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Existence of Three Solutions for a Nonautonomous Two Point Boundary Value Problem

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1. INTRODUCTION

Very recently, in [3–5], B. Ricceri proposed and developed an innovative minimax method for the study of non-linear eigenvalue problems. Let us also mention that a basic problem of the theory was solved by G. Cordaro [2], while G. Bonanno [1] gave an application of the method to the two point problem

$$\begin{cases} u'' + \lambda f(u) = 0 \\ u(a) = u(b) = 0. \end{cases}$$

The aim of the present paper is to extend the main result of [1] to the nonautonomous case.

2. RICCERI'S BASIC RESULTS

For the reader's convenience we now recall the two basic results of B. Ricceri which will be our main tools.

THEOREM 1 [4, Theorem 1]. *Let X be a separable and reflexive real Banach space; $\Phi\colon X \to R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi\colon X \to R$ a continuously Gâteaux*

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differentiable functional whose Gâteaux derivative is compact. Assume that

$$\lim_{\|u\| \to +\infty} \Phi(u) + \lambda\Psi(u) = +\infty$$

for all $\lambda \in [0, +\infty[$, *and that there exists a continuous concave function* h: $[0, +\infty[\to R$ *such that*

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then there exist an open interval $\Lambda \subseteq]0, +\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *the equation*

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X *whose norms are less than* q.

PROPOSITION 1 [5, Proposition 3.1]. *Let* X *be a non-empty set and* Φ, Ψ *two real functions on* X. *Assume that there are* $r > 0$ *and* $x_0, x_1 \in X$ *such that*

$$\Phi(x_0) = \Psi(x_0) = 0, \qquad \Phi(x_1) > r, \qquad \sup_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x) < r\frac{\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each ρ *satisfying*

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (\Psi(u)) < \rho < r\frac{\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - \Psi(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - \Psi(x))).$$

3. MAIN RESULTS

Here and in the sequel, $[a, b]$ is a compact real interval, f: $[a, b] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, and g is the real function defined by putting

$$g(t, \xi) = \int_0^{\xi} f(t, x)\, dx$$

for all $(t, \xi) \in [a, b] \times \mathbf{R}$. Moreover, X is the Sobolev space $W_0^{1,2}([a, b])$ equipped with the usual norm $\|u\| = (\int_a^b |u'(t)|^2\, dt)^{1/2}$. Our main results fully depend on the following lemma:

LEMMA 1. *Assume that there exist two positive constants d, c, with $c < \sqrt{\frac{2}{b-a}}\, d$, such that*:

(i) $g(t, \xi) \geq 0$ *for each* $(t, \xi) \in ([a, a + \frac{b-a}{4}] \cup [b - \frac{b-a}{4}, b]) \times [0, d]$

(ii) $\max_{(t,\xi) \in [a,b] \times [-c, c]} g(t, \xi) < \frac{1}{2}(\frac{c}{d})^2 \int_{a+(b-a)/4}^{b-(b-a)/4} g(t, d)\, dt.$

Then there exist $r > 0$ and $u \in X$ such that $2r < \|u\|^2$ and

$$(b - a) \max_{(t,\xi) \in [a,b] \times [-\sqrt{\frac{r}{2}}, \sqrt{\frac{r}{2}}]} g(t, \xi) < 2r \frac{\int_a^b g(t, u(t))\, dt}{\|u\|^2}.$$

Proof. We claim that the number $r = 2c^2$, and the function

$$u(t) = \begin{cases} \dfrac{4}{b-a} d(t - a), & a \leq t \leq a + \dfrac{b-a}{4} \\ d, & a + \dfrac{b-a}{4} \leq t \leq b - \dfrac{b-a}{4} \\ \dfrac{4}{b-a} d(b - t), & b - \dfrac{b-a}{4} \leq t \leq b \end{cases}$$

satisfy our conclusion. In fact $u \in W_0^{1,2}([a, b])$ and $\|u\|^2 = 8d^2/(b - a)$. Hence, taking into account that $c < \sqrt{(2/(b-a))}d$, one has $2r < \|u\|^2$. Moreover, owing to our assumptions, we have that

$$\begin{aligned} \frac{\int_a^b g(t, u(t))\, dt}{\|u\|^2} 2r &\geq \frac{b-a}{2} \left(\frac{c}{d}\right)^2 \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} g(t, d)\, dt \\ &> (b - a) \max_{(t,\xi) \in [a,b] \times [-c, c]} g(t, \xi). \end{aligned}$$

So, the proof is complete. ∎

Now, we state our main result:

THEOREM 2. *Assume that there exist four positive constants μ, d, c, s with $c < \sqrt{(2/(b-a))}d$ and $s < 2$ such that*:

(i) $g(t, \xi) \geq 0$ *for each* $(t, \xi) \in ([a, a + \frac{b-a}{4}] \cup [b - \frac{b-a}{4}, b]) \times [0, d]$

(ii) $\max_{(t,\xi)\in[a,b]\times[-c,c]} g(t,\xi) < \frac{1}{2}(\frac{c}{d})^2 \int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,dt$

(iii) $g(t,\xi) \leq \mu(1 + |\xi|^s)$ *for each* $t \in [a,b]$ *and* $\xi \in R$.

Then there exist an open interval $\Lambda \subseteq]0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *the problem*

$$\begin{cases} u'' + \lambda f(t,u) = 0 \\ u(a) = u(b) = 0 \end{cases} \tag{1}$$

admits at least three solutions belonging to $C^2([a,b])$, *whose norms in* $W_0^{1,2}([a,b])$ *are less than* q.

Proof. For each $u \in X$, we put $\Phi(u) = \frac{1}{2}\|u\|^2$ and $\Psi(u) = -\int_a^b(\int_0^{u(t)} f(t,x)\,dx)\,dt$

$$J(u) = \Phi(u) + \lambda\Psi(u).$$

It is well known that the critical points of J are the classical solutions of (1). Then, our goal is to prove that Φ and Ψ satisfy the assumptions of Theorem 1. Clearly, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, thanks to (iii) and to Poincaré inequality, one has

$$\lim_{\|u\|\to+\infty} \Phi(u) + \lambda\Psi(u) = +\infty$$

for all $\lambda \in [0,+\infty[$. We claim that there exist $r > 0$ and $u \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < r\frac{(-\Psi(u))}{\Phi(u)}.$$

To this end, taking into account that

$$\max_{t\in[a,b]} |u(t)| \leq \frac{1}{2}\|u\| \qquad \text{for each } u \in X,$$

we have, for each $r > 0$,

$$\Phi^{-1}(]-\infty,r]) \subseteq \left\{u \in X\colon |u(t)| \leq \sqrt{\frac{r}{2}} \text{ for each } t \in [a,b]\right\},$$

and so

$$\sup_{u\in\Phi^{-1}(]-\infty, r])} (-\Psi(u))$$

$$= \sup_{\|u\|^2\le 2r} \int_a^b g(t,u(t))\,dt \le (b-a) \max_{(t,\xi)\in[a,b]\times[-\sqrt{\frac{r}{2}},\sqrt{\frac{r}{2}}]} g(t,\xi).$$

Now, thanks to Lemma 1, there exist $r > 0$ and $u \in X$ such that

$$(b-a) \max_{(t,\xi)\in[a,b]\times[-\sqrt{\frac{r}{2}},\sqrt{\frac{r}{2}}]} g(t,\xi) < 2r\frac{\int_a^b g(t,u(t))\,dt}{\|u\|^2} = r\frac{(-\Psi(u))}{\Phi(u)}.$$

Finally, owing to Proposition 1, choosing $h(\lambda) = \rho\lambda$, we obtain

$$\sup_{\lambda\ge 0} \inf_{x\in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda))$$

$$< \inf_{x\in X} \sup_{\lambda\ge 0} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

Now, our conclusion follows from Theorem 1. ▮

Let $A \in C([a,b])$ and $B \in C(R)$ be two non-negative functions. Put

$$\alpha(t) = \int_a^t A(\tau)\,d\tau, \qquad \beta(\xi) = \int_0^\xi B(x)\,dx.$$

Then Theorem 2 takes the simpler form:

COROLLARY 1. *Assume that there exist four positive constants* η, d, c, s *with* $c < \sqrt{(2/(b-a))}d$ *and* $s < 2$ *such that*:

(i) $\max_{t\in[a,b]} A(t) < \frac{1}{2}(\frac{c}{d})^2 \frac{\beta(d)}{\beta(c)}[\alpha(b - \frac{b-a}{4}) - \alpha(a + \frac{b-a}{4})]$

(ii) $\beta(\xi) \le \eta(1 + |\xi|^s)$ *for each* $\xi \in R$.

Then there exist an open interval $\Lambda \subseteq]0, +\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *the problem*

$$\begin{cases} u'' + \lambda A(t)B(u) = 0 \\ u(a) = u(b) = 0 \end{cases} \tag{2}$$

admits at least three solutions belonging to $C^2([a,b])$, *whose norms in* $W_0^{1,2}([a,b])$ *are less than* q.

Proof. Put $f(t,u) = A(t)B(u)$ for each $(t,u) \in [a,b] \times \mathbf{R}$, and note that

$$\max_{(t,\xi)\in[a,b]\times[-c,c]} g(t,\xi) = \max_{t\in[a,b]} A(t)\beta(c).$$

Choosing $\mu = \eta \max_{t \in [a,b]} A(t)$, it is easy to verify that all the assumptions of Theorem 2 are satisfied. So the proof is complete. ∎

Remark 1. We explicitly observe that, if the function $A(t)$ is constant, Corollary 1 gives a version of [1, Theorem 2] for a generical interval $[a, b]$.

Now, we are going to give an example of application:

EXAMPLE 1. We consider (2) with $A(t)B(u) = t\sqrt{|u|}$ $a = 0$, $b = 20$, namely

$$\begin{cases} u'' + \lambda t\sqrt{|u|} = 0 \\ u(0) = u(20) = 0. \end{cases}$$

In this case one has $\alpha(t) = t^2/2$ and $\beta(\xi) = \frac{2}{3}\xi^{3/2}$. Moreover choosing $s = \frac{3}{2}$, $\eta = 20$, $c = 1$, $d = 4$, a simple computation shows that all assumptions of Corollary 1 are satisfied. So there exist an open interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the previous problem admits at least two non-trivial solutions belonging to $C^2([0, 20])$, whose norms in $W_0^{1,2}([0, 20])$ are less than q.

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