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Fundamental domains of cluster categories inside module categories ☆

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ABSTRACT

Let H be a finite dimensional hereditary algebra over an algebraically closed field, and let $\mathcal{C}_H$ be the corresponding cluster category. We give a description of the (standard) fundamental domain of $\mathcal{C}_H$ in the bounded derived category $D^b(H)$, and of the cluster-tilting objects, in terms of the category mod Γ of finitely generated modules over a suitable tilted algebra Γ. Furthermore, we apply this description to obtain (the quiver of) an arbitrary cluster-tilted algebra.

1. Introduction

Let k be an algebraically closed field and Q a finite acyclic quiver with n vertices. Let $H = kQ$ be the associated path algebra. The cluster category $\mathcal{C}_H$ was introduced and investigated in [7], motivated by the cluster algebras of Fomin–Zelevinsky [10]. By definition we have $\mathcal{C}_H = D^b(H)/\tau^{-1}[1]$, where τ denotes the AR-translation. An important class of objects are the cluster-tilting objects T, which are the objects T with $\mathrm{Ext}^1_{\mathcal{C}_H}(T, T) = 0$, and T maximal with this property. They are shown to be exactly the objects induced by tilting objects over some path algebra kQ' derived equivalent to kQ.

A crucial property of the cluster-tilting objects $T = T_1 \oplus \cdots \oplus T_j$ where the T_i are indecomposable, and T_i is not isomorphic to $T_{i'}$ for $i \neq i'$, is that $j = n$ and for each $i = 1, \ldots, n$ there is a unique

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indecomposable object T_i^* not isomorphic to T_i in $\mathcal{C}_H$, such that $(T/T_i) \oplus T_i^*$ is a cluster-tilting object. This is a more regular behavior than what we have for tilting modules (of projective dimension at most one) over a finite dimensional algebra A. In general there is *at most one* replacement for each indecomposable summand.

The maps in $\mathcal{C}_H$ are defined as follows, as usual for orbit categories. Choose the fundamental domain $\mathcal{D}$ of $\mathcal{C}_H$ inside $D^b(H)$, whose indecomposable objects are the indecomposable H-modules, together with $P_1[1], \dots, P_n[1]$, where the P_j are the indecomposable projective H-modules. Let X and Y be in $\mathcal{D}$. Then $\operatorname{Hom}_{\mathcal{C}_H}(X, Y) = \bigoplus_{\in Z} \operatorname{Hom}_{D^b(H)}(X, (\tau^{-1}[1])^i(Y))$.

In [2] the authors considered the triangular matrix algebra $\Lambda = \left(\begin{smallmatrix} H & 0 \\ DH & H \end{smallmatrix}\right)$, where $D = \operatorname{Hom}_k(-, k)$. They chose a fundamental domain for $\mathcal{C}_H$ inside the category $\operatorname{mod} \Lambda$ of finite dimensional Λ-modules, by using the H-modules together with $\operatorname{ind} \tau_\Lambda^{-1}(DH)$. They established a bijection between cluster-tilting objects in $\mathcal{C}_H$ and a certain class of tilting modules in $\operatorname{mod} \Lambda$, which was shown in [3] to be all tilting modules (of projective dimension at most 1).

The present paper is inspired by [2]. Instead of using the algebra Λ which normally has global dimension 3, we use a smaller triangular matrix algebra Γ which has global dimension at most 2, and is a tilted algebra. We obtain a similar connection between cluster-tilting objects in $\mathcal{C}_H$ and tilting modules in $\operatorname{mod} \Gamma$ and give an alternative proof for the special property of complements mentioned above. The projective–injective modules play a crucial role here, as in [2].

If T is a tilting H-module, a description of the quiver of $\operatorname{End}_{\mathcal{C}_H}(T)$, on the basis of the quiver of $\operatorname{End}_H(T)$, is given in [1] (see [9] for finite type). For each relation in a minimal set of relations in $\operatorname{add} T$, an arrow is added in the opposite direction. We obtain a similar description for T in the fundamental domain, but not necessarily being an H-module. Again the projective–injective modules play an essential role. Now we consider relations where we allow factoring also through the projective–injective modules, in addition to $\operatorname{add} T$. Then we obtain the same result about adding arrows in the opposite direction as before. When T is a tilting H-module, then no maps in $\operatorname{add} T$ factor through projective–injective modules.

We now describe the content section by section. In Section 2 we give some preliminary results on describing the indecomposable Λ-modules of projective dimension at most 1. In particular, we show that all predecessors of a module of projective dimension 1 have projective dimension at most 1. In Section 3 we introduce the algebra Γ which replaces Λ in our work, starting with motivation on how to choose Γ smallest possible, without losing essential information. We show that the indecomposable Γ-modules of projective dimension at most 1 are exactly the modules in the left part $\mathcal{L}_\Gamma$ of indecomposable modules where the predecessors have projective dimension at most 1. Further, this class consists of the indecomposable modules in our fundamental domain, together with the indecomposable projective–injective Γ-modules. In Section 4 we show how to describe the quiver of $\operatorname{End}_{\mathcal{C}_H}(T)$ for any T in the fundamental domain.

2. Duplicated algebras

In this section we recall work from [2] and improve the statement of the main theorem in [2]. Throughout the paper we assume that H is a basic hereditary algebra over an algebraically closed field k and Λ is the duplicated algebra of H, that is, $\Lambda = \left(\begin{smallmatrix} H & 0 \\ DH & H \end{smallmatrix}\right)$. We denote by $\operatorname{mod} \Lambda$ the category of finitely generated left Λ-modules, and we use the usual description of the left Λ-modules as triples (X, Y, f), with X, Y in $\operatorname{mod} H$ and $f \in \operatorname{Hom}_H(DH \otimes_H X, Y)$ (see [11], or [5, III, 2]). Then the full subcategory of $\operatorname{mod} \Lambda$ generated by the modules of the form $(0, Y, 0)$ is closed under predecessors and canonically isomorphic to $\operatorname{mod} H$. We will use this isomorphism to identify $\operatorname{mod} H$ with the corresponding full subcategory of $\operatorname{mod} \Lambda$ and give some alternative proofs. The opposite algebra Λ^{op} is isomorphic to the triangular matrix algebra $\left(\begin{smallmatrix} H^{op} & 0 \\ DH & H^{op} \end{smallmatrix}\right)$. Under these identifications, the duality $D : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{op}$ is given by $D(X, Y, f) = (DY, DX, Df)$, where $Df \in \operatorname{Hom}_{H^{op}}(DY, D(DH \otimes_H X)) \cong \operatorname{Hom}_{H^{op}}(DY, \operatorname{Hom}_H(DH, DX)) \cong \operatorname{Hom}_{H^{op}}(DY \otimes_H DH, DX) \cong \operatorname{Hom}_{H^{op}}(DH \otimes_{H^{op}} DY, DX)$.

We recall (see [11] or [5, III, Proposition 2.5]) that the indecomposable projective Λ-modules are given by triples isomorphic to those of the form $(0, P, 0)$ or $(P, DH \otimes_H P, 1_{DH \otimes_H P})$, with P indecomposable projective in $\operatorname{mod} H$. The former are the projective H-modules, and the latter are projective–injective Λ-modules. The remaining indecomposable injective Λ-modules are of the form $(I, 0, 0)$ with I injective in $\operatorname{mod} H$.

We denote by $pd_\Lambda M$ and $id_\Lambda M$ the projective dimension and the injective dimension of the Λ-module M, respectively. When M is in $\operatorname{mod} H$ we have $pd_H M = pd_\Lambda M$, and for that reason we will write just $pd M$. We denote by $\operatorname{rad} X$ and $\operatorname{soc} X$ the radical and the socle of the Λ-module X, respectively.

Let $\operatorname{ind} \Lambda$ denote the full subcategory of $\operatorname{mod} \Lambda$ where the objects are a complete set of representatives of the isomorphism classes of indecomposable Λ-modules.

We denote by $D^b(H)$ the bounded derived category of H, by $\mathcal{C}_H$ the cluster category of H, and by τ the Auslander–Reiten translation in $\operatorname{mod} \Lambda$ or $D^b(H)$. Note that the injective H-modules are not Λ-injective, so that $\tau_\Lambda^{-1}(I_i)$ is indecomposable for each indecomposable H-module $I_i = I_0(S_i)$, where S_i is a simple H-module. Then $\{\tau_\Lambda^{-1}(I_i)\}$ in $\operatorname{mod} \Lambda$ will play a similar role as $\{P_i[1]\}$ in the derived category $D^b(H)$. In particular, $\operatorname{add}(\operatorname{ind} H \cup \{\tau_\Lambda^{-1} DH\}) \subseteq \operatorname{mod} \Lambda$ can be considered as a fundamental domain $\mathcal{D}_\Lambda$ inside $\operatorname{mod} \Lambda$ of the cluster category $\mathcal{C}_H$ (see [2]).

We recall that given $X, Y \in \operatorname{ind} \Lambda$, a *path from Y to X* is a sequence of nonzero morphisms $Y \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_t \xrightarrow{f_t} X$, with the $X_i \in \operatorname{ind} \Lambda$. When such a path exists, Y is a *predecessor* of X, and X is a *successor* of Y. The *left part* $\mathcal{L}_\Lambda$ of $\operatorname{mod} \Lambda$, defined in [13], is the full subcategory of $\operatorname{ind} \Lambda$ consisting of the modules whose predecessors have projective dimension at most 1. That is, $\mathcal{L}_\Lambda = \{X \in \operatorname{ind} \Lambda \mid pd\, Y \leqslant 1 \text{ for any predecessor } Y \text{ of } X\}$. The main result of [2] is the following.

Theorem 2.1.

(a) *The fundamental domain $\mathcal{D}_\Lambda$ of $\mathcal{C}_H$ lies in* $\operatorname{add} \mathcal{L}_\Lambda$*, and the only other indecomposable Λ-modules in $\mathcal{L}_\Lambda$ are projective–injective.*
(b) *There is induced a bijection between cluster-tilting objects in $\mathcal{C}_H$ and tilting modules in* $\operatorname{mod} \Lambda$ *whose indecomposable non-projective–injective summands lie in $\mathcal{L}_\Lambda$.*

Note that in [3] it is shown that the bijection in (b) is with all tilting modules. Using the following results for Λ-modules with projective dimension at most one, we give a different approach to the improved version.

Proposition 2.2. *Let $X \in \operatorname{ind} \Lambda$. Then $pd_\Lambda X \leqslant 1$ if and only if $\tau_\Lambda X \in \operatorname{mod} H$. In other words, the indecomposable Λ-modules X such that $pd_\Lambda X \leqslant 1$ are those in the fundamental domain of $\mathcal{C}_H$, together with the indecomposable projective–injective Λ-modules.*

Proof. We have that $pd_\Lambda X > 1$ if and only if $\operatorname{Hom}_\Lambda(D\Lambda, \tau X) \neq 0$, and this last condition implies $\tau X \notin \operatorname{mod} H$, since the injective Λ-modules do not belong to $\operatorname{mod} H$. Conversely, if $\tau X \notin \operatorname{mod} H$, there is a projective–injective Λ-module E such that $\operatorname{Hom}_\Lambda(E, \tau X) \neq 0$, and this implies $pd_\Lambda X > 1$. $\square$

Lemma 2.3. *Let $X, Y \in \operatorname{ind} \Lambda$ be such that* $\operatorname{Hom}_\Lambda(X, Y) \neq 0$ *and $pd_\Lambda Y = 1$. Then $pd_\Lambda X \leqslant 1$.*

Proof. Let $f : X \to Y$ be a nonzero morphism. We want to show that $pd_\Lambda X \leqslant 1$. Suppose that this is not the case. Then f is not an isomorphism, and so it factors through the minimal right almost split morphism $E \to Y$. Since $f \neq 0$, we can choose an indecomposable direct summand E_0 of E, and morphisms $g_0 : E_0 \to Y$ and $h_0 : X \to E_0$ with $g_0 h_0 \neq 0$ and g_0 irreducible. Then $E_0 \notin \operatorname{mod} H$, because its predecessor X is not in $\operatorname{mod} H$. Now suppose E_0 is projective. Then it is also injective, since all indecomposable projective Λ-modules which are not in $\operatorname{mod} H$ are injective. Hence $\tau Y \cong \operatorname{rad} E_0$, because $g_0 : E_0 \to Y$ is irreducible. Now $h_0 : X \to E_0$ factors through $\operatorname{rad} E_0 \hookrightarrow E_0$, so $\operatorname{rad} E_0 \notin \operatorname{mod} H$, and we conclude that $\tau Y \cong \operatorname{rad} E_0 \notin \operatorname{mod} H$. By Proposition 2.2, this contradicts our hypothesis $pd\, Y = 1$. Therefore E_0 is not projective, and then there is an irreducible morphism $\tau E_0 \to \tau Y$. On the other hand, $pd\, Y = 1$ implies that τY is in $\operatorname{ind} H$. Hence τE_0 is in $\operatorname{ind} H$, and thus $pd\, E_0 = 1$. Therefore our original morphism $f : X \to Y$ can be replaced by $h_0 : X \to E_0$, and so we can iterate the process to obtain an arbitrarily long path of irreducible morphisms $E_m \xrightarrow{g_m} E_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_2} E_1 \xrightarrow{g_1} E_0$ with $g_1 g_2 \cdots g_m \neq 0$. But this is a contradiction, because each $E_i \in \tau^{-1} \operatorname{ind} H \setminus \operatorname{mod} H$, i.e. it is a direct summand of $\tau^{-1} DH$, and this implies that all the g_i are in $\operatorname{rad}\operatorname{End}(\tau^{-1}DH)$, which is nilpotent. □

Proposition 2.2 can be used to give the following relationship between $\mathcal{L}_\Lambda$ and the modules of projective dimension at most 1.

Proposition 2.4. *If $X \in \operatorname{ind} \Lambda$ and $pd_\Lambda X = 1$, then $X \in \mathcal{L}_\Lambda$.*

Proof. Suppose that the result does not hold, and let $Y \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_t \xrightarrow{f_t} X$ be a path in $\operatorname{ind} \Lambda$, with $pd\, X = 1$ and $pd\, Y > 1$. Then $Y \notin \operatorname{mod} H$, because H is hereditary. Since $\operatorname{mod} H$ is closed under predecessors in $\operatorname{mod} \Lambda$, then the X_i and X do not belong to $\operatorname{mod} H$ either. Now, let us choose the path so that it has minimal length among those with $pd\, X = 1$ and $pd\, Y > 1$. By Lemma 2.3, $t \geqslant 1$. By minimality, X_i is projective (and hence injective) for $1 \leqslant i \leqslant t$. The map f_{t-1} factors through $\operatorname{rad} X_t$, which is not injective, and thus not projective. Then, by minimality we must have $pd(\operatorname{rad} X_t) > 1$ and $t = 1$. Since f_1 factors through $\frac{X_1}{\operatorname{soc} X_1}$ – which is not projective – we must have $pd(\frac{X_1}{\operatorname{soc} X_1}) = 1$, by Lemma 2.3. By Proposition 2.2, $\tau(\frac{X_1}{\operatorname{soc} X_1}) \in \operatorname{mod} H$. But then $1 < pd(\operatorname{rad} X_1) = pd(\tau(\frac{X_1}{\operatorname{soc} X_1})) \leqslant 1$. This contradiction ends the proof of the proposition. □

It follows that the only indecomposable Λ-modules X with $pd_\Lambda X \leqslant 1$ and X not in $\mathcal{L}_\Lambda$, are projective–injective. Note that we do not necessarily have that $\mathcal{L}_\Lambda$ consists exactly of the indecomposable Λ-modules of projective dimension at most 1. (See example in [2].) We now have the following improvement of Theorem 2.1.

Theorem 2.5. *Let $\Lambda = \begin{pmatrix} H & 0 \\ DH & H \end{pmatrix}$ as before.*

(a) *If X is indecomposable and not projective–injective in $\operatorname{mod} \Lambda$, then X is in $\mathcal{L}_\Lambda$ if and only if $pd_\Lambda X \leqslant 1$.*
(b) *The fundamental domain $\mathcal{D}_\Lambda$ of $\mathcal{C}_H$ inside $\operatorname{mod} \Lambda$ lies in $\operatorname{add} \mathcal{L}_\Lambda$, and the remaining modules in $\mathcal{L}_\Lambda$ are projective–injective.*
(c) *There is a one-to-one correspondence between the multiplicity-free cluster-tilting objects in $\mathcal{C}_H$ and the basic tilting Λ-modules.*

It was proven in [2] that the global dimension $\operatorname{gldim} \Lambda$ of Λ is at most 3. We end this section with a more precise description of $\operatorname{gldim} \Lambda$. We will give a proof of this result using the description of Λ-modules as triples, which allows us to calculate the global dimension of Λ more precisely. This shows that Λ is normally of global dimension 3.

Proposition 2.6. $\operatorname{gldim} \Lambda \leqslant 3$. *Moreover*:

(a) $\operatorname{gldim} \Lambda = 1$ *if and only if H is semisimple.*
(b) $\operatorname{gldim} \Lambda = 2$ *if and only if $\tau_H^2 = 0$ and H is not semisimple.*

Proof. We calculate $\operatorname{gldim} \Lambda = \max\{pd\, S\colon\ S \text{ simple } \Lambda\text{-module}\} = \max\{pd(X, 0, 0), pd(0, X, 0)\colon\ X \in \operatorname{ind} H\}$.

For M in $\operatorname{mod} H$, $P_1(M) \to P_0(M) \to M \to 0$ denotes a minimal projective presentation.

Let $X \in \operatorname{ind} H$. Then $pd(0, X, 0) \leqslant 1$, for $\operatorname{mod} H$ is closed under predecessors in $\operatorname{mod} \Lambda$. Suppose X is projective in $\operatorname{mod} H$. Then the following is a minimal projective resolution:

$$0 \to \big(0, P_1(DH \otimes X), 0\big) \to \big(0, P_0(DH \otimes X), 0\big) \to (X, DH \otimes X, 1) \to (X, 0, 0) \to 0.$$

Now, if $\operatorname{gldim} \Lambda \leqslant 1$, then $P_1(DH \otimes X) = 0$ for every projective X. Since $DH \otimes -$ is the Nakayama equivalence between projective and injective H-modules, this is to say that every injective H-module is also projective, i.e. H is semisimple. This establishes (a), since Λ is clearly hereditary when H is semisimple.

Assume now that X is not projective. Then we have an exact sequence $0 \to \tau X \to D(P_1(X)^*) \to D(P_0(X)^*)$, where $(-)^* = \mathrm{Hom}_H(-, H)$. Using that for projective P there is a functorial isomorphism $DH \otimes P \simeq DP^*$, we obtain the following minimal projective resolution:

$$\begin{aligned} 0 &\to \big(0, P_1(\tau X), 0\big) \to \big(0, P_0(\tau X), 0\big) \to \big(P_1(X), DH \otimes P_1(X), 1\big) \\ &\to \big(P_0(X), DH \otimes P_0(X), 1\big) \to (X, 0, 0) \to 0. \end{aligned}$$

Thus $pd(X, 0, 0) \leqslant 2$ if and only if $P_1(\tau X) = 0$, if and only if $\tau^2 X = 0$. The proposition now follows right away. □

Corollary 2.7. *For each algebraically closed field k, there are only a finite number of basic indecomposable hereditary k-algebras H such that* gldim $\Lambda \leqslant 2$.

Proof. By Proposition 2.6, gldim $\Lambda \leqslant 1$ if and only if $\tau_\Lambda^2 = 0$, i.e. if each Λ-module is either projective or injective. Hence H is of finite representation type and so its ordinary quiver Q has no multiple arrows. In addition, $(i \to j \to k)$ is not a subquiver of Q, because the simple module S_j would be neither projective nor injective in such case. Finally, $({}_i\swarrow^j\searrow_k\swarrow^l)$ is not a subquiver of Q, because otherwise the module ${}^j_k$ would be neither projective nor injective. Therefore Q must be one of the following four quivers: A_1, A_2, A_3 with nonlinear order ($\searrow\swarrow$ and $\swarrow\searrow$). □

Denote by

$$\hat{H} = \begin{pmatrix} \ddots & & & & \\ & H & & & \\ & DH & H & & \\ & & DH & H & \\ & & & & \ddots \end{pmatrix}$$

the infinite dimensional repetitive algebra associated with the finite dimensional hereditary algebra H. As explained in [2], we have the following relationship:

$$\mathrm{mod}\, H \subset \mathcal{D} \subset \mathcal{L}_\Lambda \subset \mathrm{mod}\, \Lambda \subset \mathrm{mod}\, \hat{H} \twoheadrightarrow \underline{\mathrm{mod}}\, \hat{H} \xrightarrow{\simeq} D^b(H) \twoheadrightarrow \mathcal{C}_H.$$

The following more precise relationship will be useful.

Proposition 2.8. *Let $\alpha : (X, Y, f) \to (X', Y', f')$ be a nonzero map in* mod Λ. *Then α factors through a projective–injective Λ-module if and only if it factors through a projective module in* mod $\hat{H}$.

Proof. The projective–injective Λ-modules are additively generated by (H, DH, id). For $\hat{H}$, the projective modules, which coincide with the injective ones, are additively generated by modules of the form

$$\begin{pmatrix} \ddots & & & \\ 0 & H & 0 & \\ & DH & 0 & \\ & & & \ddots \end{pmatrix}.$$

If α factors through a projective–injective Λ-module, it is clear that it does the same when considered as a map in mod $\hat{H}$.

Conversely, assume that α factors through a projective $\hat{H}$-module. The possible projective modules must come from one or more of the following pictures:

$$(1)\quad \begin{pmatrix} X \\ Y \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} H \\ DH \end{pmatrix} \xrightarrow{\delta} \begin{pmatrix} X' \\ Y' \end{pmatrix}, \qquad (2)\quad \begin{pmatrix} 0 \\ X \\ Y \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} H \\ DH \\ 0 \end{pmatrix} \xrightarrow{\delta} \begin{pmatrix} 0 \\ X' \\ Y' \end{pmatrix},$$

$$(3)\quad \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} 0 \\ H \\ DH \end{pmatrix} \xrightarrow{\delta} \begin{pmatrix} X' \\ Y' \\ 0 \end{pmatrix}.$$

In case (2) we must have a commutative diagram

$$\begin{array}{ccc} DH & \longrightarrow & 0 \\ \| & & \downarrow \\ DH & \xrightarrow{\delta} & X', \end{array}$$

which is impossible since $\delta \neq 0$. In case (3) the diagram

$$\begin{array}{ccc} DH \otimes Y & \xrightarrow{DH\otimes\gamma} & DH \otimes H \\ \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & DH \end{array}$$

commutes. Then $DH \otimes \gamma$ must be zero. But since H is hereditary and $\gamma : Y \to H$ is nonzero, there is an indecomposable summand Y_1 of Y which is projective, with $\gamma|_{Y_1} : Y_1 \to H$ nonzero. Since $DH \otimes -$ gives an equivalence from the category of projective H-modules to the category of injective ones, then $DH \otimes \gamma|Y_1$ and hence $DH \otimes \gamma$ is nonzero. This gives a contradiction.

Hence we must have case (1), which implies that α factors through a projective–injective Λ-module. □

We end this section with some discussion about fundamental domains (see [2]). For the cluster category $\mathcal{C}_H$ we have a natural functor $\Pi : D^b H \to \mathcal{C}_H$. Let $\mathcal{D}$ be the additive subcategory of $D^b(H)$ whose indecomposable objects are the indecomposable H-modules together with the shift [1] of the indecomposable projective H-modules [7]. Then $\mathcal{D}$ is a convex subcategory of $D^b(H)$, and Π induces a bijection between the indecomposable objects of $\mathcal{D}$ and those of $\mathcal{C}_H$. In order to find other "fundamental domains", one is looking for similar properties. In particular, it is nice to use appropriate module categories rather than derived categories. A step in this direction was made in [2], by considering the duplicated algebra $\Lambda = \left(\begin{smallmatrix} H & 0 \\ DH & H \end{smallmatrix}\right)$ of a hereditary algebra H. Here there is a natural functor from $\operatorname{mod}\Lambda$ to $\mathcal{C}_H$, as discussed above, and $\operatorname{mod} H$ is naturally embedded into $\operatorname{mod}\Lambda$. In addition, the indecomposable objects $\tau_\Lambda^{-1}(I)$, for I indecomposable injective H-module, are added to $\operatorname{mod} H$ to form a fundamental domain $\mathcal{D}_\Lambda$ inside $\operatorname{mod}\Lambda$, giving a desired bijection with the indecomposables in $\mathcal{C}_H$, from our functor $\operatorname{mod}\Lambda \to \mathcal{C}_H$. Here the fundamental domain is not only convex, but is also closed under predecessors in $\operatorname{mod}\Lambda$ which are not projective–injective. We shall see that we have a similar situation when replacing the duplicated algebra Λ by a smaller algebra Γ.

3. The algebra Γ

In this section we will replace the duplicated algebra Λ by a smaller algebra Γ such that also $\operatorname{mod}\Gamma$ contains the fundamental domain $\mathcal{D}$ of $\mathcal{C}_H$.

We start with a lemma, which will be needed later.

Lemma 3.1. *Let A be a basic Artin algebra, and let S, S' be simple projective A-modules. Then*:

(a) *If I is an indecomposable injective module not isomorphic to $I_0(S)$, then* $\mathrm{Hom}_A(I, I_0(S)) = 0$.
(b) $\mathrm{End}_A I_0(S) \cong \mathrm{End}_A S$.
(c) $\mathrm{End}_A I_0(S \oplus S') \cong \mathrm{End}_A I_0(S) \times \mathrm{End}_A I_0(S')$.
(d) $S \cong \mathrm{End}_A S$ *as an* $\mathrm{End}_A S$*-vector space.*

Proof. (a) and (b) follow using the Nakayama equivalence ${}^*D$ from the category of injective Λ-modules to the category of projective Λ-modules, and (c) is a direct consequence of (a).

(d) Let ${}_A A = S \oplus Q$. Then ${}_{\mathrm{End}_A S} S \cong \mathrm{Hom}_A(A, S) = \mathrm{End}_A S \oplus \mathrm{Hom}_A(Q, S) = \mathrm{End}_A S$, since $\mathrm{Hom}_A(Q, S) = 0$, because A is basic. □

Let P be a projective Λ-module. We recall that $\mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\, P \to \mathrm{mod}(\mathrm{End}_\Lambda P)^{op}$ is an equivalence, where $\mathrm{mod}\, P$ is the full subcategory of $\mathrm{mod}\, \Gamma$ consisting of the modules with a presentation in $\mathrm{add}\, P$. Now we can take $\Gamma = (\mathrm{End}_\Lambda P)^{op}$, with the projective P such that $\mathrm{mod}\, P$ contains the fundamental domain $\mathrm{add}(\mathrm{ind}\, H \cup \{\tau_\Lambda^{-1} DH\})$ of $\mathcal{C}_H$. We want to choose P as small as possible. Since modules in $\mathrm{mod}\, P$ have their projective cover in $\mathrm{add}\, P$, it is clear that $\mathrm{add}\, P$ must contain $H \oplus P_0(\tau_\Lambda^{-1} DH)$. Next we show that this is enough.

We denote by Δ the sum of the non-isomorphic simple projective H-modules. That is, Δ is a basic Λ-module such that $\mathrm{add}\, \Delta = \mathrm{add}\, \mathrm{soc}\, H = \mathrm{add}\, \mathrm{soc}\, \Lambda$. Let ${}_\Lambda \overline{P} = H \oplus I_0^\Lambda(\Delta)$. We will prove in the next proposition that $P_0(\tau_\Lambda^{-1} DH) = I_0^\Lambda(H)$, and that the basic projective module $\overline{P}$ has the required properties. For M in $\mathrm{mod}\, \Lambda$, we denote by $\mathrm{Gen}\, M$ the full subcategory of $\mathrm{mod}\, \Lambda$ whose objects are the modules generated by M, that is, the epimorphic images of modules in $\mathrm{add}\, M$.

Proposition 3.2. *Let ${}_\Lambda \overline{P} = H \oplus I_0^\Lambda(\Delta)$. Then*:

(1) $\mathrm{add}\, \overline{P}$ *is closed under predecessors in $\mathcal{P}(\Lambda) = \{$projective Λ-modules$\}$.*
(2) $\mathrm{mod}\, \overline{P} = \mathrm{Gen}\, \overline{P} = \{(X, Y, f) \in \mathrm{mod}\, \Lambda \colon X \in \mathrm{add}\, \Delta\}$.
(3) *Let P be an indecomposable projective H-module. Then $\tau_\Lambda^{-1} D\,\mathrm{Hom}_H(P, H) = (\mathrm{soc}\, P, I_1^H(P), \pi)$, where $0 \to P \to I_0^H(P) \xrightarrow{\pi} I_1^H(P) \to 0$ is a minimal injective resolution in* $\mathrm{mod}\, H$.
(4) $I_0^\Lambda(H) = P_0(\tau_\Lambda^{-1} DH)$.
(5) $\mathrm{mod}\, H \cup \{\tau_\Lambda^{-1} DH\} \subseteq \mathrm{mod}\, \overline{P}$.
(6) *If $\overline{Q}$ is a projective Λ-module such that* $\mathrm{mod}\, H \cup \{\tau_\Lambda^{-1} DH\} \subseteq \mathrm{mod}\, \overline{Q}$, *then $\overline{P}$ is a direct summand of $\overline{Q}$.*

Proof. (1) Let $Q \to P$ be a nonzero morphism between indecomposable projective Λ-modules, with $P \in \mathrm{add}\, \overline{P}$. We have to prove that $Q \in \mathrm{add}\, \overline{P}$. We may assume that P, Q are projective–injective. Hence P is in $\mathrm{add}\, I_0^\Lambda(\Delta)$, and the result follows from Lemma 3.1(a).

(2) The first equality follows from (1). Now, $\overline{P} = (\Delta, H \oplus I_0^H(\Delta), \binom{0}{1})$, where we identify $DH \otimes_H \Delta$ with $I_0^H(\Delta)$. Thus $(X, Y, f) \in \mathrm{Gen}\, \overline{P}$ if and only if $X \in \mathrm{Gen}\, \Delta$ ($= \mathrm{add}\, \Delta$).

(3) and (4) We proceed to calculate $\tau_\Lambda^{-1} D\,\mathrm{Hom}_H(P, H) = Tr_\Lambda \mathrm{Hom}_H(P, H)$. Let $P^* = \mathrm{Hom}_H(P, H)$. Since $D\,\mathrm{Hom}_H(P, H) \cong (0, DP^*, 0)$ in $\mathrm{mod}\, \Lambda$, then $\mathrm{Hom}_H(P, H) \cong (P^*, 0, 0)$, and the following is a minimal projective presentation in $\mathrm{mod}\, \Lambda^{op}$:

$$\bigl(0, P_0\bigl(P^* \otimes_H DH\bigr), 0\bigr) \to \bigl(P^*, P^* \otimes_H DH, 1\bigr) \to \bigl(P^*, 0, 0\bigr) \to 0.$$

Applying $\mathrm{Hom}_\Lambda(-, \Lambda)$, we obtain

$$0 \to (0, P, 0) \to \Bigl(\bigl(P_0\bigl(P^* \otimes_H DH\bigr)\bigr)^*, DH \otimes_H \bigl(P_0\bigl(P^* \otimes_H DH\bigr)\bigr)^*, 1\Bigr) \to Tr_\Lambda \mathrm{Hom}_H(P, H) \to 0.$$

Since $P^* \otimes_H DH \cong DP$, then $(P_0(P^* \otimes_H DH))^* \cong (P_0(DP))^* = (DI_0(P))^* = (DI_0(\operatorname{soc} P))^* = P_0(\operatorname{soc} P) = \operatorname{soc} P$. (We used that H is hereditary in the last step.) Hence $P_0(Tr_\Lambda \operatorname{Hom}_H(P, H)) = (\operatorname{soc} P, I_0(\operatorname{soc} P), 1) = I_0^\Lambda(\operatorname{soc} P)$. The last equality follows from the description of injective Λ-modules as triples, and the fact that $\operatorname{soc} P$ is a projective H-module. Therefore the above sequence is

$$0 \to (0, P, 0) \to I_0^\Lambda(\operatorname{soc} P) \to \tau_\Lambda^{-1} D \operatorname{Hom}_H(P, H) \to 0,$$

so $\tau^{-1} D \operatorname{Hom}_H(P, H) = (\operatorname{soc} P, I_1(P), \pi)$. This establishes (3). Adding all the indecomposable projective H-modules yields the projective resolution

$$0 \to H \to I_0^\Lambda(\operatorname{soc} H) \to \tau_\Lambda^{-1} DH \to 0 \qquad (*)$$

and proves (4).

(5) We have $\operatorname{mod} H \subseteq \operatorname{mod} \overline{P}$, since H is a direct summand of $\overline{P}$ and $\operatorname{mod} H$ is closed under predecessors in $\operatorname{mod} \Lambda$. Now the projective resolution $(*)$ shows that $\tau_\Lambda^{-1} DH \in \operatorname{mod} \overline{P}$.

(6) follows from (2), (4) and (5). □

Now we define $\Gamma = \operatorname{End}_\Lambda(\overline{P})^{op}$. The next proposition describes Γ as a triangular matrix ring.

Proposition 3.3. *Let $\Gamma = \operatorname{End}_\Lambda(\overline{P})^{op}$. Then*:

(a) *Γ is isomorphic to the triangular matrix ring $\left(\begin{smallmatrix} K & 0 \\ J & H \end{smallmatrix}\right)$, where $K = \operatorname{End}_H \Delta^{op}$ is a basic semisimple algebra, and $J = I_0^H(\Delta)$. In particular, the Γ-modules can be described in terms of triples $({}_K X, {}_H Y, f)$, with $f : J \otimes_K X \to Y$.*

(b) *For $X \in \operatorname{add} \Delta$ there is an isomorphism $J \otimes_K \operatorname{Hom}_H(\Delta, X) \overset{\psi}{\simeq} DH \otimes_H X$ of H-modules which is functorial in X.*

(c) *$({}_H X, {}_H Y, f) \mapsto ({}_K \operatorname{Hom}_H(\Delta, X), {}_H Y, f\psi)$ is an equivalence from $\operatorname{mod} \overline{P}$ to $\operatorname{mod} \Gamma$.*

Proof. (a) Since ${}_\Lambda \overline{P} = H \oplus I_0^\Lambda(\Delta)$, then

$$\begin{aligned} \Gamma = \operatorname{End}_\Lambda(\overline{P})^{op} &\simeq \begin{pmatrix} \operatorname{End}_\Lambda H & \operatorname{Hom}_\Lambda(I_0^\Lambda(\Delta), H) \\ \operatorname{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & \operatorname{End}_\Lambda I_0^\Lambda(\Delta) \end{pmatrix}^{op} \\ &\simeq \begin{pmatrix} H^{op} & 0 \\ \operatorname{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & \operatorname{End}_\Lambda I_0^\Lambda(\Delta) \end{pmatrix}^{op} \simeq \begin{pmatrix} \operatorname{End}_\Lambda I_0^\Lambda(\Delta)^{op} & 0 \\ \operatorname{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & H \end{pmatrix}. \end{aligned}$$

Now, by Lemma 3.1, $\operatorname{End}_\Lambda I_0^\Lambda(\Delta) \simeq \operatorname{End}_\Lambda \Delta \simeq \operatorname{End}_H \Delta \simeq K^{op}$ is a basic semisimple algebra.

Finally, since $I_0^\Lambda(\Delta) = (\Delta, I_0^H(\Delta), 1)$, we have $\operatorname{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) \simeq \operatorname{Hom}_H(H, I_0^H(\Delta)) \simeq I_0^H(\Delta) = J$ as H–K-bimodule.

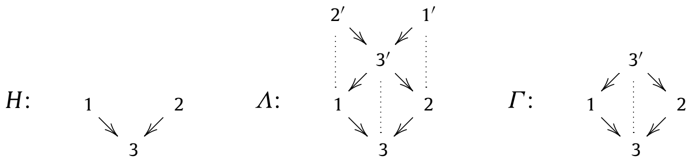
(b) Since the functors $J \otimes_K \operatorname{Hom}_H(\Delta, -)$ and $DH \otimes_H -$ are additive, we can assume that X is simple projective. By Lemma 3.1(c), we have $K \simeq \prod_{S \in \operatorname{ind} \Delta} \operatorname{End}_H(S)^{op}$, so that $J \otimes_K \operatorname{Hom}_H(\Delta, X) \simeq I_0^H(X) \otimes_{\operatorname{End}_H(X)^{op}} \operatorname{End}_H(X) \simeq \operatorname{End}_H(X) \otimes_{\operatorname{End}_H(X)} I_0^H(X) \simeq I_0^H(X)$. But $DH \otimes_H X$ is also isomorphic to $I_0^H(X)$ when X is simple projective.

(c) Let $({}_H X, {}_H Y, f) \in \operatorname{mod} \overline{P}$. By Proposition 3.2(2), we have that X is a semisimple projective H-module. Now the statement follows easily from (b). □

Note that the equivalence given in Proposition 3.3(c) is just $\operatorname{Hom}_\Lambda(\overline{P}, -) : \operatorname{mod} \overline{P} \to \operatorname{mod} \Gamma$, stated in terms of triples. We will identify $\operatorname{mod} \Gamma$ with the full subcategory $\operatorname{mod} \overline{P}$ of $\operatorname{mod} \Lambda$. Under this identification, the fundamental domain $\mathcal{D}_\Lambda$ of $\mathcal{C}_H$ in $\operatorname{mod} \Lambda$ is in $\operatorname{mod} \Gamma$, ${}_\Lambda \Gamma = {}_\Lambda \overline{P}$, and $I_0^\Gamma(H) = I_0^\Lambda(H) = P_0^\Lambda(\tau_\Lambda^{-1} DH) = P_0^\Gamma(\tau_\Gamma^{-1} DH)$. From Proposition 3.2(2), it follows easily that a minimal Λ-projective resolution of a Γ-module M is in $\operatorname{mod} \Gamma$. Hence also $pd_\Gamma M = pd_\Lambda M$ for M in $\operatorname{mod} \Gamma$.

We illustrate the situation with the following example.

Example 3.4. For the hereditary algebra H given below we indicate the corresponding algebras Λ and Γ.



By using Γ instead of Λ we get improved versions of Propositions 2.4 and 2.6.

Proposition 3.5.

(a) *Γ is a tilted algebra.*
(b) *The set of indecomposable Γ-modules with projective dimension $\leqslant 1$ is closed under predecessors, and consists of the indecomposable objects in the fundamental domain $\mathcal{D}_\Gamma$ of $\mathcal{C}_H$ plus the indecomposable projective–injective Γ-modules. In particular, $\mathcal{L}_\Gamma$ consists of the indecomposable Γ-modules of projective dimension at most one.*

Proof. (a) Let $U = DH \oplus I_0^\Gamma(\Delta)$. By [4, Lemma 2.1], it suffices to show that U is a convex tilting Γ-module.

We have $pd_\Gamma U = pd_H(DH) \leqslant 1$ because $I_0^\Gamma(\Delta)$ is projective and H is hereditary. Since $\operatorname{mod} H$ is closed under predecessors, then $\operatorname{Ext}^1_\Gamma(U,U) = \operatorname{Ext}^1_\Gamma(DH,DH) = \operatorname{Ext}^1_H(DH,DH) = 0$. Finally, $|\operatorname{ind}\operatorname{add} U| = \operatorname{rk} \mathrm{K}_0(\Gamma)$, where $\mathrm{K}_0(\Gamma)$ denotes the Grothendieck group of Γ. Hence U is tilting.

Now let us see that U is convex:

Let $T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s$ be a path in $\operatorname{ind}\Gamma$ with $T_0, T_s \in \operatorname{add} U$, where we assume that all f_i are non-isomorphisms. If $T_s \in \operatorname{mod} H$, then all T_i are H-modules, and therefore they are all H-injective. On the other hand, if $T_s \notin \operatorname{mod} H$, then $T_s \in \operatorname{add} I_0(\Delta)$. Then T_s is projective and f_s factors through $\operatorname{rad} T_s$. Since $\operatorname{rad} T_s \in \operatorname{mod} H$ and is an injective H-module, we are in the previous case, so we are done.

(b) Let $f : X \to Y$ be a nonzero map with $X, Y \in \operatorname{ind}\Gamma$ and $pd\, Y \leqslant 1$. If $pd\, Y = 1$ then, by Proposition 2.4, $pd\, X \leqslant 1$. Thus we can assume Y is projective and f is not surjective. Hence f factors through $\operatorname{rad} Y$. Since $\operatorname{rad} Y \in \operatorname{mod} H$, the proposition follows. □

Corollary 3.6. *Let $\mathcal{F} = \{X \in \operatorname{mod}\Gamma\colon\ pd\, X \leqslant 1\}$, $\mathcal{T} = \operatorname{add}(\operatorname{ind}\Gamma \setminus \mathcal{F}) = \operatorname{add}\{X \in \operatorname{ind}\Gamma\colon\ pd\, X = 2\}$. Then $(\mathcal{T}, \mathcal{F})$ is a split torsion pair in $\operatorname{mod}\Gamma$.*

Now we will prove that the global dimension of $\operatorname{End}_\Gamma(T)$ remains less than or equal to 2 when T is a tilting Γ-module. We will use results from [12], which we collect in the following lemma.

Lemma 3.7. *Let A be an Artin algebra with finite global dimension and ${}_AT$ be a tilting module such that $pd\, T = 1$ and $id\, T = s$. Let $B = (\operatorname{End}_A T)^{op}$. Then:*

(a) [12, Proposition 2.1] *We have $s \leqslant \operatorname{gldim} B \leqslant s+1$.*
(b) [12, Theorem 3.2] *If $s \geqslant 1$, then $s = \operatorname{gldim} B$ if and only if $\operatorname{Ext}^s_A(\tau T, T) = 0$.*

Proposition 3.8. *Let T be a tilting Γ-module. Then $\operatorname{gldim}\operatorname{End}_\Gamma(T) \leqslant 2$.*

Proof. By Proposition 3.5(a), Γ is a tilted algebra. Thus $\operatorname{gldim}\Gamma \leqslant 2$. Then we may assume that $pd\, T = 1$. Let $s = id\, T$. If $s \leqslant 1$, the proposition follows from Lemma 3.7(a), so assume $s = 2$. By Lemma 3.7(b), it is enough to prove that $\operatorname{Ext}^2_\Gamma(\tau T, T) = 0$. We have $\operatorname{Ext}^2_\Gamma(\tau T, T) \simeq \operatorname{Ext}^1_\Gamma(\Omega\tau T, T) \simeq$

$D\,\mathrm{Hom}_{\Gamma}(T, \tau\Omega\tau T)$. Therefore it suffices to show that $\tau\Omega\tau T = 0$. By Proposition 3.5(b), $pd\,\tau T \leqslant 1$. Hence $\Omega\tau T$ is projective and $\tau\Omega\tau T = 0$. □

We have seen that the algebra Γ is a tilted algebra and $\mathrm{mod}\,\Gamma$ contains the fundamental domain $\mathcal{D}_{\Gamma}$ of the cluster category $\mathcal{C}_H$ as a full subcategory, closed under predecessors in $\mathrm{mod}\,\Gamma$. An analogous statement to Theorem 2.5(c) also holds for tilting Γ-modules. We will use a preliminary lemma.

Lemma 3.9. *There is a bijective correspondence between the set of (isoclasses of) basic tilting Γ-modules and the set of (isoclasses of) basic tilting Λ-modules, given by ${}_{\Gamma}T \to {}_{\Lambda}T \oplus I_0^{\Lambda}(DH)/I_0^{\Lambda}(\Delta)$.*

Proof. Let T be a basic tilting Γ-module and let $\overline{Q} = I_0^{\Lambda}(DH)/I_0^{\Lambda}(\Delta)$. Then ${}_{\Lambda}\Lambda \simeq \overline{P} \oplus \overline{Q}$. Hence the basic projective–injective Λ-module $\overline{Q}$ is not in $\mathrm{mod}\,\Gamma$, and $T \oplus \overline{Q}$ is a basic partial tilting Λ-module. But $|\mathrm{ind}(T \oplus \overline{Q})| = |\mathrm{ind}\,T| + |\mathrm{ind}\,\overline{Q}| = |\mathrm{ind}\,\Gamma| + |\mathrm{ind}\,\Lambda| - |\mathrm{ind}\,\overline{P}| = |\mathrm{ind}\,\Lambda|$. Thus $T \oplus \overline{Q}$ is a basic tilting Λ-module. Conversely, let T' be a basic tilting Λ-module. Then the basic projective–injective Λ-module $\overline{Q}$ is a direct summand of T'. Now, since $pd\,T' \leqslant 1$, by Proposition 2.2, we have that $\tau_{\Lambda}T'$ is in $\mathrm{mod}\,H$. Thus T' is in $\mathrm{add}(\tau_{\Lambda}^{-1}\,\mathrm{mod}\,H \cup \{{}_{\Lambda}\Lambda\}) \subseteq \mathrm{add}(\mathrm{mod}\,\Gamma \cup \{\overline{Q}\})$, and therefore $T'/\overline{Q}$ is a basic partial tilting Γ-module, which must be a tilting Γ-module, by the counting argument used before. □

Theorem 3.10. *There is a bijective correspondence θ between the multiplicity-free cluster-tilting objects in the cluster category $\mathcal{C}_H$ of H and the basic tilting Γ-modules. For a cluster-tilting object represented by T in the fundamental domain $\mathcal{D}_{\Gamma}$ of $\mathcal{C}_H$, the corresponding tilting Γ-module is $\theta(T) = T \oplus I_0^{\Gamma}(\Delta)$.*

Proof. Let $T \in \mathcal{D}_{\Gamma}$ be a multiplicity-free cluster-tilting object in $\mathcal{C}_H$. By [2, Theorem 10], $T \oplus I_0^{\Lambda}(DH)$ is a basic tilting Λ-module. Thus, by Lemma 3.9, $\theta(T) = T \oplus I_0^{\Gamma}(\Delta) = T \oplus I_0^{\Lambda}(\Delta)$ is a basic tilting Γ-module. Conversely, if T' is a basic tilting Γ-module then, by Lemma 3.9, $T' \oplus I_0^{\Lambda}(DH)/I_0^{\Lambda}(\Delta)$ is a basic tilting Λ-module. Hence, by Theorem 2.5(c), $T'/I_0^{\Gamma}(\Delta) = T'/I_0^{\Lambda}(\Delta)$ is in the fundamental domain $\mathcal{D}_{\Lambda}$, and represents a multiplicity-free cluster-tilting object in $\mathcal{C}_H$. □

As a consequence of this result we obtain the following result of [7].

Corollary 3.11. *Let H be a hereditary algebra. Then each almost complete cluster-tilting object in $\mathcal{C}_H$ has exactly two complements.*

Proof. Let T' be an almost complete cluster-tilting object in $\mathcal{C}_H$. Then $T' \oplus I_0^{\Gamma}(\Delta)$ is an almost complete tilting module in $\mathrm{mod}\,\Gamma$. Since $\mathrm{add}\,\Delta = \mathrm{add}(\mathrm{soc}\,\Gamma)$, then $I_0^{\Gamma}(\Delta)$ is a faithful Γ-module, since $\Gamma \subseteq I_0^{\Gamma}(\Delta)$. We know from a result of Happel and Unger that then $T' \oplus I_0^{\Gamma}(\Delta)$ has exactly two complements, thus so does T' in $\mathcal{C}_H$ (see [14]). □

The following result, building upon Proposition 2.8, will be useful in the next section. Here $\underset{\sim}{\mathcal{D}}_{\Lambda}$ and $\underset{\sim}{\mathcal{D}}_{\Gamma}$ denote the categories $\mathcal{D}_{\Lambda}$ and $\mathcal{D}_{\Gamma}$ modulo the projective–injective Λ-modules, respectively the projective–injective Γ-modules.

Proposition 3.12.

(a) *A map $\alpha : (X, Y, f) \to (X', Y', f')$ in $\mathcal{D}_{\Lambda} \subset \mathrm{mod}\,\Lambda$ factors through a projective–injective Λ-module if and only if it factors through $I_0^{\Lambda}(\mathrm{soc}\,H)$.*
(b) *We have an embedding of $\underset{\sim}{\mathcal{D}}_{\Gamma}$ into $D^b(H)$ via the composition $\underset{\sim}{\mathcal{D}}_{\Gamma} = \underset{\sim}{\mathcal{D}}_{\Lambda} \subset \underset{\sim}{\mathrm{mod}}\,\Lambda \subset \underset{\sim}{\mathrm{mod}}\,\hat{H} = \underline{\mathrm{mod}}\,\hat{H} \xrightarrow{\simeq} D^b(H)$.*

Proof. The indecomposable objects in $\mathcal{D}_{\Lambda}$ are the indecomposable H-modules together with the $\tau_{\Lambda}^{-1}(I)$ for I an indecomposable injective H-module. Similarly, the indecomposable objects in $\mathcal{D}_{\Gamma}$

are the indecomposable H-modules together with the $\tau_\Gamma^{-1}(I)$ for I an indecomposable injective H-module. If $f : X \to Z$ is an irreducible map between indecomposable modules in $\operatorname{mod}\Lambda$, with Z projective–injective and X in $\operatorname{mod} H$, it is clear that $Z = \binom{S}{I_0(S)}$, where S is simple projective, since all proper predecessors of Z should be H-modules. Since then only summands of $I_0^\Lambda(\operatorname{soc} H)$ are amongst the projective–injective Λ-modules which are summands of the middle term of an almost split sequence $0 \to I \to E \to \tau_\Lambda^{-1} I \to 0$ in $\operatorname{mod}\Lambda$, we see that $\tau_\Gamma^{-1} I = \tau_\Lambda^{-1} I$, so that $\mathcal{D}_\Gamma = \mathcal{D}_\Lambda$.

It follows from the above that the only indecomposable projective–injective Λ-modules which have a nonzero map to $\tau_\Lambda^{-1} I$ are the summands of $I_0^\Lambda(\operatorname{soc} H)$. Hence no nonzero map $g : D \to D'$ in $\mathcal{D}_\Lambda$ can factor through any other projective–injective Λ-modules. (Note, however, that there might be additional projective–injective Λ-modules belonging to $\mathcal{L}_\Lambda$.) Now we conclude that $\underset{\sim}{\mathcal{D}}_\Gamma = \underset{\sim}{\mathcal{D}}_\Lambda$, and the rest follows. □

4. A description of the cluster-tilted algebras

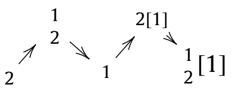
In this section our aim is to describe the quivers of cluster-tilted algebras, that is, of the endomorphism algebras of cluster-tilting objects in $\mathcal{C}_H$ [8], using the fundamental domain $\mathcal{D}_\Gamma$ for $\mathcal{C}_H$ inside $\operatorname{mod}\Gamma$. Note that a cluster-tilted algebra is determined by its quiver [6]. Let $\hat{T}$ be a cluster-tilting object in $\mathcal{C}_H$. We assume that $\hat{T}$ is represented by T in $\mathcal{D}_\Gamma \subset \operatorname{mod}\Gamma$. For X, Y in $\mathcal{D}_\Gamma$, regarded as objects in $D^b(H)$, we have that $\operatorname{Hom}_{\mathcal{C}_H}(X, Y) = \operatorname{Hom}_{D^b(H)}(X, Y) \oplus \operatorname{Hom}_{D^b(H)}(F^{-1}X, Y)$, where $F = \tau^{-1}[1]$ in $D^b(H)$ (see [6]). By Proposition 3.12 we have $\operatorname{Hom}_{D^b(H)}(X, Y) = \underset{\sim}{\mathrm{Hom}}_\Gamma(X, Y)$. We want to investigate how to describe $\operatorname{Hom}_{D^b(H)}(F^{-1}X, Y)$ in terms of $\operatorname{mod}\Gamma$, for $X, Y \in \operatorname{add} T$.

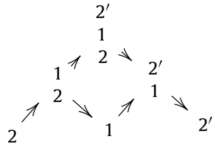
We first assume that T is an H-module. In this case $\operatorname{Hom}_{D^b(H)}(F^{-1}T, T) \simeq \operatorname{Ext}_B^2(DB, B)$, where $B = \operatorname{End}_{D^b(H)}(T)$ [1, proof of Theorem 2.3]. The top of this B–B-bimodule is generated as k-vector space by a minimal set of relations of B [1, 2.2 and 2.4]. These relations correspond to relations between projective B-modules. Since projective B-modules are of the form $\operatorname{Hom}_{D^b(H)}(T, T') = \operatorname{Hom}_H(T, T')$, such relations correspond to relations between indecomposable modules in $\operatorname{add} T$.

By this we mean the following. Let $T = T_1 \oplus \cdots \oplus T_n$ with the T_i indecomposable and pairwise non-isomorphic. We will consider maps $f : T_i \to T_j$ which are irreducible in $\operatorname{add} T$. Since $\operatorname{End}_H(T)$ is a tilted algebra, $\operatorname{add} T$ is triangular. Thus no map from T_i to T_i is irreducible in $\operatorname{add} T$ and, for $i \neq j$, maps from T_i to T_j are irreducible in $\operatorname{add} T$ precisely when they do not factor through a module in $\operatorname{add}(T/(T_i \oplus T_j))$. Let $A(i, j)$ be the space $\operatorname{Hom}_H(T_i, T_j)$ modulo the maps which factor through $\operatorname{add}(T/(T_i \oplus T_j))$. For each pair (i, j) with $i \neq j$, choose a set of irreducible maps in $\operatorname{add} T$ representing a basis of $A(i, j)$, and let $\mathcal{B}$ be the union of all these bases. To each path of maps in $\mathcal{B}$ we associate the corresponding composition map in $\operatorname{mod} H$. A linear combination of such paths is a *relation for* $\operatorname{add} T$ if the corresponding map is zero in $\operatorname{mod} H$. A set R of such relations is a *minimal set of relations for* $\operatorname{add} T$ if R is a minimal set of generators of the ideal of relations for $\operatorname{add} T$. This means that for any relation $g : T_r \to T_s$ we have $g = \sum a_i \gamma_i \rho_i \gamma_i'$, with a_i in k, ρ_i in R, and γ_i, γ_i' paths in $\operatorname{add} T$; and no proper subset of R has this property.

We will prove that a similar statement holds also when the Γ-module T is not an H-module. In this case we have to consider a minimal set of relations between indecomposable summands of T in $\operatorname{add}(T \oplus I_0^\Gamma(\Delta))$.

Consider the following example. Let $H = kQ$, where Q is the quiver $1 \to 2$. Then Γ is the path algebra of the quiver $2' \to 1 \to 2$. Let $\mathcal{D}$ be the fundamental domain of $\mathcal{C}_H$.

$\mathcal{D}$: 

AR-quiver of Γ: 

and let $T_1 = 2$, $T_2 = {}^{1}_{2}[1]$, and $T = T_1 \oplus T_2$. Then T defines a cluster-tilting object in $\mathcal{C}_H$ and is not an H-module. The Γ-module corresponding to T under the identification of $\mathcal{D}$ with $\underset{\sim}{\mathrm{mod}}\,\Gamma$ is $2 \oplus 2'$.

Moreover, $\mathrm{Hom}_{\mathcal{C}_H}(T_2, T_1) = \mathrm{Hom}_{D^bH}(\tau T_2[-1], T_1) = \mathrm{Hom}_{D^bH}(\tau \begin{smallmatrix}1\\2\end{smallmatrix}, T_1) \neq 0$, but there are no relations in $\mathrm{add}\, T$ from T_1 to T_2. However $2 \to \begin{smallmatrix}2'\\1\\2\end{smallmatrix} \to 2'$ is a zero relation from T_1 to T_2 in $\mathrm{add}(T \oplus I)$, where $I = \begin{smallmatrix}2'\\1\\2\end{smallmatrix}$ is the injective envelope of the simple 2 in $\mathrm{mod}\, \Gamma$.

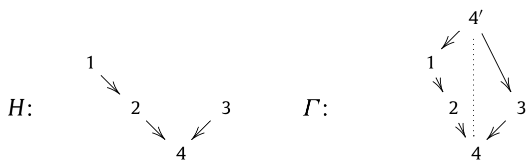
To study the general case we will define an appropriate hereditary algebra $\tilde{H}$, and use that the above mentioned result of [1] holds for tilting $\tilde{H}$-modules, to prove our desired result.

We start with defining a hereditary algebra $\tilde{H}$ such that there is an exact embedding $G : \mathrm{mod}\, \Gamma \to \mathrm{mod}(\tilde{H})$ with the property that tilting Γ-modules map to tilting $\tilde{H}$-modules.

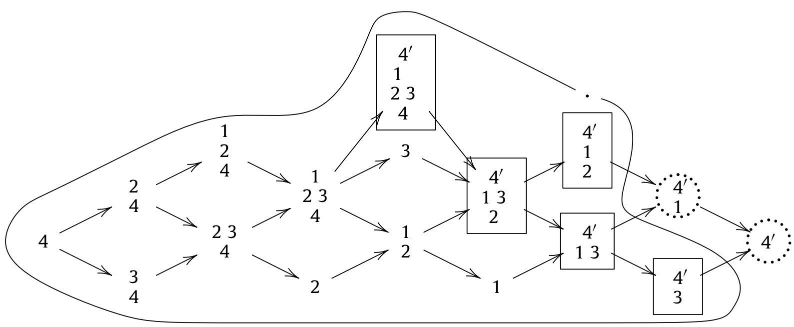
We recall from Proposition 3.5 that $U = DH \oplus I_0^{\Gamma}(\Delta)$ is a complete slice in $\mathrm{mod}\, \Gamma$. We consider another complete slice, $\Sigma = \tau_{\Gamma}^{-1} DH \oplus I_0^{\Gamma}(\Delta)$. Then $\tilde{H} = (\mathrm{End}_{\Gamma}(\Sigma))^{op}$ is a hereditary algebra of type Σ. Let $(\mathcal{T}, \mathcal{F})$ be the split torsion pair in $\mathrm{mod}\, \Gamma$ of Corollary 3.6. Then $\mathrm{ind}\, \mathcal{F}$ coincides with the predecessors of Σ. Also $D\Sigma$ is a tilting $\tilde{H}$-module, $\Gamma = \mathrm{End}_{\tilde{H}}(D\Sigma)^{op}$ and the functors $L = \mathrm{Hom}_{\tilde{H}}(D\Sigma, -)$ and $\mathrm{Ext}^1_{\tilde{H}}(D\Sigma, -) : \mathrm{mod}\, \tilde{H} \to \mathrm{mod}\, \Gamma$ induce equivalences $\mathcal{T}_{D\Sigma} \to \mathcal{F}$ and $\mathcal{F}_{D\Sigma} \to \mathcal{T}$, respectively, where $(\mathcal{T}_{D\Sigma}, \mathcal{F}_{D\Sigma})$ is the torsion pair associated to the tilting $\tilde{H}$-module $D\Sigma$.

Let $G = D\Sigma \otimes_{\Gamma} - : \mathrm{mod}\, \Gamma \to \mathrm{mod}\, \tilde{H}$. Then L and G are adjoint functors, and the restrictions $L|_{\mathcal{T}_{D\Sigma}} : \mathcal{T}_{D\Sigma} \to \mathcal{F}$ and $G|_{\mathcal{F}} : \mathcal{F} \to \mathcal{T}_{D\Sigma}$ are inverse equivalences of categories. Moreover, $G(\Sigma) = D\tilde{H}$, because $D\tilde{H} \simeq GLD\tilde{H} = G\, \mathrm{Hom}_{\tilde{H}}(D\Sigma, D\tilde{H}) \simeq G\, \mathrm{Hom}_{\tilde{H}^{op}}(\tilde{H}, \Sigma) \simeq G(\Sigma)$.

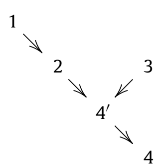
Example 4.1. We illustrate the situation for the hereditary algebra H indicated below.



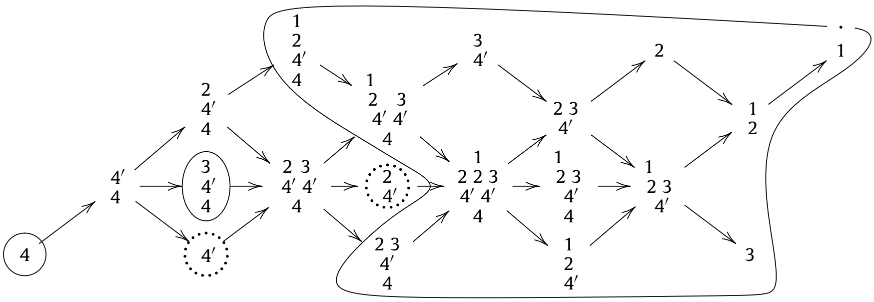
The Auslander–Reiten quiver of Γ is:



Here, ${}_{\Gamma}\Sigma$ is the sum of the five modules in frames, $\mathcal{T}$ is given by the two modules inside dotted circles, $\mathcal{F}$ is indicated by the curve, and $\tilde{H}$ is the algebra with quiver:



The Auslander–Reiten quiver of $\tilde{H}$ is



In this case ${}_H D\Sigma = 4 \oplus \begin{smallmatrix}1\\2\\4'\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4'\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\ 3\\4'\\4\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\ 3\\4'\\4\end{smallmatrix}$, $\mathcal{F}_{D\Sigma}$ consists of the two modules inside dotted circles, and $\mathcal{T}_{D\Sigma}$ consists of the 15 modules inside the regions indicated with curves.

We know from the Brenner–Butler theorem that $\mathcal{F} = \operatorname{Ker}\operatorname{Tor}_1^{\Gamma}(D\Sigma, -)$, so that $G = D\Sigma \otimes_{\Gamma} - : \operatorname{mod}\Gamma \to \operatorname{mod}\tilde{H}$ restricted to $\mathcal{F}$ is exact. In particular, $G|_{\operatorname{mod} H}$ is exact. Thus G induces an embedding

$$\hat{G} : D^b(\operatorname{mod} H) \to D^b(\operatorname{mod}\tilde{H})$$

such that $\hat{G}(M[i]) = (\hat{G}(M))[i]$.

Proposition 4.2. *Let T be a tilting Γ-module. Then $G(T)$ is a tilting $\tilde{H}$-module.*

Proof. Since Γ and $\tilde{H}$ have the same number of non-isomorphic simple modules, we only need to prove that $\operatorname{Ext}^1_{\tilde{H}}(G(T), G(T)) = 0$. Since $T \in \mathcal{F}$ because $pd\,T \leqslant 1$, it follows that $G(T) \in \mathcal{T}_{D\Sigma}$. Then $\operatorname{Ext}^1_{\Gamma}(LG(T), LG(T)) = \operatorname{Ext}^1_{\tilde{H}}(G(T), G(T))$. Thus $\operatorname{Ext}^1_{\tilde{H}}(G(T), G(T)) \simeq \operatorname{Ext}^1_{\Gamma}(T, T) = 0$, since $LG(T) \simeq T$ because $T \in \mathcal{F}$. □

We observe that in general the modules $\tau(G(X))$ and $G(\tau X)$ are not isomorphic, for $X \in \operatorname{mod}\Gamma$. We will prove next that they are isomorphic when $X = \tau_{\Gamma}^{-1} I$, for any injective H-module I. We start with three lemmas.

Lemma 4.3. *Let $f : X \to Y$ be a morphism in* mod Γ*, with $Y \in \mathcal{F}$, such that $G(f) : GX \to GY$ is* (*minimal*) *right almost split in* mod $\tilde{H}$*. Then f is* (*minimal*) *right almost split in* mod Γ*.*

Proof. The key is that $\mathcal{F}$ is closed under predecessors and $G|_{\mathcal{F}} : \mathcal{F} \to \mathcal{T}_{D\Sigma}$ is an equivalence. From this we obtain right away that $X \in \mathcal{F}$, and f is not a split epimorphism since $G(f)$ is not a split epimorphism. Now, let Z be an indecomposable Γ-module and $h : Z \to Y$ a morphism which is not a split epimorphism. Again, $Z \in \mathcal{F}$ and $G(h)$ is not an isomorphism. Hence there is a morphism $g : G(Z) \to G(X)$ such that $G(h) = G(f)g$. Using that $G|\mathcal{F} : \mathcal{F} \to \mathcal{T}_{D\Sigma}$ is an equivalence once more, we deduce there is a $g' : Z \to X$ such that $G(g') = g$ and $h = fg'$. Thus f is right almost split. The minimality is deduced in the same way. □

Let again Σ denote the complete slice in mod Γ which consists of $\operatorname{ind}(\tau_{\Gamma}^{-1} DH \oplus I_0^{\Gamma}(\operatorname{soc} H))$.

Lemma 4.4. *The indecomposable projective–injective modules* (*i.e. those in* $\operatorname{ind}(I_0^{\Gamma}(\operatorname{soc} H))$) *are sources of Σ.*

Proof. Let $P \in \operatorname{ind}(I_0(\operatorname{soc} H))$ and let $f : X \to P$ be a nonzero non-isomorphism in $\operatorname{ind} \Gamma$. Then $\operatorname{Im} f \subseteq \operatorname{rad} P$, which is an injective H-module. Thus $X \in \operatorname{mod} H$, since $\operatorname{mod} H$ is closed under predecessors in $\operatorname{mod} \Gamma$. But then $X \notin \Sigma$. □

Lemma 4.5. *Let I be an indecomposable injective $\tilde{H}$-module, and let $M \to I$ be a minimal right almost split morphism in* $\operatorname{mod} \tilde{H}$. *Then $M \in \mathcal{T}_{D\Sigma}$.*

Proof. Let M' be an indecomposable direct summand of M, and assume $M' \notin \mathcal{T}_{D\Sigma}$. Then M' is not injective and there is an irreducible morphism $I \to \tau^{-1}M'$. Since $\tilde{H}$ is hereditary, then $\tau^{-1}M'$ is injective. Thus $\tau^{-1}M' \in \mathcal{T}_{D\Sigma}$, and since $M' \notin \mathcal{T}_{D\Sigma}$ we conclude that $\tau^{-1}M'$ is Ext-projective in $\mathcal{T}_{D\Sigma}$. Therefore, there exists $N \in \operatorname{ind} \Gamma$ such that N is Ext-projective in $\mathcal{F}$ and $GN = \tau^{-1}M'$. Since $\mathcal{F}$ is closed under predecessors, then N is projective in $\operatorname{mod} \Gamma$. But we also have $N \in \operatorname{add} \Sigma$, because G maps Σ to $D\tilde{H}$, as we observed before Example 4.1. Since $\tau^{-1}DH$ contains no nonzero projective direct summands, then $N \in \operatorname{ind}(I_0(\operatorname{soc} H))$, i.e. N is projective–injective. By the preceding lemma, N is a source in Σ. Thus $\tau^{-1}M' = GN$ is a source in $\operatorname{ind}(D\tilde{H})$, which contradicts the already established existence of an irreducible morphism $I \to \tau^{-1}M'$. □

Proposition 4.6. *Let I be an indecomposable injective H-module. Then $G\tau_\Gamma^{-1} I = \tau_{\tilde{H}}^{-1} GI$.*

Proof. Since $\tau_\Gamma^{-1} I \in \Sigma$, then $G\tau_\Gamma^{-1} I$ is $\tilde{H}$-injective. Let $f : M \to G\tau_\Gamma^{-1} I$ be a minimal right almost split morphism. By Lemma 4.5, we have $M \in \mathcal{T}_{D\Sigma}$. Then there is a morphism $g : N \to \tau_\Gamma^{-1} I$ in $\operatorname{mod} \Gamma$ with $N \in \mathcal{F}$, such that $f = Gg$. By Lemma 4.3, g is minimal right almost split in $\operatorname{mod} \Gamma$. Then the almost split sequence $0 \to I \to N \xrightarrow{g} \tau_\Gamma^{-1} I \to 0$ is contained in $\mathcal{F}$, and applying G we obtain an exact sequence $(*)$ $0 \to GI \to M \xrightarrow{f} G\tau_\Gamma^{-1} I \to 0$.

Since f is minimal right almost split, then the sequence $(*)$ is almost split. Hence $GI = \tau_{\tilde{H}} G\tau_\Gamma^{-1} I$, and the result follows by applying $\tau_{\tilde{H}}^{-1}$. □

Proposition 4.7. *Let $\hat{T}$ be a cluster-tilting object in $\mathcal{C}_H$ represented by T in the fundamental domain $\mathcal{D}_\Gamma$ of $\mathcal{C}_H$, which we consider embedded in* $\operatorname{mod} \Gamma$ *as before. Let T_1, T_2 be indecomposable summands of T. Then* $\operatorname{top} \operatorname{Hom}_{D^b(H)}(F^{-1}T_1, T_2)$ *is a vector space with basis given by a minimal set of minimal relations from T_2 to T_1 in* $\operatorname{add}(T \oplus I_0^\Gamma(\Delta))$.

Proof. As we observed above, the result holds for summands T_1, T_2 of T which are H-modules. So we only need to consider the case when $T_1 \notin \operatorname{mod} H$, that is, $T_1 = \tau_\Gamma^{-1} I$, where I is an indecomposable injective module in $\operatorname{mod} H$. For if $T_1 \in \operatorname{mod} H$ and $T_2 \notin \operatorname{mod} H$, we have $\operatorname{Hom}(\tau T_1[-1], T_2) = 0$ since $T_2 = P_i[1]$ for P_i indecomposable projective [2].

Then $\tau_\Gamma^{-1} I = \tau_{D^b(H)}^{-1} I = P[1]$ in $D^b(H)$, where $\operatorname{top} P = \operatorname{soc} I$, via our identification. Then, for $X \in \operatorname{mod} H$ we have

$$\begin{aligned}\operatorname{Hom}_{D^b(H)}\big(F^{-1}\big(\tau^{-1}I\big), X\big) &= \operatorname{Hom}_{D^b(H)}\big(I[-1], X\big) \simeq \operatorname{Hom}_{D^b(\tilde{H})}\big(\hat{G}\big(I[-1]\big), \hat{G}X\big) \\ &= \operatorname{Hom}_{D^b(\tilde{H})}\big((GI)[-1], GX\big) = \operatorname{Hom}_{D^b(\tilde{H})}\big(F^{-1}\tau^{-1}(GI), GX\big).\end{aligned}$$

From Proposition 4.6 we know that $G\tau^{-1}I = \tau^{-1}GI$. Thus

$$\operatorname{Hom}_{D^b(H)}\big(F^{-1}\big(\tau^{-1}I\big), X\big) \simeq \operatorname{Hom}_{D^b(\tilde{H})}\big(\big(F^{-1}G\big(\tau^{-1}I\big)\big), GX\big).$$

Using Proposition 4.6 again we can prove that this isomorphism induces an isomorphism between the corresponding tops.

Recall that $T \oplus I_0^{\Gamma}(\Delta)$ is a tilting Γ-module (Theorem 3.10) and therefore $G(T \oplus I_0^{\Gamma}(\Delta))$ is a tilting $\tilde{H}$-module (Proposition 4.2).

Now consider $X = T_2$. Since both $G(\tau_{\Gamma}^{-1} I) = G(T_1)$ and $G(T_2)$ are modules over the hereditary algebra $\tilde{H}$, we can apply [2] to the tilting module $G(T \oplus I_0^{\Gamma}(\Delta))$ and conclude that $\operatorname{top}\operatorname{Hom}_{D^b(\tilde{H})}(F^{-1}G(T_1), GT_2)$ has a basis in correspondence with a minimal set of minimal relations from $G(T_2)$ to $G(T_1)$ in $\operatorname{add} G(T \oplus I_0^{\Gamma}(\Delta))$. Since $G|_{\operatorname{add}(T \oplus I_0^{\Gamma}(\Delta))}$ is an equivalence of categories, we obtain minimal relations as stated. $\square$

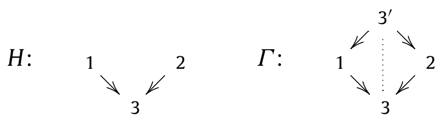
Let T and $\hat{T}$ be as in the previous proposition. We are now in the position to describe the ordinary quiver Q_C of the cluster-tilted algebra $C = \operatorname{End}_{\mathcal{C}_H}(\hat{T})$, in terms of $\operatorname{mod}\Gamma$.

Theorem 4.8. *Let $C = \operatorname{End}_{\mathcal{C}_H}(\hat{T})$, where $\hat{T}$ is a basic cluster-tilting object in $\mathcal{C}_H$ represented by $T = \bigoplus T_i$ in $\operatorname{mod}\Gamma$, with T_i indecomposable. Let $B = \underset{\sim}{\operatorname{End}}_{\Gamma}(T)$, and let i denote the vertex of Q_C associated to $\underset{\sim}{\operatorname{Hom}}_{\Gamma}(T, T_i)$. Then, for vertices i, j of Q_C the number of arrows from i to j is equal to the number of arrows from i to j in Q_B plus the cardinality of a minimal set of minimal relations from T_i to T_j in $\operatorname{add}(T \oplus I_0^{\Gamma}(\Delta)) \subset \operatorname{mod}\Gamma$.*

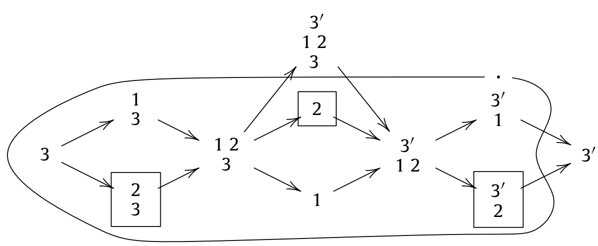
Proof. The number of arrows from i to j equals $\dim\operatorname{top}\operatorname{Hom}_{\mathcal{C}_H}(T_j, T_i) = \dim\operatorname{top}\operatorname{Hom}_{D^b(H)}(T_j, T_i) \oplus \dim\operatorname{top}\operatorname{Hom}_{D^b(H)}(F^{-1}T_j, T_i)$. Now the result follows from the previous proposition and the fact that $\dim\operatorname{top}\operatorname{Hom}_{D^b(H)}(T_j, T_i)$ is equal to the number of arrows from i to j in Q_B, because $\operatorname{Hom}_{D^b(H)}(T_j, T_i) \simeq \underset{\sim}{\operatorname{Hom}}_{\Gamma}(T_j, T_i)$, by Proposition 3.12(b). $\square$

Remark 4.9. In the above statement, for each pair of vertices i and j, only one of the summands describing the number of arrows from i to j is nonzero.

Example 4.10. For the hereditary algebra H given below we indicate the corresponding algebra Γ.



Then the AR-quiver of Γ is



and the fundamental domain of $\mathcal{C}_H$ corresponds to the region enclosed by the curve. Let $T = \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3'\\2\end{smallmatrix}$. Then $T \oplus \begin{smallmatrix}3'\\1\ 2\\3\end{smallmatrix}$ is a tilting Γ-module, so T defines a cluster-tilting object $\overline{T}$ in $\mathcal{C}_H$.

We notice that nonzero maps $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}3'\\1\ 2\\3\end{smallmatrix} \to \begin{smallmatrix}3'\\2\end{smallmatrix}$, or $\begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \to \begin{smallmatrix}3'\\2\end{smallmatrix}$ have always nonzero composition.

However, there are nonzero maps $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix} & 3' & \\ 1 & & 2 \\ & 3 & \end{smallmatrix} \oplus 2 \to \begin{smallmatrix}3'\\2\end{smallmatrix}$ with zero composition, and this relation from $\begin{smallmatrix}2\\3\end{smallmatrix}$ to $\begin{smallmatrix}3'\\2\end{smallmatrix}$ is unique, up to scalar multiples. Therefore $\dim \operatorname{Hom}_{\mathcal{C}_H}(\begin{smallmatrix}3'\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}) = 1$.

Since $\dim \operatorname{Hom}_{\Gamma}(\begin{smallmatrix}2\\3\end{smallmatrix}, 2) = 1$, and $\dim \operatorname{Hom}_{\Gamma}(2, \begin{smallmatrix}3'\\2\end{smallmatrix}) = 1$, the ordinary quiver of the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}_H}(\overline{T})$ is

$$\begin{array}{ccc} 1 & \longleftarrow & 2 \\ & \nwarrow \quad \swarrow & \\ & 3. & \end{array}$$

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