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A Computational Verification Method of Existence of Solution for Elastoplastic Torsion Problems with Uniqueness

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Abstract—We propose a numerical method to verify the existence and uniqueness of solutions to elasto-plastic torsion problems. We numerically construct a set containing solutions which satisfies the hypothesis of Banach fixed-point theorem in a certain Sobolev space. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In previous papers, we have developed numerical verification methods for the existence of solutions to variational inequalities (see [1,2]). Although the verification method enables us to find a solution, it is impossible to assure uniqueness of the solution. In this paper, we propose a numerical method to verify not only existence but also uniqueness of solutions to elastoplastic torsion problems.

2. FORMULATION AND METHODS OF VERIFICATION

Let Ω be a bounded convex domain in R^2, with piecewise smooth boundary $\partial\Omega$, and $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$. Let f be a bounded and continuous map from $H_0^1(\Omega)$ into $L^2(\Omega)$. Next, we define $K = \{v \in H_0^1(\Omega); |\nabla v| \le 1 \text{ a.e. on } \Omega\}$, where $|\nabla v| = \sqrt{(\frac{\partial v}{\partial x_1})^2 + (\frac{\partial v}{\partial x_2})^2}$. Here, $H_0^1(\Omega)$ stands for the usual Sobolev space on Ω with homogeneous boundary condition. Now, let us consider the following nonlinear elastoplastic torsion problem:

$$\text{find } u \in K \text{ such that } a(u, v-u) \ge (f(u), v-u), \qquad \forall\, v \in K, \tag{2.1}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the L^2-inner product on Ω.

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We adopt $(\nabla\phi, \nabla\psi)$ as the inner product on $H_0^1(\Omega)$, whence the associated norm is defined by $\|\phi\|_{H_0^1(\Omega)} = \|\nabla\phi\|_{L^2(\Omega)}$.

First, since $a(\,\cdot\,,\,\cdot\,)$ is a continuous bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$, for each $u \in H_0^1(\Omega)$, from the Riesz representation theorem, there exists a unique element $F(u) \in H_0^1(\Omega)$ such that $a(F(u), v) = (f(u), v)$, $\forall\, v \in H_0^1(\Omega)$. That is,

$$\exists \quad F(u) \in H_0^1(\Omega) \text{ such that } -\Delta F(u) = f(u) \text{ in } \Omega, \quad F(u) = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Then the map $F : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ is a compact operator by the above assumptions on f. In the preceding paper [1], problem (2.1) is equivalent to that of finding $u \in H_0^1(\Omega)$ such that

$$u = P_K F(u). \tag{2.3}$$

To verify the existence of a solution of (2.1) in a computer, we use the fixed-point formulation (2.3) of a compact operator $P_K F$ as above.

Now we describe a numerical verification method to verify the existence and uniqueness of solution of (2.1). First, we determine a set V for a bounded, convex, and closed subset $U \subset H_0^1(\Omega)$ as

$$V = \left\{ v \in H_0^1(\Omega) : v = P_K F(u), \forall\, u \in U \right\}.$$

From Schauder's fixed-point theorem, if $V \subset U$ holds, then there exists a solution of (2.1) in the set U. Our aim is to find a set U which includes V. A procedure to verify $V \subset U$ using a computer is as follows. Now, let V_h be a finite dimensional subspaces of $H_0^1(\Omega)$ dependent on h. We then define K_h, an approximate subset of K, by $K_h = V_h \cap K = \{v_h : v_h \in V_h, |\nabla v_h| \le 1 \text{ a.e. on } \Omega\}$. For any $u \in H_0^1(\Omega)$, we define the rounding $R(P_K F(u)) \in K_h$ as the solution of the following problem:

$$a(R(P_K F(u)), v_h - R(P_K F(u))) \ge (f(u), v_h - R(P_K F(u))), \qquad \forall\, v_h \in K_h.$$

For a set $V \subset H_0^1(\Omega)$, we define the rounding $R(P_K FV) \subset K_h$ as

$$R(P_K FV) = \{v_h \in K_h : v_h = R(P_K F(u)),\ u \in U\}.$$

Also, we define for $V \subset H_0^1(\Omega)$ the rounding error $RE(P_K FV) \subset H_0^1(\Omega)$ as

$$RE(P_K FV) = \left\{ v \in H_0^1(\Omega); \|v\|_{H_0^1(\Omega)} \le Ch \sup_{u \in U} \|f(u)\|_{L^2} \right\}. \tag{2.4}$$

From the definition, we have $V \subset R(P_K FV) + RE(P_K FV)$. Then it is sufficient to find U which satisfies $R(P_K FV) + RE(P_K FV) \subset U$. Although the verification method in the above enables us to find a solution in the set U, it is impossible to assure uniqueness of the solution in the same set. We now present a technique including the verification of uniqueness under the following additional assumption.

A1. *Suppose that there exists a* $\beta < 1$ *such that*

$$\|P_K F(u_1) - P_K F(u_2)\|_{H_0^1(\Omega)} \le \beta \|u_1 - u_2\|_{H_0^1(\Omega)}, \qquad \forall\, u_1, u_2 \in U$$

Banach fixed-point theorem gives the proof of uniqueness of solutions to variational inequality (2.1) in the set V*, and in* U*. Next let us introduce the procedure for finding such a set* U *using computers. First, we describe how to obtain a such set of* $H_0^1(\Omega)$ *on a computer. In order to find a set* U *satisfying the above condition, we use simple iterative method. The simple iteration method is as follows.*

(1) *First, we obtain an approximate solution* $u_h^{(0)} \in V_h$ *to (2.1) by some appropriate method. Set* $U_h^{(0)} = \{u_h^{(0)}\}$ *and* $\alpha_0 = 0$.

(2) *Next, we will define $R(P_KFV^{(i)})$ and $RE(P_KFV^{(i)})$ for $i \geq 0$, where $V^{(i)}$ is the set defined as follows:*

$$V^{(i)} = \left\{ v^{(i)} \in K_h : v^{(i)} = P_K F\left(u^{(i)}\right),\ u^{(i)} \in U^{(i)} \right\}.$$

$R(P_KFV^{(i)})$ is defined by the subset of K_h which consists of all elements $v_h^{(i)} \in K_h$ such that

$$a\left(v_h^{(i)}, \psi - v_h^{(i)}\right) \geq \left(f\left(u^{(i)}\right), \psi - v_h^{(i)}\right), \qquad \forall \psi \in K_h, \tag{2.5}$$

holds for some $u^{(i)} \in U^{(i)}$. Note that $R(P_KFV^{(i)})$ can be enclosed by $R(P_KFV^{(i)}) \subset \sum_{j=1}^{M} A_j \phi_j$ where $A_j = [\ \underline{A_j},\ \overline{A_j}\ ]$ are intervals, and $\{\phi\}_{j=1}^{M}$ is the basis of K_h. Next $RE(P_KFV^{(i)})$ is defined by

$$RE\left(P_KFV^{(i)}\right) = \left\{ v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq Ch \sup_{u^{(i)} \in U^{(i)}} \left\| f\left(u^{(i)}\right)\right\|_{L^2} \right\}.$$

Hence, $V^{(i)} \subset R(P_KFV^{(i)}) + RE(P_KFV^{(i)})$ holds.

(3) *Check the verification condition*

$$R\left(P_KFV^{(i)}\right) + RE\left(P_KFV^{(i)}\right) \subset U^{(i)}.$$

If the condition is satisfied, then $U^{(i)}$ is the desired set, and a solution to (2.1) exists in $V^{(i)}$, and hence, in $U^{(i)}$.

(4) *If the condition is not satisfied, we continue the simple iteration by using δ-inflation, i.e., let δ be a certain positive constant given beforehand, and take*

$$\begin{aligned}
\alpha_{i+1} &= Ch\, \underline{u^{(i)} \in U^{(i)}} \sup \left\| f\left(u^{(i)}\right)\right\|_{L^2} + \delta, \\
[\alpha_{i+1}] &= \left\{ v \in H_0^1(\Omega) : \|v\|_{H^1(\Omega)} \leq \alpha_{i+1} \right\}, \\
U_h^{(i+1)} &= \sum_{j=1}^{M} \left[\ \underline{A_j} - \delta,\ \overline{A_j} + \delta \right] \phi_j, \\
U^{(i+1)} &= U_h^{(i+1)} + [\alpha_{i+1}],
\end{aligned}$$

and then go back to the second step. The reader may refer to [1] for the details.

3. AN APPLICATION

In this section, we present an numerical example for verification according to the procedures described in the previous section. We consider the case

$$f(u) = \psi u + \zeta. \tag{3.1}$$

Here, we assume that $\psi, \zeta \in L^\infty(\Omega)$. First, in order to validate A1, we need some properties for P_KF. Let $\mathcal{L}(H_0^1(\Omega))$ be the set of bounded linear operators from $H_0^1(\Omega)$ to $H_0^1(\Omega)$. We consider the following eigenvalue problem:

$$\begin{aligned}
-\Delta u &= \lambda u, && \text{in } \Omega, \\
u &= 0, && \text{on } \partial\Omega.
\end{aligned} \tag{3.2}$$

As well known, the first eigenvalue λ_1 of (3.2) is actually equivalent to the the following problem:

$$\min_{u \in H_0^1(\Omega)} \frac{\|u\|_{H_0^1(\Omega)}^2}{\|u\|_{L^2}^2} = \lambda_1. \tag{3.3}$$

Hence, we obtain

$$\forall u \in H_0^1(\Omega), \qquad \frac{\|u\|_{H_0^1(\Omega)}}{\|u\|_{L^2}} \geq \sqrt{\lambda_1}.$$

Furthermore, by well-known results, it follows that

$$\lambda_1 = \frac{1}{2\pi^2} \tag{3.4}$$

for the unit square in two-dimensional case. In one-dimensional case, we can take λ_1 as

$$\lambda_1 = \frac{1}{\pi^2}. \tag{3.5}$$

And, by (2.2), we have $F(u_1) - F(u_2) = (-\Delta)^{-1}(\psi(u_1 - u_2))$.

Now, setting $Au := (-\Delta)^{-1}\psi u$, we consider the following inequality:

$$\begin{aligned}\|P_K F(u_1) - P_K F(u_2)\|_{H_0^1(\Omega)} &\leq \|F(u_1) - F(u_2)\|_{H_0^1(\Omega)} \\ &\leq \|A(u_1 - u_2)\|_{H_0^1(\Omega)} \\ &\leq \|A\|_{\mathcal{L}(H_0^1(\Omega))}\|u_1 - u_2\|_{H_0^1(\Omega)}.\end{aligned}$$

Here, we used the fact that $\|P_K\|_{\mathcal{L}(H_0^1(\Omega))} \leq 1$. Further, we obtain

$$\begin{aligned}\|A\|_{\mathcal{L}(H_0^1(\Omega))} &= \sup_{u \neq 0 \in H_0^1(\Omega)} \frac{\|Au\|_{H_0^1(\Omega)}}{\|u\|_{H_0^1(\Omega)}} = \sup_{u \in H_0^1(\Omega)} \frac{(\nabla Au, \nabla Au)}{\|u\|_{H_0^1(\Omega)}\|Au\|_{H_0^1(\Omega)}} \\ &= \sup_{u \in H_0^1(\Omega)} \frac{(\psi u, Au)}{\|u\|_{H_0^1(\Omega)}\|Au\|_{H_0^1(\Omega)}} \\ &\leq \|\psi\|_{L^\infty} \sup_{u \in H_0^1(\Omega)} \frac{\|u\|_{L^2}\|Au\|_{L^2}}{\|u\|_{H_0^1(\Omega)}\|Au\|_{H_0^1(\Omega)}}.\end{aligned}$$

Hence, by using (3.3) and (3.4), we have

$$\|A\|_{\mathcal{L}(H_0^1(\Omega))} \leq \frac{\|\psi\|_{L^\infty}}{2\pi^2}$$

for the unit square in two-dimensional case. Similarly, for one-dimensional case, we obtain

$$\|A\|_{\mathcal{L}(H_0^1(\Omega))} \leq \frac{\|\psi\|_{L^\infty}}{\pi^2}.$$

Therefore, we have the following result.

THEOREM 3.1. *If the function ψ in (3.1) satisfies*

$$\frac{\|\psi\|_{L^\infty}}{2\pi^2} < 1 \ (n = 2) \qquad \text{or} \qquad \frac{\|\psi\|_{L^\infty}}{\pi^2} < 1 \ (n = 1),$$

then Assumption A1 holds.

In the below, let $\Omega = (0, 1)$ and we use the approximation subspace as in the previous section, and consider the case $f(u) = 2u + A$, where A is a real constant. We choose the basis $\{\phi_i\}_{i=1}^M$ of V_h as usual hat functions, i.e., $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} means Kronecker's delta. The constant appearing in the rounding error (2.4) can be taken as $C = \sqrt{5}/\pi$ (see [1]).

The execution conditions are as follows:

$A = 3$.

Numbers of elements $= 101$. $\dim V_h = 100$.

Initial value : $u_h^{(0)} =$ Galerkin approximation (2.5), $\alpha_0 = 0$.

Extension parameter : $\delta = 10^{-3}$.

Results are as follows:

Iteration numbers : 6.

H_0^1 – error bound : 0.025340.

Maximum width of coefficient intervals in $\{A_j\} = 0.074056$.

Two free boundary points are located around $x = 0.237642$ and $x = 0.762376$. The method proposed in this paper enables us to verify solutions of elastoplastic torsion problems with uniqueness.

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