Discrete Mathematics 208/209 (1999) 387–409

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

# Recent developments on absolute geometries and algebraization by K-loops

Helmut Karzel *

*Mathematisches Institut, Technische Universität München, D-80290 München, Germany*

Received 5 March 1997; revised 3 March 1998; accepted 16 March 1998

**Abstract**

Let $(P, \mathfrak{L}, \alpha)$ be an ordered space. A spatial version of Pasch's assertion is proved, with that a short proof is given for the fact that $(P, \mathfrak{L})$ is an exchange space and the concepts *h-parallel*, *one sided h-parallel* and *hyperbolic incidence structure* are introduced (Section 2). An ordered space with hyperbolic incidence structure can be embedded in an ordered projective space $(P_p, \mathfrak{L}_p, \tau)$ of the same dimension such that $P$ is projectively convex and projectively open (cf. Property 3.2). Then spaces with congruence $(P, \mathfrak{L}, \equiv)$ are introduced and those are characterized in which point reflections do exist (Section 4). Incidence, congruence and order are joined together by assuming a compatibility axiom (ZK) (Section 5). If $(P, \mathfrak{L}, \alpha, \equiv)$ is an absolute space, if $o \in P$ is fixed and if for $x \in P, x'$ denotes the midpoint of $o$ and $x$ and $\tilde{x}$ the point reflection in $x$ then the map ${}^o$: $P \to J$; $x \to x^o := \tilde{x}$ satisfies the conditions (B1) and (B2) of Section 6, and if one sets $a + b := a^o \circ 0^o(b)$ then $(P, +)$ becomes a K-loop (cf. Theorem 6.1) and the $\mathfrak{J}$ of all lines through $o$ forms an incidence fibration in the sense of Zizioli consisting of commutative subgroups of $(P, +)$ (cf. Property 7.1). Therefore K-loops can be used for an algebraization of absolute spaces; in this way Ruoff's proportionality Theorem 8.4 for hyperbolic spaces is presented. © 1999 Published by Elsevier Science B.V. All rights reserved.

*Keywords:* Ordered space; Projective embedding; Congruence; Absolute space; Reflections; K-loops

Lecturing on foundations of geometry stimulates mostly new attempts to improve the presentations that sometimes end in new areas of research. Here we start with the concept of an ordered space $(P, \mathfrak{L}, \alpha)$ and prove a spatial version of Pasch's assertion (Property 1.3). Since the incidence closure $\overline{T \cup \{a\}}$ of a subspace $T$ and a point

* Correspondence address: Technische Universitat Munchen, FB Mathematik, Arcisstr. 21, 80333 Munchen 2, Germany.

*E-mail address:* karzel@mathematik.tu-muenchen.de (H. Karzel)


PII: S0012-365X(99)00085-0

$a \in P \backslash T$ can be decomposed in $T$ and two halfspaces $\overrightarrow{T,a}$ and $\overrightarrow{T,\bar{a}}$ (Property 1.4), the validity of the exchange axiom is an easy consequence (Property 1.5). (A rigorous proof of this fact I found only in Sörensen's paper [31] of 1986.)

The notion *hyperbolic* parallel or in short *h-parallel* in ordered spaces (introduced 1994 in [6,8]) was generalized to one sided *h-parallel* by Karzel et al. [10] and Konrad [22]; the main results are listed in Section 2.

The projective embedding of ordered spaces with hyperbolic incidence structure (Section 3) is based mainly on results of Sörensen [31] and Kreuzer [24]. The new results of the paper by Karzel et al. [11] are summarized in Properties 3.3–3.5.

In 1984, Sörensen [30] made fundamental investigations on incidence and congruence. Here we will extend notion *plane with congruence* to *space with congruence* $(P, \mathfrak{L}, \equiv)$. Also *orthogonality between lines* and *line-reflections* can be defined; but we do not know if the latter are motions (Property 4.1). Our spaces split into *regular* ones and *Lotkernspaces*. Only in certain regular spaces do point reflections exist and they turn out to be motions (Property 4.3).

Incidence, congruence and order are joined together in Section 5 by assuming the compatibility axiom ZK. The main properties of these spaces $(P, \mathfrak{L}, \alpha, \equiv)$ are found in Property 5.1. Then, by adding a further axiom WF, the notion *absolute space* [1] is established. Results recently achieved by Konrad [22], Sörensen [32], and Kroll and Sörensen [27] are collected in Properties 5.4 and 5.5.

Wefelscheid and Konrad and the present author [16,9] showed that there are close connections between hyperbolic spaces and certain reflection groups on the one hand and K-loops on the other. To find a common, most general frame which allows us to construct K-loops, is the aim of Section 6. We proceed in a manner very similar to that of Manara and Marchi in [28] (cf. Properties 6.1 and 6.2).

In 1987, Zizioli [33] gave a correct definition of a *loop with incidence fibration*, such that the usual derivation produces an incidence loop.

To each regular space $(P, \mathfrak{L}, \equiv)$ where point reflections exist, one can associate a reflection structure in the sense of Section 6, and then (via (Property 6.1) a K-loop $(P, +)$ which in addition possesses an incidence fibration $\mathfrak{F}$ (in the sense of Zizioli) and where the members of $\mathfrak{F}$ are even commutative subgroups of $(P, +)$ (Property 7.1). If $(P, \mathfrak{L}, \alpha, \equiv)$ is also an ordinary absolute space (cf. Section 5), then the corresponding loop $(P, +)$ can be used for an algebraization of the space (Property 7.2). Using this algebraization we present Ruoff's [29] astonishing proportionality Theorem (8.4) for hyperbolic spaces.

To each K-loop there corresponds the so-called structure group $\Delta$ (cf. [26]) whose properties are of interest from both the geometric and algebraic points of view. Contributions in this area were given by Im [4,5], Kiechle [17], Kiechle and Konrad [18], and Konrad [23].

[1] In order to charcterize absolute spaces which can be coordinatized by the reals, one has only to add the axiom of continuity.

The algebraization by K-loops opens a new procedure to extend the foundation results for singular spaces (achieved by Karzel and Kist [7]) now on ordinary spaces by the concept *loop with a reflection germ*. Gabrieli [1] has reported on this topic at the conference ‘Combinatorics ’96’.

## 1. Some properties of ordered spaces

Let $(P,\mathfrak{L})$ be an *incidence space* (also called a linear space), i.e. $P$ is a set and $\mathfrak{L}\subset\mathfrak{P}(P)$ is a subset of the powerset of $P$ such that

(I1) $\forall a,b\in P,\ a\neq b\,\exists_1 L\in\mathfrak{L}:a,b\in L$; let $\overline{a,b}:=L$.

(I2) $\forall L\in\mathfrak{L}:|L|\geqslant 2$.

A subset $T\subseteq P$ is called a *subspace* if for all points $x,y\in T, x\neq y$ we have $\overline{x,y}\subseteq T$. Let $\mathfrak{T}$ be the set of all subspaces of $(P,\mathfrak{L})$, and for $S\subset P$ let $\overline{S}:=\bigcap\{T\in\mathfrak{T}\mid S\subseteq T\}$ be the *incidence closure*. Each subspace $E$ which is the closure of three non-collinear points is called a *plane*. Let $\mathfrak{E}$ be the set of all planes.

For $p\in P$, $L\in\mathfrak{L}$ and $T\in\mathfrak{T}$ let

$$\mathfrak{L}(p):=\{X\in\mathfrak{L}\mid p\in X\},\quad \mathfrak{E}(L):=\{E\in\mathfrak{E}\mid L\subseteq E\}$$

and

$$\mathfrak{L}(T):=\{X\in\mathfrak{L}\mid X\subseteq T\}.$$

Now let $P^{(3)}:=\{(x,y,z)\in P^3\mid x\neq y,\ z\wedge z\in\overline{x,y}\}$. A map

$$\alpha:P^{(3)}\to\{-1,1\};\quad (x,y,z)\mapsto(x\mid y,z)$$

is called a *betweenness function*[2] and $(P,\mathfrak{L},\alpha)$ an *ordered space* if the following conditions (Z1), (Z2), (Z3) and (ZP) hold:

(Z1) For all $(a,b,c),(a,b,d)\in P^{(3)}$: $\quad (a|b,c)\cdot(a|c,d)=(a|b,d)$.

(Z2) For $a,b,c\in P$ distinct and collinear, exactly one of the values $(a|b,c),(b|c,a)$, $(c|a,b)$ equals $-1$.

(Z3) For all $a,b\in P,\ a\neq b$ there exists a $c\in P$ such that $b\in\,]a,c[\,:=\{x\in P\mid(x,a,c)\in P^{(3)}\wedge(x|a,c)=-1\}$.

(ZP) If $a,b,c\in P$ are non collinear, $x\in\,]b,c[$ and $y\in\,]a,x[$ then $\overline{c,y}\cap\,]a,b[\,\neq\emptyset$.

From now on let $(P,\mathfrak{L},\alpha)$ be an ordered space. A subset $C\subseteq P$ is called *convex* if for all $x,y\in C,\ x\neq y:\,]x,y[\,\subseteq C$. Let $\mathfrak{C}$ be the set of all convex subsets of $(P,\mathfrak{L},\alpha)$.

For $T\in\mathfrak{T}$ and $a\in P\backslash T$ let $\overset{-}{\overrightarrow{T,a}}:=\{x\in P\backslash T\mid\,]x,a]\cap T\neq\emptyset\}$, in particular, if $a$, $b\in P$ with $a\neq b$ then $\overset{-}{\overrightarrow{b,a}}:=\{x\in P\backslash\{b\}\mid b\in\,]a,x[\}$; and we set $\overrightarrow{b,a}:=\{x\in\overline{a,b}\backslash\{b\}\mid(b|a,x)=1\}$ (halfline).

$(P,\mathfrak{L},\alpha)$ has the following properties.

---

[2] $(x\mid y,z)=-1$: *x between y and z.*

**Property 1.1.** *For all* $a,b\in P,\ a\neq b,\ c\in\,]a,b[$ *and* $d\in\overset{-}{\overrightarrow{a,b}}$ *we have*:
(1) $]a,c[\,\subset\,]a,b[$,
(2) $]a,b[\,=\,]a,c[\,\dot\cup\{c\}\dot\cup\,]c,b[$,
(3) $]a,b[$ *is convex* (cf. [6], p. 146, (VII 1.5)]),
(4) $\overset{-}{\overrightarrow{a,b}}$ *is convex*,
(5) $\overline{a,b}=\overset{-}{\overrightarrow{a,b}}\dot\cup\{a\}\dot\cup\overset{-}{\overrightarrow{a,d}}$ and $\overrightarrow{a,b}=\overset{-}{\overrightarrow{a,d}}$.

**Proof.** (4) Let $x,y\in\overset{-}{\overrightarrow{a,b}},\ x\neq y$ and $z\in\,]x,y[$. By (Z1), $(a|x,y)=(a|b,x)\cdot(a|b,y)=(-1)\cdot(-1)=1$. Hence by (Z2) exactly one of the values $(x|a,y)$ and $(y|a,x)$ equals $-1$, for instance $(x|a,y)=-1$, i.e. $x\in\,]a,y[$. Then $z\in\,]x,y[\,\subset\,]a,y[$ by (1), i.e. $(z|a,y)=-1$ and hence by (Z2), $(a|y,z)=1$. Now $y\in\overset{-}{\overrightarrow{a,b}}$ implies $a\in\,]b,y[$, i.e. $(a|b,y)=-1$. Consequently by (Z1), $(a|b,z)=(a|b,y)\cdot(a|y,z)=(-1)\cdot 1=-1$, i.e. $z\in\overset{-}{\overrightarrow{a,b}}$ and so $\overset{-}{\overrightarrow{a,b}}\in\mathfrak{C}$.

(5) By definition, $a\notin\overset{-}{\overrightarrow{a,b}}\cup\overset{-}{\overrightarrow{a,d}}$ and $d\in\overset{-}{\overrightarrow{a,b}}$ implies $a\in\,]b,d[$, i.e. $(a\,|\,b,d)=-1$. Let $x\in\overline{a,b}\backslash\{a\}$. Then $-1=(a|b,d)=(a|b,x)\cdot(a|x,d)$, and so either $(a|b,x)=1$ and $(a|x,d)=-1$, i.e. $x\in\overrightarrow{a,b}$ and $x\in\overset{-}{\overrightarrow{a,d}}$ or $(a|b,x)=-1$ and $(a|x,d)=1$, i.e. $x\in\overset{-}{\overrightarrow{a,b}}$ and $x\notin\overset{-}{\overrightarrow{a,d}}$.

**Property 1.2.** *Let* $a,b,c\in P$ *be non-collinear*, $a'\in\,]b,c[,b'\in\,]c,a[,c'\in\,]a,b[,\ u\in\,]a,a'[$ and $v:=\overline{c,u}\cap\,]a,b[$ ($v$ exists by (ZP)). *Then*:
(1) $u\in\,]c,v[$ (cf. [6, VII(1.6)]),
(2) $]a,a'[\,\cap\,]b,b'[\,\neq\emptyset$ (cf. [6, VII(1.7)]),
(3) $\overline{a',b'}\cap\,]a,b[\,=\emptyset$,
(4) $\overline{\{a,b,c\}}=\overline{\{a',b',c'\}}$.

**Proof.** (3) Assume $\{z\}:=\overline{a',b'}\cap]a,b[\,\neq\emptyset$ [3] exists. If $z\in\,]a',b'[$ then $a'\in\,]c,b[$ implies by (ZP), $\emptyset\neq\overline{z,b}\cap\,]c,b'[\,\subset\overline{a,b}\cap\overline{c,a}=\{a\}$, i.e. $(a|c,b')=-1$ which is a contradiction to $b'\in\,]a,c[$. Hence $z\notin\,]a',b'[$ and we may suppose $a'\in\,]b',z[$. Then $z\in\,]a,b[$ and $a'\in\,]b',z[$ implies by (ZP), $\emptyset\neq\overline{b,a'}\cap]b',a[\,\subset\overline{b,c}\cap\overline{a,c}=\{c\}$. But $c\in\,]b',a[$ contradicts $b'\in\,]a,c[$.

(4) By (2) $\{d\}:=\,]a.a'[\,\cap\,]b,b'[\,\neq\emptyset$. $d\in\,]a,a'[$ and $c'\in\,]a,b[$ implies by (2), $\{e\}:=\,]a',c'[\,\cap\,]b,d[\,\neq\emptyset$. Therefore $b\in\overline{b',e}\subset\overline{\{a',b',c'\}}$ and so $\overline{\{a,b,c\}}\subset\overline{\{a',b',c'\}}$.

**Property 1.3** (Spatial form of the statement of Pasch). *Let* $a,b,c,d\in P$ *be distinct*, $u\in\,]a,b[,v\in\,]b,c[,w\in\,]c,d[$, *then* $]a,d[\,\cap\,\overline{\{u,v,w\}}\neq\emptyset$.

[3] If the intersection of two sets $X,Y$ consists of a single point $p$ we will write also $X\cap Y=p$ instead of $X\cap Y=\{p\}$.

**Proof.** By Property 1.2.(2), $x := ]b,w[ \cap ]v,d[ \neq \emptyset$. Now $u\in ]a,b[, x\in ]b,w[$ imply by Property 1.2(2), $y := ]x,a[\cap]u,w[\neq\emptyset$, and $x\in ]v,d[, y\in ]x,a[$ force $s := ]a,d[\cap\overline{v,y}\neq\emptyset$ by (ZP). Since $y\in ]u,w[$ one obtains $s\in\overline{v,y}\cap\overline{\{u,v,w\}}$, and so $s\in ]a,d[\cap\overline{\{u,v,w\}}$.

We extend Theorems 1.10, 1.11, and 1.12 of [22]:

**Theorem 1.4.** *Let $T\in\mathfrak{T}$ and $a,b\in P\setminus T$ such that $]a,b[\cap T\neq\emptyset$. Then*
(1) $\overrightarrow{T,a}\in\mathfrak{C}$,
(2) $\forall c\in\overrightarrow{T,b}$: $\overrightarrow{T,a}=\overrightarrow{T,c}$ and $\overrightarrow{T,a}\cap\overrightarrow{T,b}=\emptyset$,
(3) $\overline{T\cup\{a\}}=\overrightarrow{T,a}\,\dot\cup\, T\,\dot\cup\,\overrightarrow{T,b}$.

**Proof.** (1). Let $x,y\in\overrightarrow{T,a}, x\neq y, z\in ]x,y[$ and $x' := ]a,x[\cap T$, $y' := ]a,y[\cap T$. If $a,x,y$ are collinear, then $x'=y'$ and $x,y\in\overrightarrow{x',a}\subset\overrightarrow{T,a}$, hence by Property 1.1(4), $]x,y[\subset\overrightarrow{x',a}\subset\overrightarrow{T,a}$. Now let $a,x,y$ be non-collinear, i.e. $x'\neq y'$. By Property 1.2.(2), $z'' := ]a,z[\cap ]x,y'[\neq\emptyset$, and then $z''\in ]x,y'[, x'\in ]x,a[$ gives again by Property 1.2.(2), $z' := ]a,z''[\cap ]x',y'[\neq\emptyset$. From $z'\in ]a,z''[$ and $z''\in ]a,z[$ we obtain $z'\in ]a,z[$ by Property 1.1.(1). Finally $z'\in ]x',y'[\subset\overline{x',y'}\subset T$ implies $z\in\overrightarrow{T,a}$ consequently $\overrightarrow{T,a}\in\mathfrak{C}$.

(2) Let $b' := ]a,b[\cap T$, $b'' := ]c,b[\cap T$ and again $x\in\overrightarrow{T,a}$ and $x' = ]x,a[\cap T$. Then by Property 1.3, $\emptyset\neq ]c,x[\cap\overline{b',b'',x'}\subset ]c,x[\cap T$, i.e. $x\in\overrightarrow{T,c}$ and consequently $\overrightarrow{T,a}\subset\overrightarrow{T,c}$. Suppose $x\in\overrightarrow{T,a}\cap\overrightarrow{T,b}$, i.e. $x' := ]a,x[\cap T\neq\emptyset$ and $x'' := ]b,x[\cap T$. Then by Property 1.2.(4), $\overline{a,b,x}=\overline{b',x',x''}\subset T$ which contradicts $a,b\notin T$.

(3) Since $T\cup\{a\}\subset\overrightarrow{T,a}\cup T\cup\overrightarrow{T,b}=:R\subset\overline{T\cup\{a\}}$ we have to show $R\in\mathfrak{T}$. Let $x,y\in R$ with $x\neq y$. If $x,y\in T$ then $\overline{x,y}\subset T\subset R$. Therefore let $y\notin T$ and we may assume $y\in\overrightarrow{T,b}$. By (2), $\overrightarrow{T,a}=\overrightarrow{T,y}$, i.e. $R=\overrightarrow{T,y}\cup T\cup\overrightarrow{T,b}$ and $y'' := ]y,b[\cap T\neq\emptyset$. We consider the cases:

1. $x\in\overrightarrow{T,y}$. Then $b,x\in\overrightarrow{T,a}=\overrightarrow{T,y}$ implies $\overrightarrow{T,x}=\overrightarrow{T,b}$ by (2), and so $z := ]x,y[\cap T\neq\emptyset$. Hence by Property 1.1(5), $\overline{x,y}=\overrightarrow{z,x}\cup\{z\}\cup\overrightarrow{z,y}\subset\overrightarrow{T,x}\cup T\cup\overrightarrow{T,y}=R$.

2. $x\in T$. By (Z3), $\exists z\in P : x\in ]z,y[$, hence $z\in\overrightarrow{x,y}\subset\overrightarrow{T,y}=\overrightarrow{T,a}$ and by (2) $\overrightarrow{T,z}=\overrightarrow{T,b}$. Therefore $\overline{x,y}=\overrightarrow{x,z}\cup x\cup\overrightarrow{x,y}\subset\overrightarrow{T,z}\cup T\cup\overrightarrow{T,y}=R$.

3. $x\in\overrightarrow{T,b}$ and $z := \overline{x,y}\cap T\neq\emptyset$. Then $\overline{x,y}=\overline{z,y}\subset R$ by 2.

4. $x\in\overrightarrow{T,b}$ and $\overline{x,y}\cap T=\emptyset$; then $x'' := ]b,x[\cap T\neq\emptyset$. Let $z\in\overline{x,y}\setminus\{x,y\}$. If $z\in ]x,y[$ then by (1), $z\in\overrightarrow{T,b}\subset R$. If $z\notin ]x,y[$ we may assume $z\in\overrightarrow{x,y}$, i.e. $x\in ]z,y[$. From $x\in ]z,y[$ and $y''\in ]y,b[$ we obtain by Property 1.2(2), $u := ]y'',z[\cap ]b,x[\neq\emptyset$ and $u\notin T$, hence $u\neq x''$. Therefore by Property 1.1(2), $x''\in ]x,u[\,\cup\, ]u,b[$. If $x''\in ]x,u[$ then together with $u\in ]y'',z[$ we obtain $\overline{x'',y''}\cap ]x,z[\neq\emptyset$ by (ZP), and so $\emptyset\neq T\cap ]x,z[\subset T\cap\overline{x,y}$ a contradiction to $\overline{x,y}\cap T=\emptyset$. Consequently $x''\in ]u,b[$, and since $u\in ]y'',z[$ we have

$z'' := \overline{x'', y''} \cap ]b, z[ \neq \emptyset$ by (ZP). But $z'' \in \overline{x'', y''} \subset T$ and $z'' \in ]b, z[$ implies $z \in \overrightarrow{T, b}$, thus $\overline{x, y} \subset \overrightarrow{T, b} \subset R$.

From Theorem 1.4 it follows that each subspace $T \in \mathfrak{T} \setminus \{P, \emptyset\}$ and each point $a \in P \setminus T$ determines exactly one *halfspace* $\overrightarrow{T, a}$ in the following way: Let $t \in T$ and $b \in P$ such that $t \in ]a, b[$ (cf. (Z3)). Then $\overrightarrow{T, a} := \overrightarrow{T, b}$.

From Theorem 1.4 we obtain (cf. [31, (5.3)])

**Theorem 1.5.** *If* $(P, \mathfrak{L}, \alpha)$ *is an ordered space, then* $(P, \mathfrak{L})$ *is an* exchangespace, *i.e. if* $S \subset P, a, b \in P$ *such that* $b \in \overline{S \cup \{a\}} \setminus \overline{S}$, *then* $a \in \overline{S \cup \{b\}}$.

**Proof.** $b \in \overline{S \cup \{a\}} \setminus \overline{S}$ implies $a, b \notin T := \overline{S}$. Let $t \in T$ and $c \in P$ with $t \in ]a, c[$ (cf. (Z3)). Then by definition $a \in \overrightarrow{T, c}$ and by Property 1.4(3) we have $b \in (\overline{S \cup \{a\}} \setminus \overline{S}) = \overrightarrow{T, a} \cup \overrightarrow{T, c}$. If $b \in \overrightarrow{T, c}$ then $a \in \overrightarrow{T, c} \subset \overrightarrow{T, b} \dot{\cup} T \dot{\cup} \overrightarrow{T, c} = \overline{T \cup \{b\}} = \overline{S \cup \{b\}}$. If $b \in \overrightarrow{T, a}$, then $\overrightarrow{T, b} = \overrightarrow{T, c} \ni a$ by Property 1.4(2), hence $a \in \overrightarrow{T, b} \subset \overline{T \cup \{b\}} = \overline{S \cup \{b\}}$.

Since an exchange space $(P, \mathfrak{L})$ has a base B and two bases have the same cardinality (cf. e.g. [15, Section 8]) one defines $\dim(P, \mathfrak{L}) := |B| - 1$ as *dimension* of $(P, \mathfrak{L})$.

An ordered space $(P, \mathfrak{L}, \alpha)$ is called *desarguesian* if the following holds:

(D) Let $z \in P$, let $(G_1, G_2, G_3 \in \mathfrak{L}(z) := \{L \in \mathfrak{L} \mid z \in L\}$ be distinct and for $i \in \{1, 2, 3\}$ let $a_i, b_i \in G_i \setminus \{z\}$ be distinct such that the points $p_1 := \overline{a_2, a_3} \cap \overline{b_2, b_3} \neq \emptyset$, $p_2 := \overline{a_3, a_1} \cap \overline{b_3, b_1} \neq \emptyset$ and $p_3 := \overline{a_1, a_2} \cap \overline{b_1, b_2} \neq \emptyset$ exist. Then $p_1, p_2, p_3$ are collinear.

Since $(P, \mathfrak{L}, \alpha)$ is an exchange space, one obtains with the same arguments as in [15, Section 10] the following result:

**Property 1.6.** *If* $\dim(P, \mathfrak{L}) \geqslant 3$, *then* $(P, \mathfrak{L}, \alpha)$ *is desarguesian.*

**Property 1.7.** *If* $E \in \mathfrak{E}$, $A, B \in \mathfrak{L}(E)$ *with* $A \neq B$, *there are* $z \in E \setminus (A \cup B)$ *and* $C, D \in \mathfrak{L}(z)$ *such that* $C \neq D$ *and* $C \cap A, C \cap B, D \cap A, D \cap B \neq \emptyset$.

**Proof.** (1) If $s := A \cap B \neq \emptyset$, let $a \in A \setminus s$, $b \in B \setminus s$, $C := \overline{a, b}$, $z \in ]a, b[$, $a' \in A$ such that $a \in ]a', s[$ and $D = \overline{a', z}$, then $\emptyset \neq D \cap ]b, s[ \subset D \cap B$ by (ZP).

(2) If $A \cap B = \emptyset$ let $a \in A$, $b \in B$, $C := \overline{a, b}$, $z \in C$ such that $a \in ]z, b[$, $b' \in B \setminus b$ and $D := \overline{z, b'}$. Since $B \cap A = \emptyset$ we have by Property 1.4, $b' \in B \subset \overrightarrow{A, z}$ hence $\emptyset \neq ]b', z[ \cap A \subset D \cap A$.

## 2. Ordered spaces with hyperbolic incidence structure

The concepts hyperbolic parallel or in short h-parallel and one-sided h-parallel respectively, were introduced in [8] and in [10,22], respectively.

In this section let $(P, \mathfrak{L}, \alpha)$ be an ordered space. If $a, b, c, d \in P$ with $a \neq b$ and $c \neq d$, the halfline $\overrightarrow{a,b}$ is called *one-sided h-parallel* to the halfline $\overrightarrow{c,d}$, denoted by $\overrightarrow{a,b} \Vdash\!\!\rightarrow \overrightarrow{c,d}$, if

1. $\overline{a,b} \cap \overline{c,d} = \emptyset$,
2. $]a,c[ \,\cap\, ]b,d[ = \emptyset$,
3. $\forall\, x \in ]b,d[ : \overline{a,x} \cap \overrightarrow{c,d} \neq \emptyset$.

If $\overrightarrow{a,b} \Vdash\!\!\rightarrow \overrightarrow{c,d}$ and $\overrightarrow{c,d} \Vdash\!\!\rightarrow \overrightarrow{a,b}$ then we write $\overrightarrow{a,b} \,\Vert\, \overrightarrow{c,d}$ and call $\overrightarrow{a,b}$ and $\overrightarrow{c,d}$ *h-parallel*. If $G \in \mathfrak{L}$, then $\overrightarrow{a,b}$ is called *one-sided h-parallel* to $G$ – denoted by $\overrightarrow{a,b} \Vdash\!\!\rightarrow G$ – if there are $c, d \in G$ with $c \neq d$ and $\overrightarrow{a,b} \Vdash\!\!\rightarrow \overrightarrow{c,d}$. In [11,10] the following theorems are proved:

**Property 2.1** (Extended Pasch statement). *Let* $a, b, c, d \in P$ *with* $\overrightarrow{a,b} \Vdash\!\!\rightarrow \overrightarrow{c,d}$ *and* $G \in \mathfrak{L}(\overline{\{a,b,c\}}) := \{X \in \mathfrak{L} \mid X \subset \overline{\{a,b,c\}}$ with $a, c \notin G$. Then

(1) *If* $G \cap \overrightarrow{a,b} \neq \emptyset$ *then either* $G \cap ]a,c[ \neq \emptyset$ *or* $G \cap \overrightarrow{c,d} \neq \emptyset$.

(2) *If* $G \cap ]a,c[ \neq \emptyset$ *and* $G \cap \overrightarrow{a,b} = G \cap \overrightarrow{c,d} = \emptyset$ *then* $\overrightarrow{a,b} \Vdash\!\!\rightarrow G$, [10, (1.3)].

**Property 2.2.** *Let* $(P, \mathfrak{L}, \alpha)$ *be desarguesian and let* $a, b, c, d \in P$ *with* $a \neq b, c \neq d$ *and* $\overrightarrow{a,b} \Vdash\!\!\rightarrow \overrightarrow{c,d}$ *then* $\overrightarrow{c,d} \Vdash\!\!\rightarrow \overrightarrow{a,b}$, [10, (1.5)].

**Remark 1.** By Properties 1.6 and 2.2 we have: If $(P, \mathfrak{L}, \alpha)$ is an ordered space with $\dim(P, \mathfrak{L}) \geqslant 3$ or if $(P, \mathfrak{L}, \alpha)$ is a desarguesian ordered plane, then the concepts one-sided h-parallel and h-parallel coincide.

Two lines $G, H \in \mathfrak{L}$ are called *h-parallel*, denoted by $G \Vert H$ if there are $a, b \in G$, $c, d \in H$ with $a \neq b$, $c \neq d$ and $\overrightarrow{a,b} \,\Vert\, \overrightarrow{c,d}$.

**Property 2.3.** *Let* $G \in \mathfrak{L}$ *and* $p \in P \backslash G$. *Then* $|\{H \in \mathfrak{L} \mid p \in H,\ H \Vert G\}| \leqslant 2$, [10, (1.6)].

We introduce Hilbert's concept of an end as follows: A subset $\mathfrak{e} \subset \mathfrak{L}$ is called an *end* of $(P, \mathfrak{L}, \alpha)$ if

(1) $\forall X, Y \in \mathfrak{e}$ with $X \neq Y : X \Vert Y$,

(2) $\bigcup \mathfrak{e} := \bigcup_{X \in \mathfrak{e}} X = P$.

For $a, b \in P$ with $a \neq b$ let $(\overrightarrow{a,b}) := \{X \in \mathfrak{L} \mid \overrightarrow{a,b} \Vert X\} \cup \{\overline{a,b}\}$. Then $(P, \mathfrak{L}, \alpha)$ is called an *ordered space with hyperbolic incidence structure* if the 'non-euclidean parallel axiom'

(H) $\forall G \in \mathfrak{L}, \forall p \in P \backslash G \exists H_1, H_2 \in \mathfrak{L}$ with $H_1 \neq H_2, p \in H_1 \cap H_2, H_1 \Vert G$ and $H_2 \Vert G$

and the 'ends axiom'

(E) $\forall a, b \in P$ with $a \neq b$ the set $(\overrightarrow{a,b})$ is an end

are valid.

**Property 2.4.** *If* $\dim(P, \mathfrak{L}, \alpha) \geqslant 3$ *then the condition* (E) *is a consequence of* (H) [11, (3.11)].

**Remark 2.** The ordered space $(P,\mathfrak{L},\alpha)$ is an *ordered affine space* if

(P) $\forall G\in\mathfrak{L}\ \forall p\in P\backslash G\exists_1 H\in\mathfrak{L}$ with $p\in H$ and $H\Vert G$.

**Property 2.5.** *If* $(P,\mathfrak{L},\alpha)$ *has hyperbolic incidence structure then*

(1) $\overrightarrow{a,b}\ \Vert\!\!\!\rightarrow \overrightarrow{c,d} \Rightarrow \overrightarrow{c,d}\ \Vert\!\!\!\rightarrow \overrightarrow{a,b}$, [10, (1.7)]

(2) *If* $A\Vert B$ *and* $a_1,a_2\in A$ *with* $a_1\neq a_2$ *then either* $\overrightarrow{a_1,a_2}\Vert B$ or $\overrightarrow{a_2,a_1}\Vert B$, [10, (1.8)]

(3) *Let* $a,b,c\in P$ *be non-collinear, then there exists exactly one line* $(a\Vert\overrightarrow{b,c}):=A\in\mathfrak{L}$ *with* $a\in A$ *and* $\overrightarrow{b,c}\Vert A$ *and moreover* $(a\Vert\overrightarrow{b,c})\cap(c\Vert\overrightarrow{b,a})\neq\emptyset$, [10, (1.9)].

**Remark on Property 2.5.** If $a,b,c\in P$ are non-collinear then by Property 2.5(3) the points $a':=(c\Vert\overrightarrow{a,b})\cap(b\Vert\overrightarrow{a,c})$, $b':=(a\Vert\overrightarrow{b,c})\cap(c\Vert\overrightarrow{b,a})$ and $c':=(b\Vert\overrightarrow{c,a})\cap(a\Vert\overrightarrow{c,b})$ exist and we have $a\neq a', b\neq b', c\neq c'$. From Fisher and Ruoff I learned that in hyperbolic geometry (in the sense of Hilbert (cf. [15])) the statement

(C) The lines $\overline{a,a'},\overline{b,b'},\overline{c,c'}$ intersect in a common point.

is valid. Therefore the following problem arises:

**Problem.** If $(P,\mathfrak{L},\alpha)$ is an ordered space with a hyperbolic incidence structure such that (C) is valid, then is $(P,\mathfrak{L},\alpha)$ a hyperbolic geometry?

## 3. Projective embedding of ordered spaces with hyperbolic incidence structure

Let $(P,\mathfrak{L},\alpha)$ be an ordered space with $\dim(P,\mathfrak{L},)\geqslant 3$ and let $\mathfrak{E}$ be the set of all planes of $(P,\mathfrak{L})$. A subset $\mathfrak{b}\subset\mathfrak{L}$ is called *bundle* if

1. Any two lines $X,Y\in\mathfrak{b}$ are coplanar,
2. $\bigcup\mathfrak{b}:=\bigcup_{X\in\mathfrak{b}}X=P$.

With $P_p$ we denote the set of all bundles. A subset $G_p\subset P_p$ is called *b-line* if there are $\mathfrak{a},\mathfrak{b}\in G_p$ with $\mathfrak{a}\neq\mathfrak{b}$ such that for $\mathfrak{x}\in P_p$

$$\mathfrak{x}\in G_p\Leftrightarrow\forall E\in\mathfrak{E}\quad\text{holds with } \mathfrak{L}(E)\cap\mathfrak{a}\neq\emptyset,\ \mathfrak{L}(E)\cap\mathfrak{b}\neq\emptyset : \mathfrak{L}(E)\cap\mathfrak{x}\neq\emptyset.$$

Let $\mathfrak{L}_p$ be the set of all b-lines. Sörensen [31] and Kreuzer [24] proved:

**Property 3.1.** *The "bundle space"* $(P_p,\mathfrak{L}_p)$ *is a projective space with* $\dim(P_p,\mathfrak{L}_p)=\dim(P,\mathfrak{L})$ *and the map*

$$\iota : P\to P_p;\quad x\to\mathfrak{L}(x)$$

*is an injection such that* $(P,\mathfrak{L})$ *and* $(\iota(P),\mathfrak{L}_{\iota(P)})$ *with*

$$\mathfrak{L}_{\iota(p)}:=\{X\cap\iota(P)\,|\,X\in\mathfrak{L}_p : X\cap\iota(P)\neq\emptyset\}$$

*are isomorphic.*

Because of Property 3.1 we assume that $(P_p, \mathfrak{L}_p)$ is a projective space, $P$ a subset of $P_p$ and $\mathfrak{L} = \{X \cap P \mid X \in \mathfrak{L}_p : X \cap P \neq \emptyset\}$. The incidence closure of the projective space $(P_p, \mathfrak{L}_p)$ will be denoted by $\langle\rangle$.

The order $\alpha$ can be extended in a modified way to an order structure $\tau$ such that $(P_p, \mathfrak{L}_p, \tau)$ becomes an *ordered projective space* (cf., e.g. [12, p. 125]).

Let $P_p^{(4)} := \{(a_1, a_2, a_3, a_4) \in P_p^4 \mid a_1, a_2 \neq a_3, a_4$ and $a_1, a_4 \in \langle a_2, a_3 \rangle\}$ and let

$$\tau : P_p^{(4)} \to \{1, -1\}; (a_1, a_2, a_3, a_4) \mapsto [a_1, a_2 \mid a_3, a_4]$$

be a map. Then $\tau$ is called a *separation function* and $(P_p, \mathfrak{L}_p, \tau)$ an *ordered projective space* if

(T1) For all $a, b, c, d, e \in P_p$ with $a, b \neq c, d, e$ and $b, d, e \in \langle a, c\rangle : [a, b \mid c, d] \cdot [a, b \mid d, e] = [a, b \mid c, e]$.

(T2) For all $a, b, c, d \in P$ distinct with $c, d \in \langle a, b\rangle$, exactly one of the values $[a, b \mid c, d]$, $[a, c \mid d, b], [a, d \mid b, c]$ equals $-1$.

(T3) For all $(a, b, c, d) \in P_p^{(4)}$ and for each perspectivity $\pi$:
$[a, b \mid c, d] = [\pi(a), \pi(b) \mid \pi(c), \pi(d)]$.

A subset $M \subset P_p$ is called *projectively convex* in $(P_p, \mathfrak{L}_p, \tau)$ if for $a, b, c, d \in P_p$ distinct and collinear with $a, b \in M, d \notin M$ and $[a, b \mid c, d] = -1$ always $c \in M$ and *projectively open* in $(P_p, \mathfrak{L}_p, \tau)$ if $\forall c \in M\ \forall d \in P_p \backslash M$ exist $a, b \in \langle c, d\rangle \cap M$ with $[a, b \mid c, d] = -1$ and $\{x \in \langle a, b\rangle \backslash \{a, b\} \mid [a, b \mid x, d] = -1\} \subset M$.

By the papers of Sörensen and Kreuzer (cf. [31, (6.8); 24, (10.14)]) we have the following result.

**Property 3.2.** *$(P_p, \mathfrak{L}_p)$ can be changed into an ordered projective space $(P_p, \mathfrak{L}_p, \tau)$ such that*

(1) *$P$ is projectively convex and projectively open in $(P_p, \mathfrak{L}_p, \tau)$.*

(2) *If $a, b, c, d \in P$ are distinct and collinear and $u \in \langle a, b\rangle \backslash P$ then $[a, b \mid c, d] = (a \mid c, d) \cdot (b \mid c, d)$* and *$[u, a \mid b, c] = (a \mid b, c)$.*

Following Kreuzer [24,11] we define for $G \in \mathfrak{L} : s \in \langle G\rangle$ is called *limit point* of $G$ if one of the two conditions holds:

(i) $\langle G\rangle \backslash G = \{s\}$

(ii) $|\langle G\rangle \backslash G| \geqslant 2$ and for all $a, b \in G \backslash \{s\}$, for all
$x, y \in \langle G\rangle \backslash (G \cup \{s\}) : [a, b \mid x, s] = [a, s \mid x, y] = 1$.

Let $\mathrm{rd}(G)$ be the set of all limit points of $G$. $s \in P_p$ is called *limit point* of $P$ if there is a $G \in \mathfrak{L}$ such that $s \in \mathrm{rd}(G)$. Let $\mathrm{rd}(P)$ be the set of all limit points of $P$. In [11] we find the following results:

**Property 3.3.** *Let $G, H \in \mathfrak{L}, G \neq H$ with $\{s\} := \langle G\rangle \cap \langle H\rangle \neq \emptyset$ and $s \notin P$. Then*

(1) $\mathrm{rd}(G) \cap G = \emptyset$ *hence* $\mathrm{rd}(P) \cap P = \emptyset$. [11, (2.6)]

(2) $|\mathrm{rd}(G)| \leqslant 2$. [11, (2.8)]

(3) *If* $|\langle G\rangle \backslash G| = 1$, i.e. $\{s\} = \langle G\rangle \backslash G = \mathrm{rd}(G)$, *then* $\{s\} = \langle H\rangle \backslash H = \mathrm{rd}(H)$. [11, (2.9)]

(4) $s\in \mathrm{rd}(G) \Leftrightarrow s\in \mathrm{rd}(H)$, [11, (2.10)]
(5) $G \nparallel H \Leftrightarrow s\in \mathrm{rd}(G)\cap \mathrm{rd}(H)$, [11, *Eqs*. (3.4) *and* (3.6)]

**Property 3.4.** $\mathrm{rd}(P)$ *is the set of all ends of* $(P,\mathfrak{L},\alpha)$. [11, (3.9)]

**Property 3.5** (Characterization theorem) (Karzel et al. [11, (3.11)]). *Let* $(P,\mathfrak{L},\alpha)$ *be an ordered space with* $\dim(P,\mathfrak{L})\geqslant 3$. *Then*
(1) $(P,\mathfrak{L},\alpha)$ *is an ordered affine space if and only if* $\forall G\in\mathfrak{L} : |\mathrm{rd}(G)| = 1$.
(2) $(P,\mathfrak{L},\alpha)$ *has hyperbolic incidence structure if and only if for all* $G\in\mathfrak{L}$: $|\mathrm{rd}(G)| = 2$.

## 4. Incidence and congruence

Following Sörensen [30] we introduce the concept of a space with congruence $(P,\mathfrak{L},\equiv)$.

Let $(P,\mathfrak{L})$ be an incidence space and let $\equiv$ be a *congruence relation* on $P\times P$, (i.e. $\equiv$ is an equivalence relation such that for all $a,b,c\in P$ : $(a,b)\equiv(b,a)$ and $(a,a)\equiv(b,c)\Leftrightarrow b=c$) such that the following compatibility axioms (W1), (W2) and (W3) between incidence and congruence are valid:

(W1) For all $a,b,c\in P$ collinear and distinct, for all $a',b'\in P$ with $(a,b)\equiv(a',b')$ there exists exactly one $c'\in\overline{a',b'}$ such that $(a,b,c)\equiv(a',b',c')$ (i.e. $(a,b)\equiv(a',b')$, $(b,c)\equiv(b',c')$ and $(a,c)\equiv(a',c')$).

(W2) For all $a,b,x\in P$ non-collinear, for all $a',b',x'\in P$ with $(a,b,x)\equiv(a',b',x')$, for all $c\in\overline{a,b}$, for all $c'\in\overline{a',b'}$ with $(a,b,c)\equiv(a',b',c')$, $(x,c)\equiv(x',c')$ holds.

(W3) For all $a,b,x\in P$ non-collinear there exists exactly one $x'\in\overline{\{a,b,x\}}\setminus\{x\}$ with $(a,b,x)\equiv(a,b,x')$, denoted by $\widetilde{a,b}(x):=x'$.

Then we call $(P,\mathfrak{L},\equiv)$ a *weak space with congruence* and a *space with congruence*, if moreover, the following condition (F) is valid.

(F) For all $E\in\mathfrak{E}$, for all $A,B\in\mathfrak{L}(E)$ with $A\neq B$ there exists $z\in E\setminus(A\cup B)$ and $C_1,C_2\in\mathfrak{L}(z)$ with $C_1\neq C_2$ and $C_i\cap A, C_i\cap B\neq\emptyset$ for $i\in\{1,2\}$ (cf. Property 1.7)

For $G\in\mathfrak{L}$ let $a,b\in G$ with $a\neq b$ and let

$$\tilde{G}: P\to P;\quad x\mapsto\begin{cases} x & \text{if } x\in G,\\ \widetilde{a,b}(x) & \text{if } x\notin G.\end{cases}$$

By (W2), $\tilde{G}$ does not depend on the choice of $a,b\in G$. $\tilde{G}$ is called a *line-reflection*. Let $\tilde{\mathfrak{L}}:=\{\tilde{L}\mid L\in\mathfrak{L}\}$. For $A,B\in\mathfrak{L}$ and $p\in P$ let

$$A\perp B :\Leftrightarrow A\neq B \quad\text{and}\quad \tilde{A}(B)=B$$

$$(p\perp A):=\{L\in\mathfrak{L}(p)\mid L\perp A\}$$

A permutation $\varphi \in \mathrm{Sym}P$ is called a *motion* if $\varphi$ is a *collineation* (i.e., $\forall L \in \mathfrak{L}$ : $\varphi(L) \in \mathfrak{L}$ ) and if for all $a, b \in P : (a,b) \equiv (\varphi(a), \varphi(b))$.

Let $\mathfrak{A}$ be the group of all motions of $(P, \mathfrak{L}, \equiv)$. By Sörensen [30] we have

**Property 4.1.** *Let* $(P, \mathfrak{L}, \equiv)$ *be a weak space with congruence and let* $E \in \mathfrak{E}$, $G \in \mathfrak{L}(E)$, $a, b, c, d \in E, a \neq b$, $A, B, C \in \mathfrak{L}(E) \cap \mathfrak{L}(d)$, $\alpha \in \mathfrak{A}$ *and* $\sigma \in \mathrm{Sym}P$. *Then*:

(1) $\tilde{G} \circ \tilde{G} = \mathrm{id}$, $\mathrm{Fix}\tilde{G} = G$, $\tilde{G}(E) = E$, [30, (1.4)].

(2) $|\{m \in \overline{a,b} \mid (m,a) \equiv (m,b)\}| \leqslant 1$ (*i.e. there is at most one "mid-point"*), $|\{M \in \mathfrak{L}(E) \mid \tilde{M}(a) = b\}| \leqslant 1$, [30, Eq. (1.1) and (1.7)].

(3) *If* $p \notin A$ then $(p \perp A) = \{\overline{p, \tilde{A}(p)}\}$, [30, (1.6)].

(4) *If* $\alpha(d) = d$ *and* $\alpha(A) = A$ *then* $\alpha^2|_A = \mathrm{id}_A$ *and either* $A \subset \mathrm{Fix}\alpha := \{x \in P \mid \alpha(x) = x\}$ *or* $\mathrm{Fix}\alpha \cap A = \{d\}$. *If moreover* $\alpha(E) = E$ *then* $\alpha^2|_E = \mathrm{id}_E$ *and either* $\mathrm{Fix}\alpha \supset E$ *or* $\mathrm{Fix}\alpha \cap E = A$ *or* $\mathrm{Fix}\alpha \cap E = \{d\}$, *and if* $\mathrm{Fix}\alpha \cap E = A$ *then* $\alpha|_E = \tilde{A}|_E$, [30, Eqs. (1.11), (1.12) and (1.14)].

(5) *If* $\tilde{A}|_E, \tilde{B}|_E, \tilde{C}|_E \in \mathfrak{A}_E$ *then there exists a unique* $D \in \mathfrak{L}(E) \cap \mathfrak{L}(d) : \tilde{A} \circ \tilde{B} \circ \tilde{C}|_E = \tilde{D}|_E$ ('*theorem of the three reflections*'), [30, (1.15)].

*If moreover axiom* (F) *is valid* (*i.e.*, $(P, \mathfrak{L}, \equiv)$ *is a space with congruence*), *then*

(6) $\tilde{G}|_E \in \mathfrak{A}_E$, [30, (3.1)].

(7) *If* $(c,a) \equiv (c,b)$ *then there exists exactly one* $M \in \mathfrak{L}(c) \cap \mathfrak{L}(E)$ *with* $\tilde{M}(a) = b$, [30, Eqs. (1.7), (1.8) and (4.2)], *i.e.*, $|(c \perp \overline{a,b}) \cap \mathfrak{L}(E)| = 1$.

(8) *If for all* $x, y \in P : (x,y) \equiv (\sigma(x), \sigma(y))$ *then* $\sigma \in \mathfrak{A}$, [30, (1.11)].

We consider the following bifurcation: $(P, \mathfrak{L}, \equiv)$ is called *regular* if:

(R) There exists $a, b, c \in P$ with $a \neq b, c \in \overline{a,b} \backslash \{b\}$ and $(a,b) \equiv (a,c)$

and *weak Lotkernspace* resp. *Lotkernspace* if (L) resp. (L) and (F) are valid.

(L) For all $a, b, c \in P$ with $a \neq b$ and $c \in \overline{a,b} \backslash \{b\} : (a,b) \not\equiv (a,c)$.

By [32, (4.3)] we have

**Property 4.2.** *If* $(P, \mathfrak{L}, \equiv)$ *is a regular space with congruence then*

(1) *For all* $x, y \in P, x \neq y$ *there exists* $y' \in \overline{x,y} \backslash \{y\}$ *with* $(x,y) \equiv (x,y')$.

(2) *For all* $G \in \mathfrak{L}$, *for all* $c \in G$, *for all* $E \in \mathfrak{E}(G) : (c \perp G) \cap \mathfrak{L}(E) \neq \emptyset$.

For $(P, \mathfrak{L}, \equiv)$ we claim the further axiom:

(Wa) For all $x, y \in P, x \neq y : |\{z \in \overline{x,y} \backslash \{y\} \mid (x,y) \equiv (x,z)\}| \leqslant 1$.

If $(P, \mathfrak{L}, \equiv)$ is regular, then by Property (4.2) and (Wa) we can associate to each $a \in P$ the *point reflection*

$$\tilde{a} : P \to P; \quad x \mapsto \begin{cases} x & \text{if } x = a, \\ \tilde{a}(x) := \{y \in \overline{a,x} \backslash \{x\} \mid (a,x) \equiv (a,y)\} & \text{if } x \neq a. \end{cases}$$

Let $\tilde{P} := \{\tilde{p} \mid p \in P\}$.

**Property 4.3.** *Let* $(P, \mathfrak{L}, \equiv)$ *be a regular space with congruence such that* (Wa) *holds and let* $L \in \mathfrak{L}$, $a, b, c \in L$ *with* $a \neq b, d \in P, \alpha \in \mathfrak{A}$ *and* $T \in \mathfrak{T}$ *with* $a \in T$. *Then*

(1) $\tilde{a} \circ \tilde{a} = \mathrm{id}$, $\mathrm{Fix}\tilde{a} = \{a\}, \tilde{a}(T) = T$.

(2) $\mathrm{Fix}(\tilde{a} \circ \tilde{b}) = \emptyset$ by Property 4.1(2).

(3) $\sim: P \rightarrow P; p \rightarrow \tilde{p}$ *is a bijection.*

(4) $\widetilde{\alpha(a)} = \alpha \circ \tilde{a} \circ \alpha^{-1}, \widetilde{\alpha(L)} = \alpha \circ \tilde{L} \circ \alpha^{-1}$.

(5) $\tilde{P} \subset \mathfrak{A}$

(6) *If the points* $c$ *and* $\tilde{a}(\tilde{b}(c))$ *have a midpoint* $m$, *and if* $\psi := \tilde{a} \circ \tilde{b} \circ \tilde{c}$ *then* $\psi^2 = \mathrm{id}$, $\mathrm{Fix}\ \psi \in \mathfrak{T}$ *and* $\mathrm{Fix}\ \psi \cap L = \{m\}$, *i.e.* $\psi$ *is a reflection in the subspace* $\mathrm{Fix}\psi$. *If* $\mathrm{Fix}\psi \neq \{m\}, u \in \mathrm{Fix}\psi \backslash \{m\}, v := \tilde{c}(u)$, *and* $w := \tilde{b}(v)$ *then* $\{u, v, w\}$ *are non-collinear and* $c$ *resp.* $b$ *resp.* $a$ *is the midpoint of* $u, v$ *resp.* $v, w$ *resp.* $w, u$.

(7) *If there is a point* $o \in P$ *with* $\tilde{P}(o) = P$ *then any two points* $x, y \in P$, *with* $x \neq y$ *have a midpoint and* $\tilde{P}$ *acts regularly on* $P$.

**Proof.** (5) Let $a, x, y \in P$ be distinct, let $E \in \mathfrak{E}$ with $a, x, y \in E, X := \overline{a, x}, Y := \overline{a, y}$, $S \in (a \perp X) \cap \mathfrak{L}(E)$ (cf. Property 4.2(2)), and $\varphi := \tilde{S} \circ \tilde{X}$. We want to show $\tilde{a} \in \mathfrak{A}$. By Property 4.1(6) $\varphi|_E \in \mathfrak{A}_E$ and so $(x, y) \equiv (\varphi(x), \varphi(y))$. Moreover $\varphi(x) = \tilde{S} \circ \tilde{X}(x) = \tilde{S}(x) \in X$ (since $S \perp X$), $(a, x) \equiv (\varphi(a), \varphi(x)) = (a, \tilde{S}(x))$, and $\tilde{S}(x) \neq x$ (since $x \notin S$). By (Wa), $\tilde{S}(x) = \tilde{a}(x)$. By Properties 4.1(5) and (6), there exists $Z \in \mathfrak{L}(a) \cap \mathfrak{L}(E)$ with $\tilde{Z}|_E = \tilde{S} \circ \tilde{X} \circ \tilde{Y}|_E = \varphi \circ \tilde{Y}|_E$, hence $\varphi|_E = \tilde{Z} \circ \tilde{Y}|_E$, and so $\varphi(y) = \tilde{Z}(\tilde{Y}(y)) = \tilde{Z}(y)$ with $(a, y) \equiv (a, \tilde{Z}(y))$. By $S \perp X$, i.e. $S \neq X$ and $\tilde{X}|_E = \widetilde{\tilde{S}(X)}|_E = \tilde{S} \circ \tilde{X} \circ \tilde{S}|_E$ we obtain $\varphi^2|_E = \mathrm{id}_E$, hence $\varphi|_E = \tilde{Y} \circ \tilde{Z}|_E$ and so $\tilde{Z}(y) = \tilde{Y} \circ \tilde{Z} \circ \tilde{Y}(y) = \tilde{Y}(\tilde{Z}(y))$, i.e. $\tilde{Z}(y) \in Y$. From $(a, y) \equiv (a, \tilde{Z}(y))$, $a, y, \tilde{Z}(y) \in Y$ and $y \neq \tilde{Z}(y)$ we obtain by (Wa), $\tilde{Z}(y) = \tilde{a}(y)$. Therefore $(x, y) \equiv (\varphi(x), \varphi(y)) = (\tilde{S}(x), \tilde{Z}(y)) = (\tilde{a}(x), \tilde{a}(y))$, and so by Property 4.1(8), $\tilde{a} \in \mathfrak{A}$.

(6) $a, b, c \in L$ imply $\tilde{a}(\tilde{b}(c)) \in L$ and therefore $m \in L$ and $c = \tilde{m} \circ \tilde{a} \circ \tilde{b}(c) = \tilde{m} \circ \psi(c)$. By (5), $\psi, \varphi := \tilde{m} \circ \psi \in \mathfrak{A}$, $\varphi(c) = c$ and by (1), $\varphi(L) = \psi(L) = L$ and for all $E \in \mathfrak{E}(L): \varphi(E) = \psi(E) = E$. Consequently by Property 4.1(4) $\varphi^2 = \mathrm{id}$ and $\mathrm{Fix}\varphi \supset L$ or $\mathrm{Fix}\varphi \cap L = \{c\}$. Suppose $\mathrm{Fix}\varphi \cap L = \{c\}$, then $\tilde{c}|_L = \varphi|_L = \tilde{m} \circ \tilde{a} \circ \tilde{b} \circ \tilde{c}|_L$ hence $\tilde{a} \circ \tilde{b}|_L = \tilde{m}|_L$ implying $\tilde{a}(b) = \tilde{m}(b)$, and so $a = m$ by (2). This gives us $\tilde{b}|_L = \mathrm{id}_L$, a contradiction since $\mathrm{Fix}\tilde{b} = \{b\}$. Therefore $\mathrm{Fix}\varphi \supset L$, i.e. $\psi|_L = \tilde{m}|_L$ and so $\mathrm{Fix}\psi \cap L = \{m\}$. This implies, again by Property 4.1(4), $\psi^2 = \mathrm{id}$. In order to show $\mathrm{Fix}\psi \in \mathfrak{T}$, let $x, y \in \mathrm{Fix}\ \psi$ with $x \neq y$ and $z \in \overline{x, y}$. Since $\psi \in \mathfrak{A}$ we have $(x, y, z) \equiv (\psi(x), \psi(y), \psi(z)) = (x, y, \psi(z))$ and this implies $\psi(z) = z$ by (W1), hence $\overline{x, y} \subseteq \mathrm{Fix}\psi$.

(7) Let $x' \in P$ with $\tilde{x}'(o) = x$, let $z := \tilde{x}'(y)$ and $z' \in P$ with $\tilde{z}'(o) = z = \tilde{x}'(y)$. Then $\tilde{x}' \circ \tilde{z}' \circ \tilde{x}'(x) = \tilde{x}' \circ \tilde{z}'(o) = \tilde{x}'(\tilde{x}'(y)) = y$ and if $m := \tilde{x}'(z')$ by (4), $\tilde{m} = \tilde{x}' \circ \tilde{z}' \circ \tilde{x}'$. Since $\tilde{m}(x) = y, m \in \overline{x, y}$ and $(m, x) \equiv (m, y)$, i.e. $m$ is the midpoint of $x$ and $y$. Together with Property 4.1(2) this shows us that $\tilde{P}$ acts regularly on $P$.

We extend the definition of *orthogonality*: Let $L \in \mathfrak{L}, T \in \mathfrak{T}$ and $p \in P$, then $L \perp T :\Leftrightarrow L \not\subset T \wedge \tilde{L}(T) = T$. $(p \perp T) := \{L \in \mathfrak{L}(p) \mid L \perp T\}$.

**Property 4.4.** *If* $T \in \mathfrak{T}$, $p \in P \backslash T$ *and* $A \in (p \perp T)$ *with* $A \cap T \neq \emptyset$ *then* $(p \perp T) = \{A\}$.

**Proof.** Let $a:=A\cap T$ and suppose $B\in(p\perp T)\backslash\{A\}$. Then $a\notin B$, $a\neq\tilde{B}(a)$, $\tilde{B}(a)\in T$, hence $C:=\overline{a,\tilde{B}(a)}\subset T$ and $A,B\perp C$. Consequently, by Property 4.1(3) $\{A\}=(p\perp C)=\overline{p,\tilde{C}(p)}=\{B\}$.

## 5. Absolute geometry

A quadruple $(P,\mathfrak{L},\alpha,\equiv)$ is called an *ordered space with congruence* if $(P,\mathfrak{L},\alpha)$ is an ordered space, if $(P,\mathfrak{L},\equiv)$ is a weak space with congruence and if the following compatibility axiom (ZK) holds:

(ZK) For all $a,b,c,c'\in P$ with $a\neq b$, $c\neq c'$, $c'\in\overline{\{a,b,c\}}$ and $(a,b,c)\equiv(a,b,c')$ : $\overline{a,b}\cap\,]c,c'[\,\neq\emptyset$.

By Property 1.7, $(P,\mathfrak{L},\alpha)$ fulfills the axiom (F) and therefore $(P,\mathfrak{L},\equiv)$ is a space with congruence.

**Property 5.1.** *If* $(P,\mathfrak{L},\alpha,\equiv)$ *is an ordered space with congruence and* $\dim(P,\mathfrak{L})\geqslant 2$ *then*:

(1) $(P,\mathfrak{L},\equiv)$ *is regular.*

(2) *If* $a,m\in P$, $a\neq m$ *and* $b\in\overline{a,m}\backslash\{a\}$ *with* $(a,m)\equiv(b,m)$ *then* $(m|a,b)=-1$, *i.e.* $m\in\,]a,b[$.

(3) $(P,\mathfrak{L},\equiv)$ *fulfills* (Wa) *and so* $\tilde{P}\subset\mathfrak{A}$ (cf. Property 4.3(5)).

(4) *For all* $a,b,c\in P$ collinear: $\tilde{a}\circ\tilde{b}\circ\tilde{c}\in\tilde{P}$.

(5) *For all* $A,B\in\mathfrak{L}$ *with* $A\perp B : A\cap B\neq\emptyset$.

(6) *For all* $A,B,C\in\mathfrak{L}$ with $A\perp B,C$, *for all* $p\in A$, *for all* $E\in\mathfrak{E}(A)$ : $\tilde{A}|_E\in\mathfrak{A}_{\mathfrak{E}}$ (cf. *Property* 4.1(6)), $|(p\perp A)\cap\mathfrak{L}(E)|=1$ (cf. *Property* 4.1(7)) *and* $B\cap C\neq\emptyset\Rightarrow A\cap B\cap C\neq\emptyset$.

(7) $\tilde{P}\subset\mathrm{Aut}(P,\mathfrak{L},\alpha)$.

(8) *Let* $a,b\in P$ *with* $a\neq b$. *If there are* $x,y\in P\backslash\overline{a,b}$ *with* $(a,b,x)\equiv(b,a,y)$ *and* $y\in\overline{\{a,b,x\}}$ *then* $(a,b)$ *has a midpoint* (cf.[30,(5.5)]).

(9) *Let* $E\in\mathfrak{E}$, $z\in E$, $\varphi\in\mathfrak{A}_E$, *with* $\varphi(z)\neq z$ and $\varphi^2(z)=z$ *then* $(z,\varphi(z))$ *has a midpoint*, [30,(5.6)].

(9′) *For all* $o,a,b\in P$ *there exists exactly one* $c\in P$ : $\tilde{a}\circ\tilde{b}(o)=\tilde{c}(o)$.

(9″) *For all* $a,b,c\in P$ *distinct*: *If two of the segments* $]a,b[$, $]b,c[$, $]c,a[$ *have a mid-point*, *then also the third.*

(10) *For all* $E\in\mathfrak{E}$, *for all* $p\in P\backslash E$ : $|(p\perp E)|=1$ *and* $(p\perp E)\cap E\neq\emptyset$.

(11) *For all* $E\in\mathfrak{E}$, *for all* $c\in P\backslash E$, $C:=(c\perp E)$, $c_o:=C\cap E$ : $(c_o\perp E)\cap\mathfrak{L}(\overline{E\cup\{c\}})=C$.

**Proof.** (1) Since $\dim(P,\mathfrak{L})\geqslant 2$ there are $a,b,c\in P$ noncollinear and by (W3) there is a $c'\in\overline{\{a,b,c\}}\backslash\{c\}$ with $(a,b,c)\equiv(a,b,c')$. By (ZK), $m:=\overline{a,b}\cap\,]c,c'[$ exists (hence $(m|c,c')=-1$) and by (W2), $(m,c)\equiv(m,c')$.

(2) By Property 4.1(7) there is a $L\in\mathfrak{L}(m)$ with $\tilde{L}(a)=b$ and if $u,v\in L$, $u\neq v$ then $(u,v,a)\equiv(u,v,b)$ by Property 4.1(6), hence by (ZK), $\emptyset\neq L\cap\,]a,b[\,\subset L\cap\overline{a,m}=\{m\}$, i.e. $(m|a,b)=-1$.

(3) Suppose there are $x,y,y',y''\in P$ collinear and distinct with $(x,y)\equiv(x,y')\equiv(x,y'')$. Then by (2), $(x|y,y')=(x|y',y'')=(x|y,y'')=-1$ and by (Z1) $(x|y,y'')=(x|y,y')\cdot(x|y',y'')=(-1)\cdot(-1)=1$ which is a contradiction.

(4) Let $a\neq b$, $L:=\overline{a,b}$, $\psi:=\tilde{a}\circ\tilde{b}\circ\tilde{c}$, $C\in(c\perp L)$ (cf. Porperty 4.2(2)) and $x\in C\setminus\{c\}$. Then $C,\psi(C)\perp L$, $\psi(c)=L\cap\psi(C)$, $(c,x)\equiv(\psi(c),\psi(x))$ (cf. Porperty 4.3(5)) hence by [32, (3.5)] $(x,\psi(c))\equiv(c,\psi(x))$. Now $(c,\psi(c),x)\equiv(\psi(c),x,\psi(x))$ implies by [30, (5.5)] that $c$ and $\psi(c)=\tilde{a}(\tilde{b}(c))$ have a midpoint $m$. By Property 4.3(6) $\psi^2=\mathrm{id}$ and $\{m\}=L\cap\mathrm{Fix}\psi$. Suppose $\{m\}\neq\mathrm{Fix}\ \psi$. Let $u\in\mathrm{Fix}\ \psi\setminus\{m\}$, $v:=\tilde{c}(u)$, $w:=\tilde{b}(v)$ and $E:=\overline{\{u,v,w\}}$. Then $E\in\mathfrak{E}(L)$ and by Property 1.4, $E=\overrightarrow{L,u}\dot\cup L\ \dot\cup\ \overrightarrow{L,v}$. $w=\tilde{b}(v)$ implies $b\in\,]v,w[\,\cap L$ hence $w\in\overrightarrow{L,v}$. On the other hand, $u=\psi(u)=\tilde{a}\circ\tilde{b}\circ\tilde{c}(u)=\tilde{a}\circ\tilde{b}(v)=\tilde{a}(w)$, thus $a\in\,]u,w[\,\cap L$, i.e. $w\in\overrightarrow{L,u}$ which is a contradiction to $\overrightarrow{L,u}\cap\overrightarrow{L,v}=\emptyset$. Consequently $\psi=\tilde{m}$.

(5) Let $b\in B\setminus A$ and $a_1,a_2\in A$ with $a_1\neq a_2$. Then $b':=\tilde{A}(b)\subset\tilde{A}(B)=B$, $b'\neq b$ and $(a_1,a_2,b)\equiv(a_1,a_2,b')$ hence by (ZK), $\emptyset\neq A\ \cap\,]b,b'[\ \subset A\ \cap B$.

(6) If $B=C$ then $A\cap B\cap C=A\cap B\neq\emptyset$ by (5). If $B\neq C$ hence $\{x\}:=B\cap C$ then $\{\tilde{A}(x)\}=\tilde{A}(B\cap C)=\tilde{A}(B)\cap\tilde{A}(C)=B\cap C=\{x\}$ and so $x\in A$.

(7) 1. Let $o,a,b\in P$ be noncollinear and $c\in\,]a,b[$. For $C:=\overline{o,c}$, $E:=\overline{\{o,a,b\}}$ we have by Property 1.4, $E=\overrightarrow{C,a}\ \dot\cup\ C\ \dot\cup\ \overrightarrow{C,b}$ and by Property 5.1(2), $]a,\tilde{o}(a)[\,\cap C=\,]b,\tilde{o}(b)[\,\cap C=\{o\}$ hence $\tilde{o}(a)\in\overrightarrow{C,a}$, $\tilde{o}(b)\in\overrightarrow{C,b}$ and so together with Property 5.1(3), $\{\tilde{o}(c)\}=\,]\tilde{o}(a),\tilde{o}(b)[\,\cap C\neq\emptyset$, i.e. $\tilde{o}(]a,b[)=\,]\tilde{o}(a),\tilde{o}(b)[$.

2. Let $o\neq b$ and $a\in\,]o,b[$. We choose $L\in\mathfrak{L}(o)\setminus\overline{\{o,b\}}$ with $L\not\perp\overline{o,b}$ and set $B:=(b\perp L),b':=B\cap L$, $\{c\}:=(a\perp\overline{b,b'})\cap\overline{b,b'}$ (cf. (5)). Then by (6), $\overline{a,c}\cap L=\emptyset$ and so $a\in\,]o,b[$ implies $c\in\,]b,b'[$. Since $\tilde{o}\in\mathfrak{A}$ and by 1, we have $\tilde{o}(c)\in\,]\tilde{o}(b),\tilde{o}(b')[$, $\overline{\tilde{o}(a),\tilde{o}(c)}\cap\tilde{o}(L)=\emptyset$ which implies $\tilde{o}(a)\in\,]o,\tilde{o}(b)[$. This shows

3. If $o,a,b$ are collinear and distinct then $(a|o,b)=(\tilde{o}(a)|o,\tilde{o}(b))$ and for $c\in\,]a,b[$, $c\neq o$ we obtain from this formula: $-1=(c|a,b)=(c|o,a)\cdot(c|o,b)=(\tilde{o}(c)|o,\tilde{o}(a))\cdot(\tilde{o}(c)|o,\tilde{o}(b))=(\tilde{o}(c)|\tilde{o}(a),\tilde{o}(b))$, i.e. $\tilde{o}(c)\in\,]\tilde{o}(a),\tilde{o}(b)[$. For $c=o\in\,]a,b[$, i.e. $(o|a,b)=-1$, we have by (2) $(o|a,\tilde{o}(a))=(o|b,\tilde{o}(b))=-1$, hence

$$(o|\tilde{o}(a),\tilde{o}(b))=(o|a,b)\cdot(o|a,\tilde{o}(a))\cdot(o|b,\tilde{o}(b))=(-1)\cdot(-1)\cdot(-1)=-1,$$

i.e. $o\in\,]\tilde{o}(a),\ \tilde{o}(b)[$.

By 1., 2. and 3., $\tilde{o}\in\mathrm{Aut}(P,\mathfrak{L},\alpha)$ and so $\tilde{P}\subset\mathrm{Aut}(P,\mathfrak{L},\alpha)$.

(8) Let $G:=\overline{a,b}$, $E:=\overline{\{a,b,x\}}$, $x':=\tilde{G}(x)$. Then by (ZK), $]x,x'[\,\cap G\neq\emptyset$ and by Property 1.3 $E=\overrightarrow{G,x}\ \dot\cup\ G\ \dot\cup\ \overrightarrow{G,x'}$. We may assume $y\in\overrightarrow{G,x}$, i.e. $\{m\}:=\,]x,y[\,\cap G\neq\emptyset$. Now the arguments of [30] (5.5) show that $m$ is the midpoint of $(a,b)$.

(9′) If $a=b$ then $c=o$, if $b=o$ then $c=a$, if $a=o$ then $c=\tilde{o}(b)$. Let $o,a,b$ be distinct, $E\in\mathfrak{E}$ with $o,a,b\in E$, $x:=\tilde{a}\circ\tilde{b}(o)$ and $y:=\tilde{b}\circ\tilde{a}(o)$. Then $x,y\in E$, $x\neq o$ and $x=y$ would imply: $\tilde{b}(o)=\tilde{a}\circ\tilde{b}\circ\tilde{a}(o)=\widetilde{\tilde{a}(b)}(o)$, hence $\tilde{a}(b)=b$ and so $a=b$.

Therefore $x \neq y$ and moreover $(o,x) \equiv (\tilde{b}\tilde{a}(o), \tilde{b}\tilde{a}(x)) = (y,o)$. By Property 4.1(7) there exists exactly one $C \in \mathfrak{L}(o) \cap \mathfrak{L}(E)$ with $\tilde{C}(x) = y$ and by (6), $\tilde{C}|_E \in \mathfrak{A}_E$. Consequently $\varphi := \tilde{a} \circ \tilde{b} \circ \tilde{C}|_E \in \mathfrak{A}_E$, $\varphi(o) = \tilde{a} \circ \tilde{b}(o) = x \neq o$ and $\varphi^2(o) = \varphi(x) = \tilde{a} \circ \tilde{b}(y) = o$. By (9), $(o,x)$ has a midpoint $c$, hence $\tilde{c}(o) = x = \tilde{a} \circ \tilde{b}(o)$.

(9″) If, for instance, $x$ is the midpoint of $(a,b)$ and $y$ of $(b,c)$, then $\tilde{y}\,\tilde{x}(a) = \tilde{y}(b) = c$ and by (9′) there exists exactly one $z \in P$ with $\tilde{z}(a) = c$, i.e. $z$ is the midpoint of $(c,a)$.

(10) Because of (5), (6) and Property 4.1(6) we can use the same arguments of the proof of [22, (5.16)].

(11) Again, due to (6), we can take the proof of [22, (5.20)].

Since an ordered space is an exchange space (cf. Theorem 1.5) we can define the notion *reflection $\sigma$ in a subspace* $T \in \mathfrak{T} \setminus \{\emptyset\}$ by

1. $\sigma \in \mathrm{Sym}P$ and $\mathrm{Fix}\sigma = T$.
2. For all $x \in P \setminus T$, for all $t \in T : \overline{x, \sigma(x)} \cap T \neq \emptyset$ and $(x,t) \equiv (\sigma(x), t)$.

**Property 5.2.** *For all $T \in \mathfrak{T} \setminus \{\emptyset\}$ there is at most one reflection $\sigma$ in $T$ and if $\sigma$ is a reflection in $T$ then $\sigma^2 = \mathrm{id}$ and for all $x \in P \setminus T : \overline{x, \sigma(x)} \perp T$. If $T \in \mathfrak{E}$ then there is a reflection in $T$.*

**Proof.** Let $\sigma$ be a reflection in $T$. If $T = P$ then $\sigma = \mathrm{id}$. Let $T \neq P$, $x \in P \setminus T$, $X := \overline{x, \sigma(x)}$, $t_o := X \cap T$, $t \in T \setminus \{t_o\}$. Then $(x,t) \equiv (\sigma(x), t)$ and $(x, t_o) \equiv (\sigma(x), t_o)$ implies $\overline{t, t_o} \perp X$, i.e. $X \perp T$, $\tilde{X}(T) = T$ and by Property 4.4, $X = (x \perp T)$. Now $X = (x \perp T)$, $x, t_o, \sigma(x) \in X$, $x \neq \sigma(x)$ and $(x, t_o) \equiv (\sigma(x), t_o)$ yield $\sigma^2 = \mathrm{id}$ and that $\sigma$ is uniquely determined. Finally let $T = E \in \mathfrak{E}$, and let $\sigma : P \to P$ be defined by: If $x \in E$ then $\sigma(x) = x$. If $x \notin E$, let $X := (x \perp E)$, $x_o := X \cap E$ (cf. Property 5.1(10)) and $\sigma(x) := \tilde{x}_o(x)$. Then $\sigma \in \mathrm{Sym}P$, $\mathrm{Fix}\sigma = E$ and $\sigma^2 = \mathrm{id}$. For $x \in P \setminus E$ and $t \in E \setminus \{x_o\}$ we have $(x, x_o) \equiv (\sigma(x), x_o)$ and $X \perp \overline{t, x_o}$, hence $(x,t) \equiv (\sigma(x), t)$. Consequently $\sigma$ is a reflection in $E$.

**Property 5.3.** *Let $o, a, b \in P$, $\varphi := \tilde{a} \circ \tilde{b} \circ \tilde{o}$, $E \in \mathfrak{E}$ with $o, a, b \in E$ and $u := \varphi(o) = \varphi^{-1}(o)$. Then*
(1) $\varphi \in \tilde{P}$,
(2) *if $o, a, b$ are noncollinear then $\tilde{E}^3 \subset \tilde{E}$ (where $\tilde{E} := \{\tilde{x} \mid x \in E\}$).*

**Proof.** (1) If $o, a, b$ are collinear, then $\varphi \in \tilde{P}$ by Property 5.1(4). Let $o, a, b$ be noncollinear. Then $u = \varphi(o) \neq o$, $\varphi^2(o) = o$, $\varphi \in \mathfrak{A}$ (by Property 5.1(3)), $\varphi(E) = E$ (by Property 4.3(1)) and $(o, \varphi(o))$ has a midpoint $m$ (by Property 5.1(9)). For $\psi := \tilde{m} \circ \varphi$ we obtain $\psi \in \mathfrak{A} \cap \mathrm{Aut}(P, \mathfrak{L}, \alpha)$ (by Property 5.1(7)), $\psi(o) = o$, $\psi(u) = u$ hence $L := \overline{o, u} \subset \mathrm{Fix}\psi$ (by Property 4.1(4)) and either $E \subset \mathrm{Fix}\,\psi$ or $\psi|_E = \tilde{L}|_E$. By Property 5.1(2), $b \in ]o, \tilde{b}(o)[$, $a \in ]u, \tilde{b}(o)[$ hence by Property 1.4, $a, b \in \overrightarrow{L, \tilde{b}(o)}$. Therefore $x := \tilde{o} \circ \tilde{b}(o) \in \overrightarrow{L, \tilde{b}(o)}$, $\tilde{a} \circ \tilde{b} \circ \tilde{o}(x) = \tilde{a}(o) \in \overrightarrow{L, \tilde{b}(o)}$, and so $\psi(x) = \tilde{m} \circ \tilde{a}(o) \in \overrightarrow{L, \tilde{b}(o)}$ since $m \in L$. On the other hand by (ZK), $]x, \tilde{L}(x)[ \cap L \neq \emptyset$, i.e. $\tilde{L}(x) \notin \overrightarrow{L, \tilde{b}(o)}$. Consequently $\psi|_E \neq \tilde{L}|_E$, and so $E \subset \mathrm{Fix}\psi$, i.e. $\varphi|_E = \tilde{m}|_E$. If $E \neq P$, let $c \in P \setminus E$, $T := \overline{E \cup \{c\}}$, $C :=$

$(c \perp E)$, $c_o := C \cap E$ (cf. Property 5.1(10)). Then $\overrightarrow{c_o,c} = (c_o \perp E) \cap \overrightarrow{E,c}$ (by Property 5.1(11), $\psi(T) = T$ (by Porperty 4.3(1)), $\psi(\overrightarrow{E,c}) = \overrightarrow{E,c}$ (by Properties 5.1(2) and (7)), $\psi(c_o \perp E) = (\psi(c_o) \perp \psi(E)) = (c_o \perp E)$ (since $\psi \in \mathfrak{A}$) and so $\psi(\overrightarrow{c_o,c}) = \overrightarrow{c_o,c} \ni \psi(c)$. Since $(c_o,c) \equiv (\psi(c_o),\psi(c)) = (c_o,\psi(c))$, this shows, $\psi(c) = c$ and so $\varphi = \tilde{m} \in \tilde{P}$.

(2) With the same arguments as in [15, Section 21] one obtains: For all $x,y,z \in E$ there exists $s \in E$ with $\tilde{x} \circ \tilde{y} \circ \tilde{z}|_E = \tilde{s}|_E$. For $\alpha = \tilde{x} \circ \tilde{y} \circ \tilde{z}$, $\alpha(z) = \tilde{s}(z) = \alpha^{-1}(z)$ hence by (1), $\alpha = \tilde{s} \in \tilde{P}$. Consequently $\tilde{E}^3 \subset \tilde{E}$.

By Property 5.3 we can classify the planes $E \in \mathfrak{E}$ of $(P,\mathfrak{L},\alpha,\equiv)$ as follows:
$E$ is called *ordinary* if $\tilde{E}^3 \not\subset \tilde{E}$ and *singular* if $\tilde{E}^3 \subset E$.

We can also classify the ordered spaces with congruence $(P,\mathfrak{L},\alpha,\equiv)$ as follows:

$(P,\mathfrak{L},\alpha,\equiv)$ is called *singular* if $\tilde{P}^3 \subset \tilde{P}$, *ordinary* if $\tilde{P}^3 \not\subset P$ and *strictly ordinary* if each plane of $(P,\mathfrak{L})$ is ordinary.

Clearly $(P,\mathfrak{L},\alpha,\equiv)$ is singular, if and only if all planes $E$ of $(P,\mathfrak{L})$ are singular, and $(P,\mathfrak{L},\alpha,\equiv)$ is ordinary, if at least one plane of $(P,\mathfrak{L})$ is ordinary. The question whether all planes of an ordinary space are ordinary can be answered for *absolute spaces*, that is an ordered space with congruence $(P,\mathfrak{L},\alpha,\equiv)$ where the following axiom (WF) is valid:

(WF) For all $a,b,c \in P$ noncollinear there exists $d \in \overline{a,c} : (a,b) \equiv (a,d)$.

By Sörensen [32] and Konrad [22] we have:

**Property 5.4.** *Let* $(P,\mathfrak{L},\alpha,\equiv)$ *be an absolute space with* $\dim(P,\mathfrak{L}) \geqslant 2$. *Then*

(1) *The axiom* (V2) (*Streckenabtragen*) *For all* $a,b \in P$, $a \neq b$, *for all* $G \in \mathfrak{L}$, *for all* $c \in G : |\{x \in G \mid (x,c) \equiv (a,b)\}| = 2$ *of* [15] holds [32, p. 29].

(2) *For all* $E \in \mathfrak{E}$, $(E,\mathfrak{L}(E),\alpha_E,\equiv_E)$ *is an absolute plane in the sense of* [15] (cf. [32]) *and therefore any two points* $a,b \in P$ *have a midpoint*, [15, (16.11)].

(3) *For all* $E_1,E_2, \in \mathfrak{E}$, $(E_1,\mathfrak{L}(E_1),\alpha_{E_1},\equiv_{E_1})$ *and* $(E_2,\mathfrak{L}(E_2),\alpha_{E_2},\equiv_{E_2})$ *are isomorphic* [22, (5.15)].

(4) $(P,\mathfrak{L},\alpha,\equiv)$ *is ordinary if and only if all planes of* $(P,\mathfrak{L})$ *are ordinary* (*consequence of* (3)).

(5) *For all* $G \in \mathfrak{L} : \tilde{G} \in \mathfrak{A}$ [22, (5.22)].

(6) $(P,\mathfrak{L},\alpha,\equiv)$ *is a hyperbolic plane* (*in the sense of* [15, Section 26]) $\Leftrightarrow$ $\dim(P,\mathfrak{L}) = 2$ and $(P,\mathfrak{L},\alpha)$ *has hyperbolic incidence structure* [22, (6.10)].

(7) $(P,\mathfrak{L},\alpha,\equiv)$ *is* a hyperbolic space, (i.e. $\forall\, E \in \mathfrak{E} : (E,\mathfrak{L}(E),\alpha_E,\equiv_E)$ *is a hyperbolic plane*) $\Leftrightarrow$ $(P,\mathfrak{L},\alpha)$ *has a hyperbolic incidence structure.*

The axiom (W2) is the only one which is not restricted to coplanar points. Therefore the statement

(W2′) Let $a,b,x \in P$ be noncollinear, $a',b',x' \in \overline{\{a,b,x\}}$, $c \in \overline{a,b}$, $c' \in \overline{a',b'}$ with $(a,b,x) \equiv (a',b',x')$ and $(a,b,c) \equiv (a',b',c')$ then $(x,c) \equiv (x',c')$.

is a weakening of (W2). Recently Kroll and Sörensen (cf. [27]) proved:

**Property 5.5.** *Let* $(P, \mathfrak{L}, \equiv)$ *be given such that* $(P, \mathfrak{L}, \alpha)$ *is an ordered space and such that the axioms* (W1), (W2′), (W3), (ZK) and (WF) *are valid. If for all* $E \in \mathfrak{E}$, $(E, \mathfrak{L}(E), \alpha_E, \equiv_E)$ *is a hyperbolic plane, then* $(P, \mathfrak{L}, \alpha, \equiv)$ *is a hyperbolic space.*

## 6. Reflection structures and loops

Let $P \neq \emptyset$ be a set, $0 \in P$ be fixed, Sym $P$ be the group of all permutations of $P$, $J := \{\sigma \in \mathrm{Sym} P \mid \sigma^2 = \mathrm{id}\}$, $J^\times := J \backslash \{\mathrm{id}\}$ and let ${}^o : P \to J;\ x \mapsto x^o$ be a map such that

(B1) For all $x \in P : x^o(0) = x$.

For all $a, b \in P$ we set $a^+ := a^o \circ 0^o$, $a+b := a^+(b)$, $-a := 0^o(a)$, $\delta_{a,b} := ((a+b)^+)^{-1} \circ a^+ \circ b^+ = 0^o \circ (a+b)^o \circ a^o \circ 0^o \circ b^o \circ 0^o$ and moreover $P^o := \{x^o \mid x \in P\}$, $P^+ := P^o \circ 0^o = \{x^+ \mid x \in P\}$ and $\Delta := \langle \{\delta_{a,b} \mid a, b \in P\} \rangle$ the subgroup of Sym $P$ generated by the permutations $\delta_{a,b}$. Then:

**Theorem 6.1.** $(P, +)$ *is a right loop*, 0 *the neutral element and for all* $a \in P : a + (-a) = -a + a = 0$. *Moreover*:

(1) *The following statements are equivalent*:

(a) $P^o$ *acts regularly on* $P$ (*i.e. for all* $a, b \in P$ *there exists exactly one* $x \in P$ : $x^o(a) = b$).

(b) $(P, +)$ *is a loop.*

(2) *The following statements are equivalent*:

(a) (B2a) *For all* $a \in P : a^o \circ 0^o \circ a^o \in P^o$.

(b) $\forall a \in P : \delta_{a,a} = \mathrm{id}$, *i.e.* $(a + a)^+ = a^+ \circ a^+$.

(3) *The following statements are equivalent*:

(a) (B2b) $0^o \circ P^o \circ 0^o = P^o$.

(b) (FK2a) $\forall a \in P : \delta_{a,-a} = \mathrm{id}$, *i.e.* $(-a)^o = 0^o \circ a^o \circ 0^o$.

(c) (FK2b) $0^o \in \mathrm{Aut}(P, +)$, *i.e.* $\forall x, y \in P : -(x + y) = -x + (-y)$.

(4) *The following statements are equivalent*:

(a) (B2) $\forall a \in P : a^o \circ P^o \circ a^o = P^o$

(b) (Bol-Identity) $\forall a, b \in P : a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$.

(c) $(P, +)$ *is a K-loop, i.e. a loop with* $\Delta \leqslant \mathrm{Aut}(P, +)$, (FK2a), (FK2b) *and* (FK2c)$\forall a, b \in P : \delta_{a,b} = \delta_{a,b+a}$.

(d) $(P, +)$ *is a Bruck-loop*; *i.e. a loop with Bol-Identity and* (FK2b).

*Furthermore, if* $\delta \in \Delta$ *then*:

(5) $\delta \in \mathrm{Aut}(P, +) \Leftrightarrow \forall x \in P : \delta \circ x^o \circ 0^o \circ \delta^{-1} \circ 0^o \in P^o$.

(6) *If* $\delta \circ x^o \circ \delta^{-1} \in P^o$ *for all* $x \in P$ *then* $\delta \in \mathrm{Aut}(P, +)$ *and* $\delta \circ x^o \circ \delta^{-1} = (\delta(x))^o$.

(7) *If* (B2) *holds then* $\Delta \leqslant \mathrm{Aut}(P, +)$.

**Proof.** Since $a^+ = a^o \circ 0^o \in \mathrm{Sym} P$ and $P^o \subset J$, the equation $a + x = b$ has the unique solution $x := 0^o \circ a^o(b)$ and $0^+ = 0^o \circ 0^o = \mathrm{id}$, hence $0 + b = b$. Since $a + 0 = a^o \circ 0^o(0) =$

$a^o(0)=a$, 0 is neutral. By (B1) and $P^o\subset J$, $a+(-a)=a+0^o(a)=a^o\circ 0^o\circ 0^o(a)=a^o(a)=0$ and $-a+a=(-a)^o\circ 0^o(a)=(-a)^o(-a)=0$.

(1) The equation $b=x+a=x^o\circ 0^o(a)=x^o(-a)$ has exactly one solution iff $P^o$ acts regularly on $P$.

(2) By $a^+\circ a^+=a^o\circ 0^o\circ a^o\circ 0^o$, $(a+a)^+=(a+a)^o\circ 0^o$, $(a+a)^o(0)=a+a$, $a^o\circ 0^o\circ a^o(0)=a^+(a)=a+a$ and (B1) we have:

$$(a+a)^+=a^+\circ a^+\Leftrightarrow (a+a)^o=a^o\circ 0^o\circ a^o\Leftrightarrow a^o\circ 0^o\circ a^o\in P^o.$$

(3) Since $0^o\circ a^o\circ 0^o(0)=0^o\circ a^o(0)=0^o(a)=-a=(-a)^o(0)$ we have by (B1): $(a^+)^{-1}=(-a)^+\Leftrightarrow 0^o\circ a^o\circ 0^o=(-a)^o\Leftrightarrow \forall b\in P: 0^o(a+b)=0^o\circ a^o\circ 0^o(b)=(-a)^o(b)=(-a)^o\circ 0^o\circ 0^o(b)=-a+(-b)=0^o(a)+0^o(b)\Leftrightarrow 0^o\in \mathrm{Aut}(P,+)$.

(4) $a^+\circ b^+\circ a^+=a^o\circ 0^o\circ b^o\circ 0^o\circ a^o\circ 0^o$, $(a+(b+a))^+=(a+(b+a))^o\circ 0^o$, $a^o\circ 0^o\circ b^o\circ 0^o\circ a^o(0)=a+(b+a)$ and (B1) show the equivalence of (B2) and the Bol-Identity. By Kreuzer [25, (3.5)] we have the equivalence of the third and fourth statement. Finally (B2) implies (B2b), hence (FK2b) by (3). Consequently the Bol-Identity implies the fourth statement.

(5) Since $\delta(a+b)=\delta\circ a^+(b)=\delta\circ a^o\circ 0^o(b)$, $\delta(a)+\delta(b)=\delta(a)^o\circ 0^o\circ\delta(b)$ and $\delta\circ a^o\circ 0^o\circ\delta^{-1}\circ 0^o(0)=\delta\circ a^o\circ 0^o\circ\delta^{-1}(0)=\delta\circ a^o\circ 0^o(0)=\delta\circ a^o(0)=\delta(a)=(\delta(a))^o(0)$ (take into consideration that each map $a^+$ is determined by the image of 0 by (B1), hence $a^+(0)=a^o\circ 0^o(0)=a^o(0)=a$ and therefore $\delta_{a,b}(0)=0$ by definition of $\delta_{a,b}$) we obtain by (B1): $\delta\in\mathrm{Aut}(P,+)\Leftrightarrow \forall a\in P: \delta\circ a^o\circ 0^o\circ\delta^{-1}\circ 0^o=(\delta(a))^o\Leftrightarrow\forall a\in P: \delta\circ a^o\circ 0^o\circ\delta^{-1}\circ 0^o\in P^o$.

(6) $\delta\circ x^o\circ\delta^{-1}\in P^o$ and $\delta\circ x^o\circ\delta^{-1}(0)=\delta\circ x^o(0)=\delta(x)=(\delta(x))^o(0)$ imply by (B1) that $\delta\circ x^o\circ\delta^{-1}=(\delta(x))^o$, in particular, $\delta\circ 0^o\circ\delta^{-1}=(\delta(0))^o=0^o$, and so $\delta\circ x^o\circ 0^o\circ\delta^{-1}\circ 0^o=\delta\circ x^o\circ\delta^{-1}\circ\delta\circ 0^o\circ\delta^{-1}\circ 0^o=(\delta(x))^o\circ 0^o\circ 0^o=(\delta(x))^o\in P^o$, i.e., $\delta\in\mathrm{Aut}(P,+)$ by (5).

(7) Since each $\delta$ is a product of six elements of $P^o$, (7) follows from (6).

**Theorem 6.2.** *For $a\in P$ let $\tilde{\tilde{a}}:=a^o\circ 0^o\circ a^o$ and let $\tilde{\tilde{P}}:=\{\tilde{\tilde{a}}\mid a\in P\}$. Then*

(1) $a\in\mathrm{Fix}\,\tilde{\tilde{a}}=a^o\,(\mathrm{Fix}\,0^o), \tilde{\tilde{0}}=0^o$, $\tilde{\tilde{a}}\circ\tilde{\tilde{0}}=a^+\circ a^+$.

(2) *If* $\mathrm{Fix}\,0^o=\{0\}$ *then* $\approx: P\to\tilde{\tilde{P}}$ *is a bijection and* $\mathrm{Fix}\,\tilde{\tilde{a}}=\{a\}$.

(3) $\tilde{\tilde{P}}\subset P^o\Leftrightarrow$ (B2a).

(4) $\tilde{\tilde{P}}=P^o\Leftrightarrow$ (B2a) *and* $\tilde{\tilde{P}}(0)=P$.

**Remark.** In [28] Manara and Marchi (cf. also [13,2,3]) introduced a class of reflection geometries by starting with a map $\approx: P\to \mathrm{Sym}P\cap J^\times$ such that $\tilde{\tilde{P}}$ acts regularly on $P$ and $a\in\mathrm{Fix}\,\tilde{\tilde{a}}$ for each $a\in P$.

## 7. Geometric K-loops

The results of Section 6 will be applied to geometric structures. First of all let $(P,\mathfrak{L},\equiv)$ be a regular space with congruence such that (Wa) is true. Then to each

$p \in P$ the point-reflection $\tilde{p}$ can be associated (cf. Section 4). Moreover we claim:

(C1) $\exists 0 \in P$ such that $\forall x, y \in P,\ \exists z \in P : \tilde{x}(\tilde{y}(0)) = \tilde{z}(0)$

and we set $F := \{\tilde{x}(0) \mid x \in P\}$, $\mathfrak{L}_F := \{L \cap F \mid L \in \mathfrak{L} : |L \cap F| \geqslant 2\}$ and $(\tilde{x}(0))^o := \tilde{x}|_F$.

**Property 7.1.** $(P, \mathfrak{L}, \equiv; 0)$ *and* $(F, \mathfrak{L}_F, \equiv_F)$ *have the properties*:

(1) $\forall p \in P : \tilde{p}(F) = F$

(2) *The maps*

$$\begin{cases} P \xrightarrow{\sim} \tilde{P} \ \rightarrow \ F \xrightarrow{\circ} F^o = \tilde{P}|_F \subset \mathrm{Sym} F \cap J \\ x \mapsto \tilde{x} \mapsto \tilde{x}(0) \mapsto (\tilde{x}(0))^o = \tilde{x}|_F \end{cases}.$$

*are bijections.*

(3) $F^o$ *acts regularly on* $F$ *and* $\forall a \in F$: $a^o \circ F^o \circ a^o = F^o$.

(4) $(F, +)$ *with* $a + b := a^o \circ 0^o(b)$ *is a K-loop* (cf. *Property* 6.1(4)).

(5) $(F, \mathfrak{L}_F, \equiv_F)$ *is a regular weak space with congruence fulfilling* (Wa) *and* $F^+ \subset \mathfrak{A}_F$.

(6) $\mathfrak{F} := \mathfrak{L}_F(0)$ *is an incidence fibration of the loop* $(F, +)$ *in the sense of Zizioli* [33], *i.e.*

(Fb1) $\forall X \in \mathfrak{F}$: $X \leqslant (F, +)$ (*i.e.* $X$ *is a subloop*) *and* $|X| \geqslant 2$.

(Fb2) $\cup \mathfrak{F} := \bigcup_{L \in \mathfrak{F}} L = F$, $\forall A, B \in \mathfrak{F},\ A \neq B : A \cap B = \{0\}$.

(Fb3) $\forall a, b \in F,\ \forall X \in \mathfrak{F}$: $\delta_{a,b}(X) \in \mathfrak{F}$, *and* $\mathfrak{L}_F = \{a + X \mid a \in F,\ X \in \mathfrak{F}\}$.

(7) $\forall X \in \mathfrak{F}$, $(X, +)$ *is a commutative group.*

(8) *If in* $(P, \mathfrak{L}, \equiv)$ *any two points* $a, b$ *have a midpoint*, *then* (C1) *is fulfilled and* $F = P$.

**Proof.** (1) follows from (C1).

(2) Let $x, y \in P$ with $\tilde{x}|_F = \tilde{y}|_F$. Then $\tilde{x}(0) = \tilde{y}(0)$, hence by Property 4.3(2), $x = y$.

(3) Let $\tilde{a}(0)$, $\tilde{b}(0) \in F$ be given. By (C1), $\exists c \in P$ with $\tilde{a}(\tilde{b}(0)) = \tilde{c}(0)$. Then by Property 4.3(4), $\widetilde{\tilde{a}(c)} = \tilde{a} \circ \tilde{c} \circ \tilde{a}$ and $\widetilde{\tilde{a}(c)}(\tilde{a}(0)) = \tilde{a} \circ \tilde{c}(0) = \tilde{a} \circ (\tilde{a} \circ \tilde{b}(0)) = \tilde{b}(0)$. Since $\widetilde{\tilde{a}(c)}|_F \in \tilde{P}|_F = F^o$, $F^o$ acts transitively and so by Property 4.3(2) it acts regularly on $F$.

(4) The map '$\circ$' fulfills (B1) by (2) and (B2) by (3) so that by Property 6.1(4), $(F, +)$ is a K-loop.

(5) (W1) Let $a, b, c \in F$ be collinear and distinct and $a', b' \in F$ with $(a, b) \equiv (a', b')$. By (3) we may assume $a = a' = 0$ and moreover $b \neq b'$. Let $m \in P$ such that $c = \tilde{m}(0)$ and $L := (0 \perp \overline{b, b'})$ (cf. Property 4.1(7)). Then $m \in \overline{0, c} = \overline{0, b}$, $\tilde{L}(b) = b'$, $(0, b, c, m) \equiv (0, b', \tilde{L}(c), \tilde{L}(m))$ and so $\tilde{L}(m)$ is the midpoint of 0 and $\tilde{L}(c)$, i.e. $\tilde{L}(c) \in F$.

(W3) Again let $a = 0$. Let $L := \overline{a, b} = \overline{0, b}$ and $m \in P$ with $\tilde{m}(0) = x$. Then $\tilde{L}(x) \in \{\overline{a, b, x}\} \backslash \{x\}$, $(a, b, x) \equiv (a, b, \tilde{L}(x))$, and since $\widetilde{\tilde{L}(m)}(0) = \tilde{L} \circ \tilde{m} \circ \tilde{L}(0) = \tilde{L} \circ \tilde{m}(0) = \tilde{L}(x)$, we have $\tilde{L}(x) \in F$.

(W2) and (Wa) are clear.

(6) and (7). Let $X \in \mathfrak{L}_F(0)$, $a = \tilde{a}'(0)$, $b = \tilde{b}'(0) \in F \backslash \{0\}$, and $c' \in P$ with $\tilde{c}'(0) = c := a + b$. Then by Property 4.3(5) $\delta_{a,b} = (c^+)^{-1} \circ a^+ \circ b^+ = \tilde{0} \circ \tilde{c}'_o \circ \tilde{a}' \circ \tilde{0} \circ \tilde{b}'_o \circ \tilde{0} \in \mathfrak{A}$,

by (1), $\delta_{a,b}(F)=F$ and since $\delta_{a,b}(0)=0$, $\delta_{a,b}(X)\in\mathfrak{L}_F(0)$. Moreover: $a+X=a^o\circ 0^o(X)=\widetilde{a'}(X)=X\Leftrightarrow a'\in\overline{X}\Leftrightarrow a\in X$. This shows $X\leqslant(F,+)$.

If $a,b\in X$, then $c\in X$ and $0,a',b',c'\in\overline{X}$. By (C1), there exists $z\in P$ with $\widetilde{a'}(\widetilde{b'}(0))=\tilde{z}(0)$ hence by Property 4.3(6), $\widetilde{a'}\circ\widetilde{b'}\circ\tilde{0}=\tilde{0}\circ\widetilde{b'}\circ\widetilde{a'}$ and so $\widetilde{b'}\circ\tilde{0}\circ\widetilde{a'}=\widetilde{a'}\circ\tilde{0}\circ\widetilde{b'}$ implying $a+b=b+a$. In the same way, $\widetilde{a'}\circ\widetilde{c'}\circ\tilde{0}=\tilde{0}\circ\widetilde{c'}\circ\widetilde{a'}$ and $\widetilde{c'}\circ\widetilde{b'}\circ\tilde{0}=\tilde{0}\circ\widetilde{b'}\circ\widetilde{c'}$. Consequently $\delta_{a,b}=(\tilde{0}\circ\widetilde{c'}\circ\widetilde{a'})\circ\tilde{0}\circ\widetilde{b'}\circ\tilde{0}=\widetilde{a'}\circ\widetilde{c'}\circ\tilde{0}\circ\tilde{0}\circ\widetilde{b'}\circ\tilde{0}=\widetilde{a'}\circ\widetilde{c'}\circ\widetilde{b'}\circ\tilde{0}=\widetilde{a'}\circ\tilde{0}\circ\widetilde{b'}\circ\widetilde{c'}$, thus $\delta_{a,b}(c)=\widetilde{a'}\circ\tilde{0}\circ\widetilde{b'}(0)=\widetilde{a'}\circ\tilde{0}(b)=a+b=c$. By $\delta_{a,b}\in\mathfrak{A}$, $0,c\in\mathrm{Fix}\delta_{a,b}$ and Property 4.1(4) we obtain $\overline{0,c}\subset\mathrm{Fix}\delta_{a,b}$. If $c\neq 0$ then $X\subset\overline{0,c}\subset\mathrm{Fix}\delta_{a,b}$ hence $\delta_{a,b|X}=\mathrm{id}_X$. If $c=0$, then $b=-a$ and by (3) and Property 6.1(3), $\delta_{a,b}=\mathrm{id}$. Consequently $\forall a,b,x\in X$ : $a+(b+x)=(a+b)+\delta_{a,b}(x)=(a+b)+x$.

By adding the concept of order we obtain connections with results of the papers [9,16].

**Property 7.2.** *Let $(P,\mathfrak{L},\alpha,\equiv)$ be an ordered space with congruence. Then*

(1) *$(P,\mathfrak{L},\equiv)$ is a regular space with congruence fulfilling* (Wa) *and* (C1) *for each* $0\in P$ (cf. *Property* 5.1(1), (3) and (9′)).

(2) *For $\mathfrak{D}:=\tilde{P}$ and $\Gamma:=\mathfrak{A}$, the pair $(\Gamma,\mathfrak{D})$ is a reflection group fulfilling the axiom*

(S3) *For all $a,b,c\in\mathfrak{D}$ with $acbcbca=bcacacb$ : $abc\in J$,*

*of* [9] (*by Property* 5.3), *if in addition $(P,\mathfrak{L},\alpha,\equiv)$ is singular or strictly ordinary, then the general three reflection axiom* (cf. [12, p. 181; 15, p. 117])

(S1) *For all $a,b,x,y,z\in\mathfrak{D}$ with $a\neq b$ and $abx$, $aby$, $abz\in J^\times$ : $xyz\in\mathfrak{D}$ is valid, and if moreover any two points of $P$ have a midpoint then $(\Gamma,\mathfrak{D})$ is a reflection group with midpoints in the sense of* [9].

(3) *If $(P,\mathfrak{L},\alpha,\equiv)$ is strictly ordinary, if for $a\in F\backslash\{0\}$, $Z(a):=\{x\in F\,|\,x+a=a+x\}$ and $\mathfrak{Z}:=\{Z(a)\,|\,a\in F\backslash\{0\}\}$, then $\mathfrak{Z}=\mathfrak{F}=\mathfrak{L}_F(0)$ and so $\mathfrak{L}_F=\{b+Z(a)|b,a\in F,\ a\neq 0\}$.*

(4) *If $(P,\mathfrak{L},\alpha,\equiv)$ is an ordinary absolute space then in* (3) $F=P$.

## 8. Quasidilatations and Ruoff's proportionality theorem

In this section let $(P,\mathfrak{L},\alpha,\equiv)$ be an absolute space, $0\in P$ be fixed and $(P,+)$ the corresponding K-loop where 0 is the neutral element (cf. Property 7.1). For each $X\in F=\mathfrak{L}(0)$ we can make the following observations:

1. $(X,+,\alpha|_X)$ is a commutative group (cf. Property 7.1(7)) with a *betweenness relation*, i.e. (Z1) and (Z2) are valid.

2. If $x_0$, $x_1\in X$, $x_0\neq x_1$ are fixed, then there is exactly one total order "$<$" on $X$ such that (for $a,b\in X$ with $a\neq b$ we set $(a|b)=1$ if $a<b$ and $(a|b)=-1$ if $b>a$):

(i) $(x_0|x_1)=1$

(ii) $\forall(a,b,c)\in X^{3'}:=\{(x,y,z)\in X^3|x\neq y,z\}$: $(a|b,c)=(a|b)(a|c)$.

In fact '$<$' is defined by $(a|b)=-(a|x_0,b)\cdot(x_0|x_1,a)$ if $a\neq x_0$ and $(x_0|b)=(x_0|x_1,b)$ if $a=x_0$ (cf. [15, Eq. (13.3); 14, Eq. (1.1)]).

3. A *betweenness preserving* map $\varphi:X\to X$, i.e. $\forall(a,b,c)\in X^{3'}$: $(a|b,c)=(\varphi(a)|\varphi(b),\varphi(c))$ is either *isotone*, i.e. $\forall(a,x_0,b),(x_0,x_1,a)\in X^{3'}$:

$$(a|x_0,b)\cdot(x_0|x_1,a)=(\varphi(a)|x_0,\varphi(b))\cdot(x_0|x_1,\varphi(a))$$

or *antitone*, i.e.

$$(a|x_0,b)\cdot(x_0|x_1,a) = -(\varphi(a)|x_0,\varphi(b))\cdot(x_0|x_1,\varphi(a))$$

(cf. [14, (1.3)]). By Property 5.1(7), for each $p\in X$, $\tilde{p}|_X$ is a betweenness preserving map, but for $q\in X\backslash p$, $(\tilde{p}(q),p,p)$, $(p,q,\tilde{p}(q))\in X^{3'}$ and $\varphi:=\tilde{p}$ we have

$$(\tilde{p}(q)|p,p)\cdot(p|q,\tilde{p}(q)) = 1\cdot(-1) = -1$$

by Property 5.1(2), and

$$(\tilde{p}(\tilde{p}(q))|p,\tilde{p}(p))\cdot(p|q,\tilde{p}(\tilde{p}(q))) = (q|p,p)\cdot(p|q,q) = 1\cdot 1,$$

i.e. $\tilde{p}|_X$ is antitone.Therefore for each $x\in X$ the map $x^+|_X = x^0\circ 0^0|_X$ is isotone and so $(X,+,\beta)$ is an ordered commutative group.

4. Each $x\in X$ is the midpoint of $(0,x+x)$ and since by Property 5.4(2) there is exactly one midpoint $x'$ of $(0,x)$ we have $x{=}x'{+}x'$. Therefore, the ordered commutative group $(X,+,\beta)$ is divisible by 2, and so $(X,+)$ is a module over the ring $\mathbb{Z}_{(2)}:=\{m\cdot 2^{-u}|m\in\mathbb{Z},u\in\mathbb{N}\cup\{0\}\}$ of all dual fractions.

Since each $x\in P\backslash\{0\}$ is contained in exactly one $X\in\mathfrak{F}$, we have by 4 a well defined operation $\cdot:\mathbb{Z}_{(2)}\times P\to P:(\lambda,x)\mapsto\lambda\cdot x$. For each $\lambda\in\mathbb{Z}_{(2)}\backslash\{0\}$ the map $\lambda^{\cdot}:P\to P;x\mapsto\lambda\cdot x$ is called *quasidilatation* with center 0.

We can state the results:

**Property 8.1.** $(P,+,\mathfrak{F},\mathbb{Z}_{(2)},\cdot)$ *is a structure where* $(P,+,\mathfrak{F})$ *is a loop with an incidence fibration and* $\cdot:\mathbb{Z}_{(2)}\times P\to P$ *is a map such that for all* $\lambda,\mu\in\mathbb{Z}_{(2)}$ *for all* $X\in\mathfrak{F}$ *for all* $a,b\in P$ *hold the following*:

(1) $\lambda\cdot a=0\Leftrightarrow\lambda=0$ *or* $a=0$.

(2) *If* $\lambda\neq 0$, *then* $\lambda\cdot P=P$ *and* $\lambda\cdot X=X$.

(3) $(\lambda\cdot\mu)\cdot a=\lambda\cdot(\mu\cdot a)$, $(\lambda+\mu)\cdot a=\lambda\cdot a+\mu\cdot a$.

(4) *If* $a,b\in X$ *then* $\lambda\cdot(a+b)=\lambda\cdot a+\lambda\cdot b$.

**Property 8.2.** *For* $\lambda\in\mathbb{Z}_{(2)}\backslash\{0,1\}$ *the quasidilatation* $\lambda^{\cdot}$ *is a collineation of* $(P,\mathfrak{L})$ *if and only if* $(P,\mathfrak{L},\alpha,\equiv)$ *is singular* (*i.e.* $\tilde{P}^3=\tilde{P}$).

For our absolute space we introduce a distance. Let $U\in\mathfrak{F}$ and $e\in U\backslash\{0\}$ be fixed and $U_+:=\overrightarrow{0,e}\cup\{0\}$. By Property 5.4(1) there is a surjection

$|\,|:P\to U_+;\ p\mapsto|p|$, where $|p|$ is the uniquely determined point of $U_+$ with $(0,p)\equiv(0,|p|)$, called *absolute value*, and so a *distance* $\mathrm{d}:P\times P\to U_+;\ (p,q)\mapsto|-p+q|$.

Since $(P,+)$ is a K-loop and $P^+\subset\mathfrak{A}$ we have

**Property 8.3.** *For all* $a,b,c,d \in P$, *for all* $\varphi \in \mathfrak{A}$ :

(1) $(a,b) \equiv (c,d) \Leftrightarrow d(a,b) = d(c,d)$.

(2) *If* $b \in ]a,c[$ *then* $d(a,c) = d(a,b) + d(b,c)$.

(3) $d(a,b) = d(\varphi(a),\varphi(b))$.

Two distinct lines $A,B \in \mathfrak{L}$ can have one of the properties:

1. $A \cap B \neq \emptyset$.
2. $A \cap B = \emptyset$, $\exists L \in \mathfrak{L} : L \perp A,B$ and

(a) $\overline{A \cup B} \in \mathfrak{E}$, this case we denote by $A \perp\!\!\!\perp B$ and if $(P,\mathfrak{L},\alpha,\equiv)$ is hyperbolic, we call $A,B$ *overparallel*.

(b) $\overline{A \cup B} \notin \mathfrak{E}$

3. $A \cap B = \emptyset$, $A,B$ do not have a common perpendicular and

(a) $\overline{A \cup B} \in \mathfrak{E}$; (if $(P,\mathfrak{L},\alpha,\equiv)$ is hyperbolic, then $A \nparallel B$, i.e. $A,B$ are hyperbolic parallel).

(b) $\overline{A \cup B} \notin \mathfrak{E}$.

For the second case let $a := A \cap L, b := B \cap L$ and set $d(A,B) := d(a,b)$. If $(P,\mathfrak{L},\alpha,\equiv)$ is ordinary then $L$ is unique, and in the singular case $d(a,b)$ has the same value for all common perpendiculars.

Recently Ruoff [29] proved the following interesting proportionality theorem for which we can give two formulations:

**Property 8.4.** *Let* $(P,\mathfrak{L},\alpha,\equiv)$ *be a hyperbolic geometry and let* $0,a,b \in P$ *be non-collinear.*

(1) *If* $(P,+)$ *is the corresponding K-loop with respect to* $0$ *and* $\lambda \in \mathbb{Z}_{(2)} \backslash \{0,1\}$ *then* $\overline{a,b} \perp\!\!\!\perp \overline{\lambda \cdot a, \lambda \cdot b}$.

(2) *Let* $c \in \overrightarrow{0,a} \backslash \{a\}$ *such that there is* $\lambda \in \mathbb{Z}_{(2)}$ *with* $d(0,c) = \lambda \cdot d(0,a)$ *and let* $d \in \overrightarrow{0,b}$ *with* $d(0,d) = \lambda \cdot d(0,b)$, *then* $\overline{a,b} \perp\!\!\!\perp \overline{c,d}$.

## 9. For further reading

The following references are also of interest to the reader: [19–21].

## References


[1] E. Gabrieli, Loops with reflection germ: a characterization of absolute planes.

[2] H. Hotje, M. Marchi, S. Pianta, On a class of point-reflection geometries, Discrete Math. 129 (1994) 139–147.

[3] H. Hotje, S. Pianta, E. Zizioli, Symmetric incidence groupoids.

[4] B. Im, H. Karzel, Determination of the automorphism group of a hyperbolic K-loop, J. Geom. 49 (1994) 96–105.

[5] B. Im, H. Karzel, K-loops over dual numbers, Resultate Math. 28 (1995) 75–85.

[6] H. Karzel, Raum-Zeit-Welt und hyperbolische Geometrie, Beiträge Geom Algebra 29 (1994), TUM-M9412, Techn. Univ. München, Vorlesungsausarbeitung von A. Konrad.

[7] H. Karzel, G. Kist, Zur Begründung metrischer affiner Ebenen, Abh. Math. Sem. Univ. Hamburg 49 (1979) 234–236.
[8] H. Karzel, A. Konrad, Eigenschaften angeordneter Räume mit hyperbolischer Inzidenzstruktur I, Beiträge Geom. Algebra 28 (1994) 27–36, TUM-M 9415 Techn. Univ. München.
[9] H. Karzel, A. Konrad, Reflection groups and K-loops, J. Geom. 52 (1995) 120–129.
[10] H. Karzel, A. Konrad, A. Kreuzer, Eigenschaften angeordneter Räume mit hyperbolischer Inzidenzstruktur II, Beiträge Geom. Algebra 33 (1995) 7–14, TUM-M 9509, Techn. Univ. München.
[11] H. Karzel, A. Konrad, A. Kreuzer, Zur Einbettung angeordneter Räume mit hyperbolischer Inzidenzstruktur, Beiträge Geom. Algebra 30 (1995) 17–27, TUM-M 9502, Techn. Univ. München.
[12] H. Karzel, H.-J. Kroll, Geschichte der Geometrie seit Hilbert, Wissenschaftliche Buchgesellschaft, Darmstadt 1988.
[13] H. Karzel, M. Marchi, S. Pianta, On commutativity in point-reflection geometries, J. Geom. 44 (1992) 102–106.
[14] H. Karzel, C.J. Maxson, Archimedeisation of some ordered geometric structures which are related to kinematic spaces, Resultate Math. 19 (1991) 290–318.
[15] H. Karzel, K. Sörensen, D. Windelberg, Einführung in die Geometrie, Vandenhoeck, Göttingen, 1973.
[16] H. Karzel, H. Wefelscheid, A geometric construction of the K-loop of a hyperbolic space, Geom. Dedicata 58 (1995) 227–236.
[17] H. Kiechle, K-loops from classical groups over ordered fields, J. Geom. 61 (1998) 105–127.
[18] H. Kiechle, A. Konrad, The structure group of certain K-loops, in: G. Saad, M. Thomsen (Eds.), Proceedings of the Conference on Nearrings, Nearfields and K-loops (Hamburg, 30.7.–6.8. 1995) Kluwer Acad. Press, Dordrecht, 1997, pp. 287–294.
[19] G. Kist, Theorie der verallgemeinerten kinematischen Räume, Beiträge Geom. Algebra 14 (1986) TUM-M8611, Habilitationsschrift, Techn. Univ. München.
[20] E. Kolb, A. Kreuzer, Geometry of kinematic K-loops, Abh. Math. Sem. Univ. Hamburg 65 (1995) 189–197.
[21] A. Konrad, Hyperbolische Loops über Oktaven und K-Loops, Resultate Math. 25 (1994) 331–338.
[22] A. Konrad, Nichteuklidische Geometrie und K-Loops, Ph.D. Thesis, Technische Universität München, 1995.
[23] A. Konrad, Zur Strukturgruppe einiger K-loops, Beiträge Geom. Algebra 30 (1995) 13–16, TUM-M 9502, Techn. Univ. München.
[24] A. Kreuzer, Zur Theorie der Bündelräume, Beiträge Geom. Algebra 22 (1993) TUM-M 9311, Techn. Uni. München.
[25] A. Kreuzer, Inner mappings of Bol loops, Math. Proc. Cambridge Philos. Soc. 123 (1998) 53–57.
[26] A. Kreuzer, H. Wefelscheid, On K-loops of finite order, Resultate Math. 25 (1994) 79–102.
[27] H.-J. Kroll, K. Sörensen, Hyperbolische Räume, Beiträge Geom. Algebra 34 (1996) 37–44, TUM-M 9608, Techn. Univ. München.
[28] C.F. Manara, M. Marchi, On a class of reflection geometries, Rend. Sci. Mat. Appl. A 125 (2) (1991) 203–217.
[29] D. Ruoff, Ein nichteuklidischer Proportionalitätssatz, Beiträge Geom. Algebra 34 (1996) 1–13, TUM-M 9608, Techn. Univ. München.
[30] K. Sörensen, Ebenen mit Kongruenz, J. Geom. 22 (1984) 15–30.
[31] K. Sörensen, Projektive Einbettung angeordneter Räume, Beiträge Geom. Algebra 15 (1986) 8–35, TUM-M 8612, Techn. Univ. München.
[32] K. Sörensen, Eine Bemerkung zu absoluten Ebenen, Beiträge Geom. Algebra 24 (1993) 23–30, TUM-M 9315, Techn. Univ. München.
[33] E. Zizioli, Fibered incidence loops and kinematic loops, J. Geom. 30 (1987) 144–156.