(e, n) \) and an \( \text{RPF}(e', n) \) exist then an \( \text{RPF}(e + e', n) \) exists.

**Proof.** Obviously. \( \Box \)

Now we can proceed in proving the theorem. Since there are latin squares of any order \( n \geq 1 \) Observation 5.3 together with 5.5 shows the existence of \( \text{RPF}(e, n) \) in the case that \( e \) is an even integer \( >0 \). Now we consider the cases with odd \( e \). The cases with \( e = 1 \) or \( n = 2 \) can be seen immediately. Thus we may assume \( e \) odd, \( e \geq 3 \), and \( n \neq 2 \).

Since there are, with exception of the cases \( n = 2 \) or \( n = 6 \), always pairs of orthogonal \( \text{LQ}(n)s \) [2], the existence of \( \text{RPF}(3, n) \) is established if \( n \) is not 2 or 6. As there is no pair of orthogonal latin squares of order 6 [7] no \( \text{RPF}(3, 6) \) exists.

However, there is a \( \text{RPF}(5, 6) \) (we will construct it in the next lemma). Thus together with Observation 5.5. and the existence of \( \text{RPF}(e, n) \) for even \( e \) we have \( \text{RPF}(e, n)s \) for all odd \( e > 1 \) except in the cases \( n = 2 \) or \((e, n) = (3, 6)\). \( \Box \)

**Lemma 5.6.** An \( \text{RPF}(5, 6) \) exists.

**Proof.** More generally, we construct \( \text{RPF}(p, p + 1) \) for all primes \( p > 2 \). (In this way the construction of the \( \text{RPF}(5, 6) \) will be much more transparent and the necessary calculations to check the construction are in fact even easier.)

Let \( M_p := \mathbb{N}_p \cup \{\infty\} \) (\( \infty \) may be any element which is not in \( \mathbb{N}_p \)), and let \( \mu \) be a
primitive root mod $p$ and $\sigma$, $\tau$, $\Phi_i$ permutations on $M_p$ defined by

$$
\sigma(\infty) := 0, \quad \sigma(0) := \infty, \quad \sigma(x) := mx \quad \text{for } x \in M_p - \{0, \infty\};$

$$
\tau(\infty) := \infty, \quad \tau(x) := x + 1 \quad \text{for } x \in M_p - \{\infty\};$

$$
\Phi_j := \tau^j \sigma \tau^{-j} \quad \text{for all integers } j,
$$

where the arithmetics should be done modulo $p$.

These permutations have the following properties:

(P1) Each $\Phi_i$ is a fixed-point-free permutation since $\sigma$ has no fixed point.

(P2) If $j \neq 0 \mod p$ then $\tau^i \sigma^{-1} \tau^{-j} / \sigma$ has no fixed point since this permutation maps $\infty$ to $j(1 - \mu^{-1})$, 0 to $j$, $\mu^{-1}j$ to $\infty$ and the remaining elements $x$ to $x + j(1 - \mu^{-1})$. Hence, if $k \neq 1 \mod p$ the mapping

$$
\Phi_{k-1} \Phi_i = \tau^k \sigma^{-1} \tau^{-k+1} \sigma \tau^{-l} = \tau^l (\tau^{(k-1)} \sigma^{-1} \tau^{-(k-1)} \sigma) \tau^{-l}
$$

has no fixed point.

(P3) $\tau$ and $\tau^{-1} \sigma$ are of order $p$. To calculate the order $\tau^{-1} \sigma$ one observes that $\tau^{-1} \sigma$ fixes $(\mu - 1)$ and (one needs the fact that $\mu$ is a primitive root) the remaining elements form the cycle $(\infty, -1, -\mu - 1, -\mu^2 - \mu - 1, \ldots, 0)$ of length $p$. For all integers $k$ it follows

$$
\Phi_{p-1+k} \Phi_{p-2+k} \cdots \Phi_{0+k} = \tau^{p-1+k} \sigma \tau^{-1} \sigma \cdots \sigma \tau^{-(k-1)} \sigma = \tau^k (\tau^{-1} \sigma)^p \tau^{-k} = \text{id}.
$$

Now consider the graph $G := \text{path}(0, 1, \ldots, p)(M_p)$. Obviously, $G$ is a $P_p(p + 1)$. For $x \in M_p$ let $F_x$ be the factor of $G$ with

$$
E(F_0) := \{(i, x), (i + 1, x)\} \mid 0 \leq i < p, x \in M_p\},
$$

$$
E(F_x) := \{(i, x), (i + 1, \Phi_{i+k}(x))\} \mid 0 \leq i < p, x \in M_p\} \quad \text{for all } k \in M_p - \{\infty\}.
$$

We claim that $(F_x)_{x \in M_p}$ is an RPF($p$, $p + 1$).

(1) Each $F_x$ is a vertex-disjoint union of $p + 1$ paths $U_{xy}$ which join $(0, y)$ with $(p, y)$ (this is obvious in the case $x = \infty$ and follows from (P3) in the other cases).

(2) $F_x$ and $F_k$ with $k \in \mathbb{N}_p$ have no common edge (this follows from (P1)); $F_k$ and $F_l$ with $l, k \in \mathbb{N}_p$ with $k \neq l$ have no common edge because $\Phi_i^{-1} \Phi_{i+k} \Phi_{i+l}$ is a fixed-point-free permutation (see (P2) above).

(3) Finally each edge of $G$ is an edge of some $F_x$ (count the edges). $\Box$

With the technique of identifying vertices (Lemma 3.6), from Theorem 5.2 we have the following.

**Corollary 5.7.** Let $k \geq 1$, $e \geq 3$. If $a_1, a_2, \ldots, a_k$ are positive integers which are divisible by $e$ and $ne = \sum a_i$, then $C_{a_1} \cdots C_{a_k} \mid C_e(n)$ except in the cases:

(i) $n = 2$ and $e$ odd,

(ii) $n = 6$, $e = 3$ and $(a_1, \ldots, a_k) = (6*3)$

in which the contrary is true.
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Fig. 3.

Proof. The hypothesis implies that \((a_1, \ldots, a_k)\) is a contraction of the list \((n \ast e)\).

By Lemma 3.6, we have \(C_{a_1, \ldots, a_k} \parallel C_n(e)\) if an RPF\((e, n)\) exists. Because of Theorem 5.2 it remains to treat the cases \(n = 2\) with odd \(e\) and \(n = 6\) with \(e = 3\). In the case \(n = 2\) Lemma 1.3 will work.

In order to settle the case \(n = 6\) firstly one observes that the existence of an RPF\((e, n)\) with odd \(e\) is indeed equivalent to \(F \parallel C_n(e)\) if \(F\) is a union of \(n\) vertex-disjoint cycles of length \(e\). Thus the non-existence of an RPF\((3, 6)\) (Theorem 5.2) implies \(C_{a_1, \ldots, a_k} \parallel C_6(6)\).

Secondly, we have to show \(C_{a_1, \ldots, a_k} \parallel C_6(6)\) if \(L = (a_1, \ldots, a_k)\) is a contraction of \((6 \ast 3)\) and \(L \neq (6 \ast 3)\). A rotational pathlike factor of a \(P_6(6)\) with the type \(((4 \ast 1), (2), (0))\) can be easily found, see Fig. 3. Hence, by Lemma 2.8, a pathlike factorisation in a \(P_6(6)\) with the type \(((4 \ast 3), (4), (2))\) exists. Because of Lemma 3.6 we have \(C_{a_1, \ldots, a_k} \parallel C_6(6)\) whenever \((a_1, \ldots, a_k)\) is a contraction of the list \((4 \ast 3, 1 \ast 6)\). But these are the remaining cases. \(\square\)

Especially, we have shown \(C_{a_k} \parallel C_4(n/4)\) whenever 4 divides \(n\) (this result was already stated in Lemma 1.4 in order to prove Theorem 1.1 in the case \(n = 0 (4)\)).

6. An application to the original Oberwolfach problem

We finish this paper with an application of pathlike factorisations to the original Oberwolfach problem.

Definition 6.1. Let \(G\) be a graph, \(v_0 \in V(G)\) and \(m \geq 1\) (we assume here and later \(v_0 \notin (V(G) \times \mathbb{N}_m)\)—otherwise rename \(v_0\)). Then \(G_{v_0}(m)\) shall denote the graph with

\[V(G_{v_0}(m)) = \{v_0\} \cup ((V(G) - \{v_0\}) \times \mathbb{N}_m)\]

and the edges

(i) \(\{v_0(a, x), (a, x), (a, y)\}\) for all \(a\) with \(\{v_0, a\} \in E(G)\) and all \(x, y \in \mathbb{N}_m, x \neq y\),

(ii) \(\{(a, x), (b, y)\}\) for all \(\{a, b\} \in E(G - \{v_0\})\) and all \(x, y \in \mathbb{N}_m\).

Lemma 6.2. Let \(\Phi = ((\sigma_i)_{i \in I}, F)\) be a factorisation of \(G\) into isomorphic parts and \(v_0 = \sigma_i(w_0) \in V(G)\) a constant vertex of \(\Phi\). Then \(\tilde{\Phi} = ((\tilde{\sigma}_i)_{i \in I}, F_{v_0}(m))\) with \(\sigma_i(w_0) = v_0, \sigma_i(x, y) = (\sigma_i(x), y)\) for \((x, y) \in (V(G) - v_0) \times \mathbb{N}_m\) is a factorisation of \(G_{v_0}(m)\) into isomorphic parts for each \(m \geq 1\) and \(v_0\) is a constant vertex of \(\tilde{\Phi}\).
Proof. By verification of the details. □

To have shorter notations for the graphs which we will study we introduce the following.

**Definition 6.3.** $Z(e) := \text{cycle}(\infty, 0, \ldots, e - 2)$, $Z(e, m) := (Z(e),)(m)$ (for $e \geq 3$, $m \geq 1$).

**Lemma 6.4.** $Z(e, m) \parallel K_{1 + (e - 1)m}$ if $e$ is odd, $e \geq 3$, and $m \geq 1$.

**Proof.** Let $H$ denote the complete graph on $\{\infty, 0, \ldots, e - 2\}$. Since the automorphism group of a cycle is transitive, the well-known fact $Z(e) \simeq C_e \parallel K_e = H$ for odd $e \geq 3$ implies the existence of a factorisation $\Phi$ of $H$ into graphs $Z(e)$ which is constant on $\infty$.

Because of Lemma 6.2 and $H_{\infty}(m) = K_{1 + \infty l_j}$, now the assertion follows. □

By Lemma 6.4 and the transitivity $\parallel$, we have $F \parallel K_{1 + (e - 1)m}$ whenever $F \parallel Z(e, m)$ with odd $e \geq 3$. So it is promising to investigate factorisations of $Z(e, m)$ into isomorphic 2-regular graphs in order to get solutions of the Oberwolfach problem.

For odd $m$ a lot of factorisations of $Z(e, m)$ can be found in the following way: Let $e = \alpha + \beta + 2$, $\alpha, \beta \geq 2$, $m \geq 1$ and $m$ odd. For $i \in \mathbb{N}_m$ define bijections $\sigma_i$ on $V(Z(e, m))$ by $\sigma_i(v) = v$ for $v \in \{\infty\} \cup \{\alpha - 1, \alpha + 1\} \times \mathbb{N}_m$ and $\sigma_i(x, y) = (x, y + i)$ for the remaining vertices of $Z(e, m)$ where $y + i$ is calculated mod $m$.

Next, choose any pathlike factor $B$ of path($\alpha - 1, \alpha, \alpha + 1$) $\subseteq Z(e, m)$ and define $F = F(B)$ as the factor of $Z(e, m)$ with the edges from $B$ together with the edges

(i) \{\infty, (a, 0)\}, \{(a, y), (a, -y)\} for $a = 0$ and $a = e - 2$, $y \in \mathbb{N}_m - \{0\}$,

(ii) \{(x, y), (x + 1, -y)\} for $x = 0, 1, \ldots, \alpha - 2, \alpha + 1, \ldots, \alpha + \beta - 1$ and $y \in \mathbb{N}_m$, (where $-y$ is always taken mod $m$). Then $F$ is 2-regular and it is easy to verify that $((\sigma_i)_{i \in \mathbb{N}_m}, F)$ is a factorisation of $Z(e, m)$ (recall that $m$ is assumed to be odd). Hence $F \parallel Z(e, m)$ and, by Lemma 6.4, $F \parallel K_{1 + (e - 1)m}$ if $e$ is odd.

**Lemma 6.5.** Let $\alpha, \beta \geq 2$, $n \geq 0$, and let $\tilde{L} = (a_0, \ldots, a_k)$ be a contraction of the list

$L = (1 \ast (\alpha + \beta + 2), n \ast (2\alpha + 1), n \ast (2\beta + 1))$.

Then $C_{a_0, \ldots, a_k} \parallel Z(\alpha + \beta + 2, 2n + 1)$.

**Proof.** Consider the construction given above for $F \subseteq Z(e, m)$ with $e = \alpha + \beta + 2$, $m = 2n + 1$. First we take as $B$ the graph which consists of the union of the paths path($\alpha - 1, 0$), $(\alpha, 0), (\alpha + 1, 0))$, path($\alpha - 1, i$), $(\alpha, i), (\alpha - 1, m - i)$ and path($\alpha + 1, i$), $(\alpha, n - i), (\alpha + 1, m - i)$ for $i = 1, \ldots, (m - 1)/2$. Then
$F = F(B)$ consists of one cycle of length $\alpha + \beta + 2$, $n = (m - 1)/2$ cycles of length $2\alpha + 1$ and $n = (m - 1)/2$ cycles of length $2\beta + 1$. Thus we have already proved Lemma 6.5 in the special case $L = L$. Now let $B$ be an arbitrary pathlike factor of path $(\alpha - 1, \alpha, \alpha + 1)(\mathbb{N}_{m})$. Let $J_1, J_2$ be two vertex-disjoint cycles of $F = F(B)$; we may assume that $F$ is a $C_{b_0}, \ldots, b_k$ and $J_1 = C_{b_0}, J_2 = C_{b_1}$. Clearly there are edges $\{(a, x_1), (\mu_1, y_1)\} \in E(J_1) \cap E(B)$ for $i = 1, 2$. Replace these two edges in $B$ by $\{(a, x_1), (\mu_2, y_2)\}$ and $\{(a, x_2), (\mu_1, y_1)\}$. The resulting graph $B$ is again a pathlike factor of path$(\alpha - 1, \alpha, \alpha + 1)(\mathbb{N}_n)$ and $\overline{F} = F(\overline{B})$ will be a $C_{b_0+b_1}, \ldots, b_k$.

Now Lemma 6.5 follows immediately. \(\square\)

Lemma 6.5 (together with Lemma 6.4) means for the Oberwolfach problem that $F \parallel K_n$ holds for infinitely many $n$ and at least for all 2-regular factors $F$ of the $K_n$ without ‘small’ cycles.

**Corollary 6.6.** If $t, a_0, a_1, \ldots, a_k$ are integers with $t \geq 1$, $k \geq 0$, $a_i > (2t + 1)(4t + 5)$, and $\sum a_i = 2t + 3 \mod(4t + 4)$ then $C_{a_0}, \ldots, a_k \parallel K_{\sum a_i}$.

**Proof.** Let $t, a_0, \ldots, a_k$ be as in the hypothesis. Since $\gcd(2t + 1, 2t + 3) = 1$ there is a solution of the system of equations

$$a_i = x_i(2t + 1) + y_i(2t + 3) \quad (i = 0, \ldots, k)$$

in integers $x_0, \ldots, x_k, y_0, \ldots, y_k$. Since $a_i$ are assumed to be sufficiently large we can find a solution of (*) in nonnegative integers (to show this one needs only $a_i \geq (2t)(2t + 2)$).

But it is even possible to find a solution of (*) in nonnegative integers such that additionally

$$D = \sum y_i - \sum x_i = 1$$

is satisfied: If we take any solutions of (*) and set $D := \sum y_i - \sum x_i$ then

$$D(2t + 3) = (\sum x_i)(2t + 1) \mod(4t + 4)$$

follows from the hypothesis $\sum a_i = 2t + 3 \mod(4t + 4)$; hence, since $\gcd(2t + 3, 4t + 4) = 1$, we have already $D = \sum y_i - \sum x_i = 1 \mod(4t + 4)$ instead of (**). Now let a nonnegative solution of (*) be given with $D > 1$, i.e. with $D \leq 4t - 3$. Then there is at least one $i_0$ with $y_{i_0} < x_{i_0}$ because of $D = \sum(y_i - x_i) < 0$. It must be $x_{i_0} \geq (2t + 3)$ since otherwise $a_{i_0} = x_{i_0}(2t + 1) + y_{i_0}(2t + 3)$ would be $\leq(2t + 1)(4t + 5)$.

So in the case $D < 1$ we can replace a certain $x_{i_0}$ by $x_{i_0} - (2t + 3) \geq 0$, and $y_{i_0}$ by $y_{i_0} + (2t + 1)$ which gives a new nonnegative solution of (*) and increments $D$ by $4t + 4$. In a similar way the case $D > 1$ can be treated.

In our nonnegative solution of (*) and (***) at least one $y_i$, say $y_0$, must be greater than 0 and by (*) we have with $\alpha := t$, $\beta := t + 1$

$$a_0 = \alpha + \beta + 2 + x_0(2\alpha + 1) + (y_0 - 1)(2\beta + 1),$$

$$a_i = x_i(2\alpha + 1) + y_i(2\beta + 1) \quad (i = 1, \ldots, k)$$

(***)
such that $n := \sum x_i (y_0 - 1) + y_1 + \cdots + y_k$ holds with nonnegative summands by (**). But this means that Lemma 6.5 is applicable and Lemma 6.4 gives the assertion. □

References