Multidimensional Stochastic Matrices and Error-Correcting Codes*

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Abstract

The exact value of $7(n, 9, r)$, the largest possible number of codewords in a code of length $n$, distance $r + 1$, where the entries of the codewords are members of an alphabet of $q$ elements, is determined in large number of cases. Also, a new upper bound is given for this function $7$. These exact values and estimations are used to answer a question of Jurkat and Ryser concerning the existence of higher dimensional stochastic $(0, 1)$ matrices in several cases.

I. INTRODUCTION

It is well known that the permutation matrices are the extremal elements of the convex and compact set of doubly stochastic matrices of order $n$. In other words, the extreme elements are the $(0, 1)$ matrices and only those. In [10], Jurkat and Ryser introduced the notion of stochastic matrices in a higher dimensional setting. For completeness these definitions will be repeated here. The presentation of [8] will be followed.

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Let $q$ and $n$ be fixed positive integers. Let $J_{n,q} = \{(i_1, \ldots, i_n) : i_l \in \mathbb{N}, 1 \leq i_l \leq q \text{ for } l = 1, \ldots, n\}$. A subset $T$ of $J_{n,q}$ is said to be an affine subspace of dimension $r$ of $J_{n,q}$ if

$$T = \{(i_1, \ldots, i_n) \in J_{n,q} :$$

$$i_{l_1} = \text{const}, \ldots, i_{l_{n-r}} = \text{const}, \text{ where } 1 \leq l_1 < l_2 < \cdots < l_{n-r} \leq n\}.$$ 

We shall say that a matrix $A$ has dimension $n$ and order $q$ if

$$A : J_{n,q} \to \mathbb{C}.$$ 

**Definition 1.** Let $n \geq 2$ and $r < n$ be positive integers. A stochastic matrix $A$ of dimension $n$, degree $r$, and order $q$ [denoted by $A \in S(n, q, r)$] is an $n$-dimensional matrix with nonnegative entries of order $q$ with the property that $\sum a_{i_1, \ldots, i_n} = 1$ for every $r$-dimensional affine subspace $T$ of $J_{n,q}$.

It is obvious that if $A$ is a $(0, 1)$ matrix and if $A \in S(n, q, r)$, then $A$ is an extremal element of $S(n, q, r)$. It follows from [10], [2], and [9] that the convex and compact sets $S(3, q, r)$ where $q$ and $r$ are fixed positive integers, $q \geq 2$, $r = 1$ or $r = 2$, have $(0, 1)$ extremal elements, and except for the case $q = 2$ there are also extremal elements having entries different from zero or one.

If $n \geq 4$, in general, a given class of stochastic matrices has no $(0, 1)$ extremal elements. In [10], Jurkat and Ryser asked to characterize those stochastic classes (in terms of $n, q$, and $r$) which have $(0, 1)$ matrices.

If $\xi$ and $\eta$ are $n$-tuples, then $d_H(\xi, \eta)$ will stand for the Hamming distance between $\xi$ and $\eta$. The following result appears in [8]:

**Theorem 1.** $S(n, q, r)$ has a $(0, 1)$ extremal element if and only if $J_{n,q}$ has a subset $M$ such that

$$\text{for all } \xi, \eta \in M, \xi \neq \eta, \text{ we have } d_H(\xi, \eta) \geq r + 1, \quad (1)$$

and

the cardinality of $M$ is $q^{n-r}$. \quad (2)
If a set \( M \subset J_{n,q} \) has the property (1) of Theorem 1, then we shall say that \( M \) is an \( \{n, q, r\} \) code. Theorem 1 suggests the consideration of the following function \( \tau \), where

\[
\tau(n, q, r) = \max\{\text{card } M : M \subset J_{n,q}, M \text{ is an } \{n, q, r\} \text{ code}\}.
\]

This function has already been introduced in the literature (see for example [1], [5], [12], [13], or [15]). In our notation \( \tau(n, 2, r) = A(n, r + 1) \), where \( A(n, s) \) is the function defined in [12, p. 42]. Note that the determination of \( \tau(n, q, r) \) is considered as the main problem of the theory of error-correcting codes [12, 15]. Thus the existence of a \((0,1)\) matrix of a class \( S(n, q, r) \) is equivalent to the existence of a \( \{n, q, r\} \) code \( M \) such that \( \text{card } M = q^{n-r} \).

We shall call such a code a combinatorial MDS code or simply a CMDS code, since a linear CMDS code is called an MDS code [12, 15]. If \( \{n, q, r\} \) is a CMDS code, then such a code will be also referred as a code of length \( n \), order \( q \), and dimension \( d \), where \( d = n - r \).

In a recent paper of Brualdi and Csima a sufficient and necessary condition concerning the existence of higher dimensional stochastic \((0,1)\) matrices has been given with the aid of orthogonal arrays. (See Theorem 3 of [3].) For the definition of orthogonal arrays, we refer to [3]. Using Theorem 3 of [3], a complete classification of higher dimensional stochastic \((0,1)\) matrices has been obtained in [3] in the case when \( 2 \leq q \leq 6 \). Equivalently, it follows that [3] gives a complete description of CMDS codes in the same cases.

In this paper it will be always assumed that if \( M \) is an \( \{n, q, r\} \) code consisting of \( N \) codewords, then \( M \) is written as an \( N \times n \) matrix, where the different codewords of \( M \) are the rows of this matrix.

One of the topics of this paper is the study of the function \( \tau(n, q, r) \). It will be shown that the exact values of this function can be determined in a large number of cases. Inequalities concerning this function will be derived both in the case when \( q = 2 \) and when \( q \geq 3 \). These exact values and estimations will show the nonexistence of CMDS codes with certain parameters. A new method will be introduced in the study of the function \( \tau \) by considering \( n \) as a function of \( l = n - r \).

In the next section inequalities of the Plotkin type will be established for the case when \( q \geq 3 \). In the third and the fourth section the exact values of \( \tau \) will be given for large numbers of cases. In the last section we derive some inequalities for the function \( \tau \) and present an application of these inequalities to the problem of the existence of both CMDS codes and higher dimensional \((0,1)\) matrices.
II. INEQUALITIES OF PLOTKIN TYPES

The following result is known as a Plotkin inequality.

**Theorem 2.** If \( d \) is even and \( 2d > n \), then

\[
\tau(n, 2, d - 1) = \Lambda(n, d) \leq 2 \left[ \frac{d}{2d - n} \right].
\]  \hspace{1cm} (3)

If \( d \) is odd and \( 2d + 1 > n \), then

\[
\tau(n, 2, d - 1) = \Lambda(n, d) \leq 2 \left[ \frac{d + 1}{2d + 1 - n} \right].
\]  \hspace{1cm} (4)

Equations (3) and (4) can be generalized in the following way:

**Theorem 3.** If \( dq > n(q - 1) \) and \( \tau(n, q, d - 1) \not\equiv 0 \) (mod \( q \)), then

\[
\tau(n, q, d - 1) \leq \left[ \frac{dq}{dq - n(q - 1)} - 1 \right].
\]  \hspace{1cm} (5)

If \( dq > n(q - 1) \) and \( \tau(n, q, d - 1) \equiv 0 \) (mod \( q \)), then

\[
\tau(n, q, d - 1) \leq q \left[ \frac{d}{dq - n(q - 1)} \right].
\]  \hspace{1cm} (6)

**Remark.** Theorem 2 has been generalized already (see [1] or [11]), but our generalization yields a sharper result.

**Proof.** Write \( \tau(n, q, d - 1) = N = sq + \kappa \), where \( s \) is a positive integer and \( 0 \leq \kappa \leq q - 1 \). We separate the cases \( \kappa = 0 \), \( \kappa = 1 \), \( 2 \leq \kappa \leq q - 2 \), and \( \kappa = q - 1 \). The proof of this theorem is similar to the proof of Theorem 2 as it is given in [12]. Hence, only one case will be considered here, where there are some additional complications. Thus assume that \( \tau(n, q, d - 1) = N = sq + \kappa \), where \( s \) and \( \kappa \) are positive integers with \( 2 \leq \kappa \leq q - 2 \). By definition there exists an \( \{n, q, d - 1\} \) code \((\xi_1, \ldots, \xi_N)\). Consider the
STOCHASTIC MATRICES AND CODES

$N \times n$ matrix $A$ whose rows are the $\xi_i$. By calculating the sum

$$\sum_{i=1}^{N} \sum_{j=1}^{N} d_H(\xi_i, \xi_j)$$

in two different ways, the following inequality can be obtained:

$$\frac{N(N - 1)d}{2n} \leq \left[ \left( q - \kappa \right) \left( \frac{N - \kappa}{q} \right)^2 + \left( \frac{\kappa}{2} \right) \left( \frac{N - \kappa + q}{q} \right)^2 \right. \left. + \kappa(q - \kappa) \frac{(N - \kappa)(N - \kappa + q)}{q^2} \right].$$

It is easy to see that the previous inequality can be written as

$$N(N - 1)d \leq \frac{n}{q} \left[ (q - 1)N^2 - (q - \kappa)\kappa \right] \leq \frac{n}{q} \left[ (q - 1)N^2 - q + 1 \right].$$

Hence,

$$qNd \leq n(q - 1)(N + 1),$$

from which the result follows.

III. SOME EXACT VALUES OF $\tau$

The following obvious lemma will serve for easy reference.

**Lemma 1.** $\tau(n, q, r) \geq q$.

**Theorem 4.** $\tau(n, q, n - l) = q$, provided that

$$q \geq 2, \quad l \geq 2, \text{ and } n \geq \left( \frac{q + 1}{2} \right)(l - 1) + 1.$$

**Proof.** A direct proof will be given. In view of Lemma 1, the only thing that remains to be shown is that $\tau(n, q, n - l) \leq q$. Thus, assume that $H = (\xi_1, \ldots, \xi_q)$ is a given $\{n, q, n - l\}$ code and it is to be shown that there
is no \( \xi_{q+1} \in J_{n,q} \) such that \( H_1 = H \cup \xi_{q+1} \) is also a \( \{n,q,n-l\} \) code. Since any two different members of \( H \) can agree at no more than \( l-1 \) coordinate places, there are at most \( \binom{q}{2}(l-1) \) coordinate places such that at least two members of \( H \) agree, and at the rest of the coordinate places no two members of \( H \) can agree, i.e., the members of \( H \) take every value between one and \( q \). Thus if

\[
n \geq \left( \frac{q}{2} \right)(l-1) + q(l-1) + 1,
\]

then clearly there is no \( \xi_{q+1} \) such that \( H_1 = H \cup \xi_{q+1} \) is a \( \{n,q,n-l\} \) code.

It is worthwhile to mention the following two special cases of the previous theorem:

\[
\tau(n,2,n-l) = 2 \quad \text{provided that} \quad l \geq 2 \quad \text{and} \quad n \geq 3l - 2,
\]

\[
\tau(n,3,n-l) = 3 \quad \text{provided that} \quad l \geq 2 \quad \text{and} \quad n \geq 6l - 5.
\]

**Theorem 5.** Let \( n \) and \( l \) be positive integers, with \( 2 \leq l \leq n-1 \). Then

(a) \( \tau(n,2,n-l) = A(n,n-l+1) = 4 \quad \text{provided that} \quad l \geq 2 \quad \text{and} \quad 3l - 4 < n < 3l - 3 \),

\[
\tau(n, q, n-l) = q + 1 \quad \text{provided that}
\]

\[
l \geq 2, \quad q \geq 3, \quad \text{and} \quad n = \left( \frac{q+1}{2} \right)(l-1).
\]

**Proof.** We prove first (a). It is well known (see [1] or [12]) that

\[
A(n,2r-1) = \tau(n,2,2r-2) = A(n+1,2r) = \tau(n+1,2,2r-1).
\]

Hence, \( \tau(3l-4,2,2l-4) = \tau(3l-3,2,2l-3) \). Therefore, one has to show that there is a common value, which is 4, for all \( l \geq 2 \). It is enough to consider the case when \( n = 3l - 3 \). The following example will show that \( \tau(3l-3,2,2l-3) \geq 4 \). Consider the code

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1
\end{array}
\]
where each digit 1 (2) represents a block of \( l - 1 \) ones (twos) and where the rows represent the different codewords. The converse now follows easily by (3).

To prove (b), assume that

\[
q \geq 3 \quad \text{and} \quad n = \left( \frac{q + 1}{2} \right)(l - 1).
\]

First, with the aid of an example, it will be shown that \( \tau(n, q, n - l) \geq q + 1 \).

Indeed, let \( H = (\xi_1, \ldots, \xi_{q+1}) \), where each \( \xi_i, 1 \leq i \leq q + 1, \) consists of \( \left( \frac{q + 1}{2} \right) \) blocks of \( l - 1 \) elements with the property that each element of every block takes the same value. In the representation, such a block will be denoted by a single element, which is the common element of the block. Let \( \xi_1 \) (the first codeword) consist of \( \left( \frac{q + 1}{2} \right) \) ones (i.e. \( \left( \frac{q + 1}{2} \right) \) blocks of length \( l - 1 \), \( \xi_2 \) consist of a one followed by \( \left( \frac{q + 1}{2} \right) - 1 \) twos, and \( \xi_3 \) consist of the sequence 2, 1, 2 followed by \( \left( \frac{q + 1}{2} \right) - 3 \) threes. In general, \( \xi_i \) (3 \( \leq i \leq q + 1 \)) consists of \( i - 1 \)'s occurring \( \left( \frac{i - 1}{2} \right) \) times, followed by 1, 2, \ldots, \( i - 1 \), with the remaining elements, if any, being all \( i \)'s. It is easy to see that

\[
d_h(\xi_i, \xi_j) = \left( \left( \frac{q + 1}{2} \right) - 1 \right)(l - 1) \quad \text{for} \quad i \neq j.
\]

Conversely, Theorem 3 implies that

\[
\tau\left( \left( \frac{q + 1}{2} \right)(l - 1), q, \left( \left( \frac{q + 1}{2} \right) - 1 \right)(l - 1) - 1 \right) \leq q + 1
\]

if \( \tau(\cdot, q, \cdot) \neq 0 \pmod{q} \).

Now, if \( \tau(\cdot, q, \cdot) \) were congruent to 0 (mod \( q \)), then Theorem 3 would imply that

\[
\tau\left( \left( \frac{q + 1}{2} \right)(l - 1), q, \left( \left( \frac{q + 1}{2} \right) - 1 \right)(l - 1) - 1 \right) \leq q \left[ \frac{q + 2}{q} \right] = q.
\]

This last inequality shows that \( \tau(\cdot, q, \cdot) \) is strictly less than 2\( q \); but the example given earlier shows that it is strictly greater than \( q \). Hence \( \tau(\cdot, q, \cdot) \neq 0 \pmod{q} \).
The following two examples illustrate the method described by the previous theorem.

**Example 1.** The case $q = 3$ yields the code

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 2 & 3 & 3 & 3 \\
3 & 3 & 3 & 1 & 2 & 3 \\
\end{array}
\]

where each digit represents a block of length $l - 1$. Thus, for example, when $q = 3$ and $l = 3$, the following $\{12, 3, 9\}$ code is obtained:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

**Example 2.** When $q = 4$, we have the code

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

The next result describes an interesting pattern of the function $\tau$.

**Theorem 6.**

\[
\tau(n, 2, n - l) = \begin{cases} 
8 & \text{for } 3 \leq l \leq 4, \\
6 & \text{for } l = 5, \\
4 & \text{for } l \geq 6,
\end{cases}
\]

provided that $3l - 6 \leq n \leq 3l - 5$.

**Proof.** The cases $3 \leq l \leq 4$ and $l = 5$ are well known [12]. Thus, one has to consider only the case $l \geq 6$. It will be shown first that $\tau(n, 2, n - l) \geq 4$. Without loss of generality one can assume that $n = 3l - 6$. Then the example given in (9) yields this fact, provided that the length of the block is chosen to be $l - 2$. 
On the other hand, (4) implies that

$$\tau(3l - 6, 2, 2l - 6) \leq 2 \left[ \frac{2l - 4}{4l - 9 - (3l - 6)} \right] = 4 \quad \text{if} \quad l \geq 6.$$  

The following two theorems are generalizations of the previous result.

**Theorem 7.** $\tau(n, 2, n - l) = 4$ provided that $3l - 2s \leq n \leq 3l - 2s + 1$, with $l \geq 4s - 6$, $l \geq s + 1$, $s \geq 3$.

**Proof.** It will be shown first that $\tau \geq 4$. Three cases will be considered.

**Case I:** $s = 3c$. Assume that $n = 3l - 2s$. The example given in (9) yields this fact, provided that the length of each block is chosen to be $l - 2c$. Note that

$$d_H(\xi_i, \xi_j) = 2l - 4c > 2l - 6c + 1 \quad \text{for} \quad i \neq j.$$  

**Case II:** $s = 3c + 1$. Let $n = 3l - 2s$. The example given in Case I shows this fact provided that the length of each of the first two blocks is chosen to be $l - 2c - 1$ and the length of the third block is chosen to be $l - 2c$. Notice that $d_H(\xi_i, \xi_j) \geq 2l - 4c - 3 \geq 2l - 6c - 1$.

**Case III:** $s = 3c - 2$. Let $n = 3l - 2s$. Again, in this case, the example given in (9) will be considered, with the following modifications. Each of the first two blocks has length $l - 2c - 1$, and the length of the third block is $l - 2c - 2$. Therefore

$$d_H(\xi_i, \xi_j) \geq 2l - 4c - 3 \geq 2l - 6c - 3,$$

from which the statement follows.

The converse statement can be established with the aid of the Plotkin inequality (4). Indeed,

$$\tau(3l - 2s, 2, 2l - 2s) \leq 2 \left[ \frac{2(l - s + 1)}{l - 2s + 3} \right] = 4,$$

since $2l - 2s + 2 < 3l - 6s + 9$ when $4s - 7 < l$. 

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THEOREM 8. Let \( l = 4s - (7 + x) \), where \( x \) is a nonnegative integer with \( s > 2 + x/2 \). Let \( n = 3l - 2s \). Then

\[
\tau(n, 2, n - l) = 6 \quad \text{when} \quad s > 2 + x, \quad (10)
\]

while

\[
\tau(n, 2, n - l) = 8 \quad \text{when} \quad s = 2 + x, \quad s \geq 3. \quad (11)
\]

Proof. Another application of the Plotkin inequality (4) shows that \( \tau(n, 2, n - l) \leq 6 \) when \( s > 2 + x \), and \( \tau(n, 2, n - l) \leq 8 \) when \( s = 2 + x \), \( s \geq 3 \).

To prove the converse, assume first that \( s > 2 + x \). To simplify the exposition, a \( \{10, 2, 5\} \) code \( M \) consisting of six codewords will be considered first.

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\
\end{array} \quad (12)
\]

This example shows, using the already proven part of this theorem, that \( \tau(9, 2, 4) = A(9, 5) = 6 \). Next, consider the special case \( l = 4s - 7 \), \( s \geq 4 \), \( x = 0 \). Notice that the case \( s = 3 \) has just been discussed. Hence, \( n = 10s - 21 \) and \( d = n - l + 1 = 6s - 13 \). Consider the example discussed in (12) with the following modifications. Let the length of the block of the first column be \( s - 3 \), and let \( s - 2 \) be the length of the block of each of the following nine columns. Now, (12) shows that \( d_H(\xi_i, \xi_j) \geq 6s - 13 \) for \( i \neq j \), which proves (10) when \( l = 4s - 7 \). Basically, the same example can be considered when \( l = 4s - (7 + 10t) \), \( t \) is a nonnegative integer, and \( s = 3 + 10t \). In that case, (12) will have the following modifications. The length of the block of the first column is \( s - (3 + 3t) \), and the length of each of the remaining nine blocks is \( s - (2 + 3t) \). Here, \( n = 10s - 21 - 30t \) and \( d = 6s - 13 - 20t \). Now

\[
d_H(\xi_i, \xi_j) \geq 6s - 18t - 13 \quad \text{for} \quad 1 \leq i < j \leq 6.
\]

This proves (10) when \( l = 4s - (7 + 10t) \).

The next case is \( l = 4s - 8 \), \( s \geq 4 \). Then \( n = 10s - 24 \), \( d = 6s - 15 \). In that case, consider (12) with the following modifications. Let the length of
each of the first four columns be $s - 3$, and let the length of each of the remaining six columns be $s - 2$. Since any two codewords agree at least once in the positions of the first four columns, we see that $d_H(\xi_i, \xi_j) \geq 6s - 15$ for $i \neq j$. This proves (10) when $l = 4s - 8$.

Notice that the case $l = 4s - 8$ when $s = 3$ yields $r = 8$. This fact has been presented in [12]. This case can be illustrated by the following example:

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 1 & 1 & 2 \\
1 & 2 & 2 & 2 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 \\
\end{array}
$$

The case $l = 4s - (8 + 10t)$, where $t$ is a nonnegative integer, can be handled in a similar fashion. There are eight more cases to consider, and only those cases will be discussed where there are some additional complications.

The first such case is when $l = 4s - 9, s \geq 5, n = 10s - 27, d = 6s - 17$. Consider (12) with the following modifications. Let the length of each of the first seven columns be $s - 3$, and let the length of each of the remaining three columns be $s - 2$. Notice that in (12) any two codewords are different in at least one of the last three positions. Hence

$$d_H(\xi_i, \xi_j) \geq 6s - 17 \quad \text{for} \quad 1 \leq i < j \leq 6.$$

Finally, consider the case $l = 4s - 12, s \geq 8$. Here, $n = 10s - 36, d = 6s - 23$. Once again, consider (12) with the following modifications. Let the length of each of the first six columns be $s - 4$, and let the length of each of the remaining four columns be $s - 3$. By repeating the ideas used in the proof of the case $l = 4s - 8$, one can conclude that

$$d_H(\xi_i, \xi_j) \geq 6s - 23 \quad \text{for} \quad 1 \leq i < j \leq 6.$$

We now discuss the case $s = 2 + x$. Let $n = 3l - 2s$, where $l = 4s - (7 + x), d = n - l + 1$, and $s \geq 3$. Hence, $n = 7s - 15$ and $d = 4s - 9$. It will be shown that there exists a $\{7s - 15, 2, 4s - 10\}$ code $M$ having eight codewords. Indeed, consider a code such that the first codeword, $\xi_1$ consists of $7s - 15$ ones; $\xi_2$ consists of $3s - 6$ ones followed by $4s - 9$ twos; $\xi_3$ consists of $s - 2$ ones, then $2s - 4$ twos, then $2s - 4$ ones, followed by
2s - 5 twos; \( \xi_4 \) consists of \( s - 2 \) ones, and \( 4s - 8 \) twos, then \( 2s - 5 \) ones; 
\( \xi_5 \) consists of \( s - 2 \) twos, \( s - 2 \) ones, \( 2s - 4 \) twos, and \( 2s - 4 \) ones, then 
\( s - 3 \) twos; \( \xi_6 \) consists of \( s - 2 \) twos, \( s - 2 \) ones, \( s - 2 \) twos, and \( 2s - 4 \) twos, then \( s - 3 \) ones; \( \xi_7 \) consists of \( 2s - 4 \) twos and \( s - 2 \) ones, followed by an alternating sequence of twos and ones, starting with a one; and finally, \( \xi_8 \) consists of \( 2s - 4 \) twos and \( s - 2 \) ones, followed by an alternating sequence of twos and ones, starting with a two.

It is easy to see that \( M \) is a \( \{7s - 15, 2, 4s - 10\} \) code. We illustrate that with the following example, in the case when \( s = 5 \):

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\
\end{array}
\]

\( \blacksquare \) (14)

**Theorem 9.** Let \( l = 4s - 7 - x \), where \( s = 1 + x \), and let \( n = 3l - 2s \). Then \( \tau(n, 2, n - l) = 8 \) when \( s \geq 6 \).

**Proof.** The case \( s = 4 \) is a trivial one, and it is well known that \( \tau(n, 2, n - l) = 10 \) when \( s = 5 \). Plotkin's inequality shows that the condition \( s \geq 6 \) implies that \( \tau(n, 2, n - l) < 8 \). Hence, it has to be shown only that \( \tau(n, 2, n - l) \geq 8 \). This fact can be shown by a construction which is similar to (14). In this case \( n = 7s - 18 \) and \( d = 4s - 11 \). Let \( C \) be a code consisting of the following eight codewords: \( \xi_1 \) consists of \( 7s - 18 \) ones; \( \xi_2 \) consists of \( 3s - 7 \) ones, followed by \( 4s - 11 \) twos; \( \xi_3 \) consists of \( s - 2 \) ones, then \( 2s - 5 \) twos, then \( 2s - 5 \) ones, followed by \( 2s - 6 \) twos; \( \xi_4 \) consists of \( s - 2 \) ones, then \( 2s - 5 \) twos, then \( 2s - 5 \) twos, followed by \( 2s - 6 \) ones; \( \xi_5 \) consists of \( s - 2 \) twos and \( s - 2 \) ones, then \( s - 3 \) twos, then \( s - 2 \) twos, then \( s - 3 \) ones, then \( s - 3 \) ones, followed by \( s - 3 \) twos; \( \xi_6 \) consists of \( s - 2 \) twos, \( s - 2 \) ones, and \( s - 3 \) twos, then \( s - 2 \) ones, then \( s - 3 \) twos, then \( s - 3 \) ones, followed by \( s - 3 \) ones; \( \xi_7 \) consists of \( 2s - 4 \) twos and \( s - 3 \) ones, followed by an alternating sequence of twos and ones, starting with a one; and finally, \( \xi_8 \) consists of \( 2s - 4 \) twos and \( s - 3 \) ones, followed by an alternating sequence of twos and ones starting with a two. It is easy to see that \( C \) is a \( \{7s - 18, 2, 4s - 12\} \) code. Hence, the proof is complete. \( \blacksquare \)
The previous result can be extended in the following way:

**Theorem 10.** Let \( l = 4s - (7 + x) \), where \( s = x - r \) and where \( r \) is a nonnegative integer, and let \( n = 3l - 2s \). Then \( \tau(n, 2, n - l) = 8 \) when \( s \geq 9 + 3r \).

**Proof.** In this case \( n = 3l - 2s = 7s - (21 + 3r) \), and \( d = 4s - (13 + 2r) \). Plotkin's inequality implies that it need only be shown that \( \tau(n, 2, n - l) \geq 8 \). A code \( C \) will be constructed, which will imply this inequality. In this construction the inequality \( s \geq 9 + 3r \) will not be used. Hence, this example will show that 8 is a lower bound for the function \( \tau \), provided that the remaining conditions of Theorem 10 are satisfied. To simplify our proof, we will introduce the following notation exclusively for this proof. Let \( z = 5 + r - \lfloor r/3 \rfloor \), \( u = \lfloor r/3 \rfloor \), and let \( v = \lfloor z/2 \rfloor \). The codeword \( \xi_1 \) of \( C \) consists of \( 7s - (21 + 3r) \) ones; \( \xi_2 \) consists of \( 3s - (8 + r) \) ones, followed by \( 4s - (13 + 2r) \) twos; \( \xi_3 \) consists of \( s - (3 + u) \) ones, followed by \( 2s - z \) twos, then \( 2s - (7 + r) \) twos, then \( 2s - (6 + r) \) ones; \( \xi_4 \) consists of \( s - (3 + u) \) ones, then \( 2s - z \) twos, then \( 2s - (7 + r) \) one, then \( 2s - (6 + r) \) twos; \( \xi_5 \) consists of \( s - (3 + u) \) twos, \( s - v \) ones, then \( s - (z - v) \) twos, then \( s - \lfloor (7 + r)/2 \rfloor \) twos, then \( s - \{7 + r - \lfloor (7 + r)/2 \rfloor \} \) ones, then \( s - \lfloor (6 + r)/2 \rfloor \) one, then \( s - \{6 + r - \lfloor (6 + r)/2 \rfloor \} \) twos; \( \xi_6 \) consists of \( s - (3 + u) \) ones, then \( s - (z - v) \) twos, then \( s - \lfloor (7 + r)/2 \rfloor \) ones, then \( s - \{7 + r - \lfloor (7 + r)/2 \rfloor \} \) twos, then \( s - \lfloor (6 + r)/2 \rfloor \) twos, then \( s - \{6 + r - \lfloor (6 + r)/2 \rfloor \} \) ones; \( \xi_7 \) consists of \( s - (3 + u) \) twos, then \( s - v \) twos, then \( s - \lfloor (7 + r)/2 \rfloor \) one, then an alternating sequence of ones and twos starting with a one; \( \xi_8 \) consists of \( s - (3 + u) \) twos, then \( s - v \) twos, then \( s - (z - v) \) ones, then an alternating sequence of ones and twos starting with a two.

Then with the aid of a lengthy computation one can see that \( C \) is a \( \{7s - (21 - 3r), 2, 4s - (12 + 2r)\} \) code. 

The next result is an extension of the second part of Theorem 5 when \( q \geq 3 \).

**Theorem 11.** Let

\[
n = \left( \frac{q + 1}{2} \right)(l - 1) - \kappa,
\]

where \( \kappa \) is a nonnegative integer, \( q \geq 3 \). Then \( \tau(n, q, n - l) = q + 1 \) when

\[
l > 1 + \frac{6\kappa}{q(q - 1)}.
\]
Proof. A simple computation gives \( dq > n(q - 1) \). To show that
\[
\tau(n, q, n - l) \leq q + 1,
\]
consider first the case when \( \tau(n, q, n - l) \neq 0 \pmod{q} \). It is enough to show that
\[
\frac{dq}{dq - n(q - 1)} < q + 3,
\]
or equivalently that
\[
(n - d)(q^2 + 2q) < 3n.
\]
Notice that
\[
n - d = l - 1 \quad \text{and} \quad n = \binom{q + 1}{2}(l - 1) - \kappa.
\]
Therefore, it has to be shown only that \( 3\kappa < q(q - 1)(l - 1)/2 \). But this inequality holds obviously in view of (15). Thus we have shown that \( d/[dq - n(q - 1)] < 1 + 3/q \leq 2 \). In other words, if \( \tau(n, q, n - l) \equiv 0 \pmod{q} \), then \( \tau(n, q, n - l) = q \). Hence, in both cases \( \tau(n, q, n - l) \leq q + 1 \).

In order to show the converse inequality, notice that in the proof of Theorem 5 an \( \{n_1, q, n_1 - l\} \) code was constructed consisting of \( q + 1 \) codewords, with
\[
n_1 = \binom{k + 1}{2}(l - 1).
\]
Now, if the last \( \kappa \) columns of this code are deleted, then we obtain a code of \( \{n, q, n - l\} \) type.

IV. SOME PROPERTIES OF CMDS CODES

In [3] a complete description of higher dimensional stochastic \((0, 1)\) matrices has been obtained in the case when \( 2 \leq q \leq 6 \), using several properties of orthogonal arrays. The same description can be obtained by using properties of CMDS codes. In this section we describe some of these
properties of CMDS codes. For some further properties, we refer to [3], where those results are presented in a different, but equivalent form. Of course, the results which will be discussed in this section will be also useful for obtaining a complete description of higher dimensional stochastic \((0, 1)\) matrices in the case when \(q \geq 7\). For reference the following simple result is stated.

**Lemma 2.** If \(M\) is an \(\{n, q, r\}\) code and if \(\xi_1\) and \(\xi_2\) are different codewords of \(M\), then they cannot agree at \(n - 1\) coordinate places.

The obvious proof follows from the fact that \(d_H(\xi_1, \xi_2) \geq r + 1\).

The next result is an immediate consequence of Lemma 2.

**Lemma 3.** If \(M\) is a CMDS \(\{n, q, r\}\) code, then at each coordinate place each member of the alphabet occurs exactly \(q^{n-r-1}\) times.

From a given CMDS code one can obtain new CMDS codes with different parameters as follows.

**Lemma 4.** If \(M\) is a CMDS code with parameters \(\{n, q, r\}\), and if \(n > r + 2\), then there is also a CMDS code with parameters \(\{n - 1, q, r\}\).

**Proof.** Consider \(M\) in its matrix form. Construct a new code \(A\) consisting the first \(n - 1\) coordinates of those codewords of \(M\) whose last coordinate is one. Then, by Lemma 3, \(A\) has \(q^{n-r-1}\) members, and the Hamming distance between any two different codewords of \(A\) is at least \(r + 1\). Finally, the cardinality of this code implies that it is a CMDS code.

**Lemma 5.** Let \(n > 3\) and \(r > 2\). If \(M\) is a CMDS code with parameters \(\{n, q, r\}\), then there is also a CMDS code with parameters \(\{n - 1, q, r - 1\}\).

**Proof.** Let \(M\) be in its matrix form. By deleting the last column of \(M\) one obtains a new code \(A\) consisting of all the codewords of \(M\) without the last coordinates. Obviously, the Hamming distance between any two codewords of \(A\) is at least \(r\), and the cardinality of \(A\) is \(q^{n-r} = q^{(n-1)-(r-1)}\). This shows that \(A\) is a CMDS code.

The method used in the proof of the previous lemma yields the following inequality for the function \(\tau\).

**Lemma 6.** Let \(n > 3\) and \(r > 2\). Then

\[
\tau(n, q, r) \leq \tau(n - 1, q, r - 1).
\]  

(17)

We shall need the following result. It can be found in [14].
THEOREM 12. Let \( n \geq 4, r \geq 2 \). There is no CMDS \( \{n, q, r\} \) code when \( n - r \geq 2 \) and \( r \geq q \).

It will be useful to introduce some new functions.

Assume that \( M \) is a CMDS code with parameters \( \{n, q, r\} \), and \( \xi \) is a codeword of \( M \),

\[
\xi = (a_1, a_2, \ldots, a_r, a_{r+1}, a_{r+2}, \ldots, a_n).
\]

Then we define the following functions on the last \( n - r \) coordinates of the codewords:

\[
\kappa(a_{r+1}, a_{r+2}, \ldots, a_n) = (a_1, \ldots, a_r),
\]

and

\[
\kappa_i(a_{r+1}, a_{r+2}, \ldots, a_n) = a_i \quad \text{for} \quad 1 \leq i \leq r.
\]

The next result deals with the case \( n - r = 2 \), and it can be found in [7].

THEOREM 13. Assume that \( r \geq 2 \). Let \( n \geq 4 \), and let \( n - r = 2 \). Then there exists an \( \{n, q, r\} \) CMDS code if and only if there exist \( r \) mutually orthogonal latin squares of order \( q \).

The next result deals with the case when \( n - r \geq 3 \), and it was obtained by Singleton [14]. Since our proof is essentially different from the one given in [14], it will be given here.

THEOREM 14. Let \( r \geq 2 \). There is no \( \{n, q, r\} \) CMDS code when

\[
q - 1 + r < n. \quad (18)
\]

Proof. Let \( M \) be a CMDS code with parameters \( \{n, q, r\} \). Let \( \xi \) be the codeword of \( M \) each of whose last \( n - r \) coordinates is equal to one. Let \( \kappa_1(1, 1, \ldots, 1) = \alpha, \kappa_2(1, 1, \ldots, 1) = \beta \) where \( 1 \leq \alpha \leq q \) and \( 1 \leq \beta \leq q \).

As was discussed earlier, any codeword of \( M \) is uniquely determined and can be uniquely described (in \( M \)) by its last \( n - r \) coordinates. Let \( M_i \) be the subset of \( M \) consisting of the following \( (q - 1)(n - r) \) codewords, which
are here described only by their last \( n - r \) coordinates:

\[
\begin{array}{cccc}
211...1 & 121...1 & \cdots & 111...2 \\
311...1 & 131...1 & \cdots & 111...3 \\
\vdots & \vdots & \ddots & \vdots \\
q11...1 & 1q1...1 & \cdots & 111...q
\end{array}
\]

(19)

Let \( a_{r+1}, \ldots, a_n \) and \( b_{r+1}, \ldots, b_n \) be two different members of \( M_1 \), once again described by their last \( n - r \) coordinates. One can see easily that

\[ d_H((a_{r+1}, \ldots, a_n), (b_{r+1}, \ldots, b_n)) < 2. \]

Thus

\[ d_H((a_{r+1}, \ldots, a_n), (1, \ldots, 1)) = 1, \]

it follows that \( \kappa_1(a_{r+1}, \ldots, a_n) \neq \kappa_1(h_{r+1}, \ldots, h_n) \)

and also that \( \kappa_2(a_{r+1}, \ldots, a_n) \neq \beta \).

Since \( d_H((a_{r+1}, \ldots, a_n), (1, \ldots, 1)) = 1, \) it follows that \( \kappa_1(a_{r+1}, \ldots, a_n) \neq \alpha \)

and also that \( \kappa_2(a_{r+1}, \ldots, a_n) \neq \beta \). Now, there are at most \((q - 1)^2\) codewords of \( M \) such that the first coordinate of the codeword is not \( \alpha \) and the second coordinate of the codeword is not \( \beta \), and the codeword differs at exactly one of the last \( n - r \) coordinates from \( \xi \). Since each element of the set \( M_1 \) has this property, it follows that the inequality \((q - 1)^2 \geq (q - 1)(n - r)\) must hold. Hence the assertion follows.

**Remark.** Since there is an MDS code with parameters \( (5, 4, 2) \), it follows that the inequality (18) or, equivalently, Theorem 14 is the best possible.

The following result can be found in [4].

**Theorem 15.** If \( C \) is a CMDS code with parameters \( \{q + 2, q, q - 1\} \), then \( q \) is divisible by 4, and it is the order of a finite projective plane.

As a corollary of the previous result, we obtain the following result of Casse [6].

**Theorem 16.** A linear MDS code with parameters \( \{q + 2, q, q - 1\} \) exists if and only if \( q = 2^m \) for some positive integer \( m \).

**Proof.** Theorem 15 implies that \( q \) must be even, and since \( q \) also must be the order of a Galois field, it follows that \( q \) must be a power of 2. On the
other hand, there is a well-known method to generate linear codes with these parameters [12]. Our method will be slightly different, so it is included here.

Assume that \( q = 2^m \) is the order of a Galois field \( F \), where \( m \) is a positive integer. It will be assumed that the elements of \( F \) are labeled from 0 to \( q - 1 \), with the number 0 to correspond to the zero element of \( F \). The linear code \( C \) will be defined as the set of all \( q + 2 \)-tuples where the first three entries are the coefficients of a polynomial \( p \) over \( F \) of degree at most two (arranged always in the same order) and the remaining \( q - 1 \) entries consist of this \( p \) evaluated at the elements labeled from 1 to \( q - 1 \) in this order. The cardinality of \( C \) is \( q^3 \). By separating cases, one can see that the Hamming distance between any two different codewords is at least \( q \), or, equivalently, they can agree at no more than two coordinate places. Here, we discuss only one case. Assume that we have two different codewords of \( C \), say \( \xi_1 \) generated by \( p \), and \( \xi_2 \) generated by \( q \), with the property, that \( p \) and \( q \) have only one coefficient in common: that of the first order term. Then \( p \) and \( q \) cannot agree more at than one position among the elements numbered from 1 to \( q - 1 \), since in the given Galois field every element is its own additive inverse.

VI. INEQUALITIES AND STOCHASTIC MATRICES

In this section some new inequalities will be derived about the function \( \tau \) in the case when \( q = 2 \) and also in the case when \( q \geq 3 \). These inequalities will yield an alternative method to solve the problem of the existence of higher dimensional stochastic matrices in the case when \( q = 2 \) and when \( q = 3 \), and it will yield a partial solution in the rest of the cases. The equivalent problem about CMDS codes in the case when \( q = 2 \) has a few different solutions; a solution of it, which was given in [3], has been referred to already earlier in this paper. The solution presented in [1] uses results from matroid theory. In this paper we present a further alternative approach when \( q = 2 \) and \( q = 3 \) by giving an upper bound for the function \( \tau \). To our knowledge, these inequalities yield the best upper bounds at the present time when \( q \geq 3 \).

It is known that [1, 12]

\[
A(n, d) \leq 2A(n - 1, d) \quad \text{if} \quad d \leq n - 1.
\]

This last inequality can be rewritten as follows:

\[
A(n, d) = \tau(n, 2, d - 1) \leq 2A(n - 1, d) = 2\tau(n - 1, 2, d - 1),
\]
or equivalently

\[ \tau(n, 2, n - l) \leq 2\tau(n - 1, 2, (n - 1) - (l - 1)) \quad \text{if} \quad l \geq 2. \quad (20) \]

In view of (20) and Theorem 5, we have that

**Theorem 17.** \( \tau(n, 2, n - l) \leq 2^s \quad \text{if} \quad n = 3l - 2s, \quad l \geq 2, \quad 1 \leq s \leq l. \)

A \((n, 2, n - l)\) code is CMDS if and only if \( \tau(n, 2, n - l) = 2^l \). Thus it follows from the previous result, when \( l \geq 2 \), that \( \tau(n, 2, n - l) \) can be \( 2^l \) only when \( 3l - 2l \leq n \leq l + 1 \). On the other hand, \( \tau(n, 2, 0) = 2^n \), and it is well known that \( \tau(n, 2, 1) = 2^{n-1} \). The case \( l = 1 \) yields \( \tau(n, 2, n - 1) = 2 \).

Thus, we have shown that when \( q = 2 \), there are only three different types of CMDS codes, namely the codes \( \{n, 2, 0\}, \{n, 2, 1\}, \{n, 2, n - 1\} \). Since the code \( \{n, 2, 0\} \) does not correspond to a higher dimensional stochastic matrix, the following result yields a solution of the problem of Jurkat and Ryser in the case when \( q = 2 \).

**Theorem 18.** The classes \( S(n, 2, 1) \) and \( S(n, 2, n - 1) \) for all \( n \geq 2 \), and only those, have \((0, 1)\) extremal elements when \( q = 2 \).

Using Theorem 6, the following improvement of Theorem 17 can be obtained.

**Theorem 19.** Let \( n = 3l - 2s \), with \( l \geq 2 \), \( 1 \leq s \leq l \). Then

\[
\tau(n, 2, n - l) \leq \begin{cases} 
2^s & \text{for} \quad s \leq l \leq s + 1, \\
32^{s-2} & \text{for} \quad l = s + 2, \\
2^{s-1} & \text{for} \quad l \geq s + 3.
\end{cases}
\]

This last inequality can be improved considerably by using the results proven in the first part of this paper. In this paper, we just formulate one such improvement.

**Theorem 20.** Let \( n = 3l - 2s \), with \( 4s - 3t \leq l \leq 4s - 3t + 2 \), where \( 2 \leq t \leq s - 1 \), \( s \geq 3 \). Then \( \tau(n, 2, n - l) \leq 2^t \).

Based upon Theorem 6, one would expect the function \( \tau \) to be monotone in the following sense:

\[ \tau(n_1, 2, n_1 - l_1) \geq \tau(n_2, 2, n_2 - l_2) \]
when \( n_1 = 3l_1 - m \) and \( n_2 = 3l_2 - m \), where \( 2 \leq l_1 \leq l_2 \), and where \( m \) is a positive integer. However, this is not the case. One can give the following counterexample:

\[
A(9, 4) = 20 = \tau(9, 2, 3) < A(12, 6) = 24 = \tau(12, 2, 5).
\]

Next, inequalities will be derived for the function \( \tau \) in the case when \( q \geq 3 \). To simplify the discussion, the case \( q = 3 \) will be discussed first. It is easy to see that

\[
\tau(n, 3, n - l) \leq 3\tau(n - 1, 3, n - l)
\]

\[
= 3\tau(n - 1, 3, (n - 1) - (l - 1)) \quad \text{if} \quad l \geq 2. \tag{21}
\]

In view of (21), Theorem 4, and Theorem 5, the following inequalities can be derived.

THEOREM 21. \( \tau(n, 3, n - l) \leq 3^s \) if \( n = 6l - 5s, \quad l \geq 2, \quad 1 \leq s \leq l; \) and

\[
\tau(n, 3, n - l) \leq 43^{s-1} \quad \text{if} \quad n = 6l - 5s - 1, \quad l \geq 2, \quad 1 \leq s < l.
\]

Using Theorem 11, the previous inequality can be improved in the following fashion:

\[
\tau(n, 3, n - l) \leq 43^{s-1}, \quad \text{if} \quad n = 6l - 5s - 1 - t, \quad l \geq 2 + t,
\]

\[
1 \leq s \leq s + t \leq l - 1. \tag{22}
\]

This last inequality, together with Theorem 12, implies that when \( q = 3 \), the only nontrivial CMDS code (i.e. code with dimension different from 1 and from \( n - 1 \)) is the code \( \{4, 3, 2\} \).

The case \( q \geq 4 \) can be treated in an entirely similar fashion.

THEOREM 22. \( \tau(n, q, n - l) \leq q^s \) if

\[
n = \left(\frac{q + 1}{2}\right)l - \left\{\left(\frac{q + 1}{2}\right) - 1\right\}s, \quad l \geq 2, \quad 1 \leq s \leq l,
\]
and \( \tau(n, q, n - l) \leq (q + 1)q^{s-1} \) if

\[
n = \left( \binom{q+1}{2} \right) l - \left( \left( \binom{q+1}{2} - 1 \right) s - 1, \quad 1 \leq s < l.
\]

Similarly, the inequality (22) can be generalized the following way:

\[
\tau(n, q, n - l) \leq (q + 1)q^{s-1} \quad \text{if}
\]

\[
n = \left( \binom{q+1}{2} \right) l - \left( \left( \binom{q+1}{2} - 1 \right) s - 1 - t,
\]

where

\[
l \geq 2 + \frac{6t}{q^2 - q}, \quad 1 \leq s \leq s + \left[ \frac{6t}{q(q-1)} \right] \leq l - 1.
\]

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