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On Lipschitz-type maximal functions and their smoothness spaces

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ABSTRACT

In a recent monograph (cf. No. 293 of the Memoirs of the Amer. Math. Soc. 47 (1984)) DeVore and Sharpley study maximal functions of integral type and their related smoothness spaces. One of their central results gives an embedding theorem for the smoothness spaces in terms of Besov spaces. In this paper we consider the related problem when the Besov spaces are substituted by the so-called A-spaces introduced by Popov (take the τ-modulus instead of the ω-modulus). We will define Lipschitz-type maximal functions whose smoothness spaces satisfy a corresponding embedding theorem in terms of A-spaces. By well-known results new insights can only be expected for functions satisfying low order smoothness conditions and, therefore, only function spaces generated by first order differences are considered.

1. INTRODUCTION

To get an impression of the problem considered in this paper we first of all state some well-known results concerning the ω- and τ-moduli and their related function spaces. For sake of brevity we restrict ourselves to the onedimensional trigonometric case.

Let L_p, $1 \le p < \infty$, be the space of all 2π-periodic functions f with $|f|^p$ Lesbesgue integrable on $[0, 2\pi]$ and C the space of all 2π-periodic continuous functions. The spaces may be normed in the usual way by

$$\|f\|_p := \{ \int_0^{2\pi} |f(x)|^p dx\}^{1/p}, \quad f \in L_p,\ 1 \le p < \infty,$$

$$\|f\|_\infty := \max \{|f(x)| : x \in [0, 2\pi]\}, \quad f \in C.$$

Finally, let AC be the space of all 2π-periodic absolutely continuous functions on $\mathbb{R}$.

For $f: \mathbb{R} \to \mathbb{R}$ the r-th Riemann difference is defined by

$$\Delta_h^r f(x) := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh), \quad r \in \mathbb{N},\ h>0,\ x \in \mathbb{R},$$

and the r-th local ω-modulus by

$$\omega_r(f, x, \delta) := \sup \left\{ |\Delta_h^r f(t)| : t, t+rh \in \left[x - \frac{r\delta}{2}, x + \frac{r\delta}{2} \right] \right\}, \quad \delta > 0.$$

Now, for $f \in L_p$, $1 \leq p < \infty$, or $f \in C$ the r-th ω-modulus of smoothness is given by

$$\omega_{r,p}(f, \delta) := \sup \{ \|\Delta_h^r f\|_p : 0 < h \leq \delta \}, \quad \delta > 0,$$

and the r-th τ-modulus by

$$\tau_{r,p}(f, \delta) := \|\omega_r(f, \cdot, \delta)\|_p, \quad \delta > 0.$$

A detailed discussion of these moduli, especially the last one, may be found in [9].

Associated with these moduli of smoothness we have the Besov spaces $B_{p,q}^{\theta,r}$, $1 \leq p, q \leq \infty$, $r \in \mathbb{N}$, $0 < \theta < r$, which are defined as the collection of all functions $f \in L_p$, $1 \leq p < \infty$, or $f \in C$ for which the integral

$$\int_0^\infty (t^{-\theta} \omega_{r,p}(f, t))^q \frac{dt}{t}, \quad 1 \leq q < \infty,$$

or the supremum

$$\sup_{0<t} (t^{-\theta} \omega_{r,p}(f, t)), \quad q = \infty,$$

respectively, are finite (cf. [2], pp. 228, 229). Moreover, we have the A-spaces $A_{p,q}^{\theta,r}$, $1 \leq p, q \leq \infty$, $r \in \mathbb{N}$, $0 < \theta < r$, which are defined as those functions $f \in L_p$, $1 \leq p < \infty$, or $f \in C$ for which

$$\int_0^\infty (t^{-\theta} \tau_{r,p}(f, t))^q \frac{dt}{t}, \quad 1 \leq q < \infty,$$

or

$$\sup_{0<t} (t^{-\theta} \tau_{r,p}(f, t)), \quad q = \infty,$$

respectively, are finite (cf. Popov [7]). In general, we have $A_{p,q}^{\theta,r} \subset B_{p,q}^{\theta,r}$ but for $p = \infty$ or $\theta p > 1$, $1 \leq p < \infty$, the A-spaces coincide with the Besov spaces (cf. [7], [5]). By means of well-known reduction theorems concerning the smoothness characterization of functions belonging to Besov spaces (cf. [2], pp. 228, 229) we therefore can expect new specific results only for integral A-spaces which are generated by differences of first order (the crucial case $\theta = p = 1$, where second order differences may be involved, will not be considered in detail, here). So, from now on we put $1 \leq p < \infty$ and $r = 1$.

In [4] DeVore and Sharpley study integral maximal functions of type f_θ^*, $0<\theta<1$,

$$f_\theta^*(x) := \sup_{t\neq x} \frac{1}{|x-t|^{1+\theta}} \int_{\min\{x,t\}}^{\max\{x,t\}} |f(\xi)-f_{x,t}|d\xi,$$

$$f_{x,t} := \frac{1}{|x-t|} \int_{\min\{x,t\}}^{\max\{x,t\}} f(\tau)d\tau,$$

and the related smoothness spaces $C_p^\theta := \{f\in L_p : f_\theta^* \in L_p\}$ (see also Calderón/Scott [3], where maximal functions of this type seem to appear for the first time). Among various other results they prove that C_p^θ are Banach spaces with respect to their corresponding norms

$$|||f|||_{p,\theta} := \|f\|_p + \|f_\theta^*\|_p$$

(cf. [4], p. 37, Lemma 6.1) and that C_p^θ are embedded by the Besov spaces in the form

$$B_{p,p}^{\theta,1} \subset C_p^\theta \subset B_{p,\infty}^{\theta,1}$$

with nontrivial inclusions (cf. [4], p. 48, Theorem 7.1).

Now, it is convenient to consider the corresponding situation in case of the A-spaces, i.e., we want to define maximal functions whose related smoothness spaces are embedded by $A_{p,p}^{\theta,1}$ and $A_{p,\infty}^{\theta,1}$. It is the aim of this paper to show that the Lipschitz-type maximal functions $f_\theta^\sim$, $0<\theta\leq 1$,

$$(1.1)\qquad f_\theta^\sim(x) := \sup_{t\neq x} \frac{|f(x)-f(t)|}{|x-t|^\theta},$$

f bounded and measurable, are appropriate. Let us note that $f_\theta^* \leq f_\theta^\sim$ and that $f_\theta^\sim$ may be interpreted as the limiting case ($q=\infty$) of the maximal functions $N_q^\theta(f)$ considered in Chapter 5 of [4]. Moreover, we mention that the essential difference between these *integral free* maximal functions and the *integral* maximal functions f_θ^* consists in the fact that removable points of discontinuity of f are recognized by $f_\theta^\sim$ in form of a singularity of order θ while they are ignored by f_θ^*. This behaviour corresponds exactly to the different sensitivity of the τ-modulus resp. ω-modulus in case of pointwise changes of f (cf. [5]).

Now, coming back to the functions $f_\theta^\sim$, let us note that they are measurable (see Theorem 1) and, therefore, their corresponding smoothness spaces D_p^θ may be defined to consist of those functions $f\in L_p$, $1\leq p<\infty$, which are bounded and satisfy

$$(1.2)\qquad \|f_\theta^\sim\|_p = \{ \int_0^{2\pi} (f_\theta^\sim(t))^p dt\}^{1/p} < \infty.$$

It should be noticed that D_p^θ are normed linear subspaces of L_p with respect to the norms

$$(1.3)\qquad \|f\|_{p,\theta} := \|f\|_p + \|f_\theta^\sim\|_p$$

and that – in contrast to L_p or C_p^θ – two functions $f_1, f_2 \in D_p^\theta$ satisfy $\|f_1 - f_2\|_{p,\theta} = 0$ if and only if $f_1(x) = f_2(x)$ for all $x \in [0, 2\pi]$. Therefore, D_p^θ are normed linear spaces without having any concept of equivalence classes in mind.

II. MAIN RESULTS

First of all we want to prove the measurability of the new Lipschitz-type maximal functions. The central ideas of the following proof may be found in the classical book of Saks (cf. [8], pp. 113, 114, Theorem (4.3)).

THEOREM 1. *Let $f: \mathbb{R} \to \mathbb{R}$ be bounded and measurable. Then the functions $f_\theta^{\sim}$ are measurable for all $\theta \in (0, 1]$.*

PROOF. Fix $\theta \in (0, 1]$ and define

$$(2.1) \qquad f_{\theta,k}^{\sim}(x) := \sup_{|t-x|>1/k} \frac{|f(x)-f(t)|}{|x-t|^\theta}, \quad x \in \mathbb{R},\ k \in \mathbb{N}.$$

Obviously, we have

$$(2.2) \qquad \lim_{k\to\infty} f_{\theta,k}^{\sim}(x) = f_\theta^{\sim}(x), \quad x \in \mathbb{R},$$

$$(2.3) \qquad f_{\theta,k+1}^{\sim}(x) \geq f_{\theta,k}^{\sim}(x), \quad x \in \mathbb{R},\ k \in \mathbb{N}.$$

Now, let $\alpha \in \mathbb{R}$ be given arbitrarily. We assume that f is constant on a measurable set $\dot{M} \subset \mathbb{R}$ and consider the subset

$$M_\alpha^{(k)} := \{x \in M : f_{\theta,k}^{\sim}(x) > \alpha\},$$

$k \in \mathbb{N}$ arbitrarily but fixed. Some easy continuity arguments show that for each $x \in M_\alpha^{(k)}$ there exists an $\varepsilon(x) > 0$ and a point $t^* = t^*(x) \in \mathbb{R}$ such that for all

$$\xi \in U_{\varepsilon(x)}(x) := \{y \in \mathbb{R} : |x-y| < \varepsilon(x)\}$$

we have

$$(a) \qquad |\xi - t^*| > \frac{1}{k},$$

$$(b) \qquad \frac{|f(x)-f(t^*)|}{|\xi - t^*|} > \alpha.$$

This implies $M \cap U_{\varepsilon(x)}(x) \subset M_\alpha^{(k)}$ and, moreover, by doing this for each point $x \in M_\alpha^{(k)}$:

$$M_\alpha^{(k)} = \bigcup_{x \in M_\alpha^{(k)}} (M \cap U_{\varepsilon(x)}(x)) = M \cap \Big(\bigcup_{x \in M_\alpha^{(k)}} U_{\varepsilon(x)}(x) \Big).$$

Therefore, $M_\alpha^{(k)}$ is measurable, i.e., $f_{\theta,k}^{\sim}$, $k \in \mathbb{N}$, are measurable if $f(\mathbb{R})$ is a finite subset of $\mathbb{R}$ (f is a simple function). Now, each bounded measurable function f may be written as the limit of a uniformly convergent sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ with $f_n(\mathbb{R})$ finite for each $n\in\mathbb{N}$. By means of the uniform convergence of this sequence for each $\delta>0$ there exists a constant $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and all $x,t\in\mathbb{R}$, $|t-x|>1/k$, we have

$$\frac{|f_n(x)-f_n(t)|}{|x-t|^\theta}-\delta\leq\frac{|f(x)-f(t)|}{|x-t|^\theta}\leq\frac{|f_n(x)-f_n(t)|}{|x-t|^\theta}+\delta. \tag{2.4}$$

Taking the supremum over all $t\in\mathbb{R}$ with $|t-x|>1/k$ (2.4) implies that $f_{\theta,k}^{\sim}$ is the uniform limit of a sequence of measurable functions for each fixed $k\in\mathbb{N}$. Therefore, $f_{\theta,k}^{\sim}$ are measurable for all $k\in\mathbb{N}$ and, finally, by (2.2) and (2.3) $f_\theta^{\sim}$ is measurable, too. □

By the above theorem the smoothness spaces D_p^θ are well-defined and we may start to examine them (again, in the trigonometric case). First of all, we will take a look at the saturation case $\theta=1$. Here we expect for $1<p<\infty$, i.e., $\theta p>1$, that D_p^θ will coincide with some known smoothness spaces.

THEOREM 2. *For $1<p<\infty$ and $\theta=1$ we have $D_p^1=B_{p,\infty}^{1,1}$. For $p=\theta=1$ we only have $D_1^1\subset B_{1,\infty}^{1,1}\cap C$.*

REMARK. Let us first mention that $B_{p,\infty}^{1,1}$ is given by all functions $f\in L_p$ satisfying

$$\sup_{t>0}\{t^{-1}\omega_{1,p}(f,t)\}<\infty.$$

We remember that in case $1<p<\infty$ $B_{p,\infty}^{1,1}$ consists of those functions $f\in L_p$ which coincide almost everywhere with a function $g\in AC$ satisfying $g'\in L_p$. As usual in this context we identify each equivalence class of functions in $B_{p,\infty}^{1,1}$ with its absolutely continuous representative, i.e.,

$$B_{p,\infty}^{1,1}:=\{f\in AC: f'\in L_p\},\quad 1<p<\infty.$$

Finally, we note that $B_{1,\infty}^{1,1}$ is the space of functions which coincide almost everywhere with a 2π-periodic function of bounded variation on $[0,2\pi]$ (for details compare [2], p. 230, Theorem 4.1.6).

PROOF OF THEOREM 2. Let us first point to the fact that by (1.1) each discontinuity of a bounded function f implies a singularity of $f_1^{\sim}$ of order 1. Therefore, by (1.2) we immediately have $D_p^1\subset C$ for $1\leq p<\infty$.

Moreover, in case $1\leq p<\infty$ $f\in D_p^1$ implies (the norms always taken with respect to x):

$$\begin{aligned}\sup_{t>0}(t^{-1}\omega_{1,p}(f,t))&=\sup_{t>0}(t^{-1}\sup_{0<h\leq t}\|f(x+h)-f(x)\|_p)\\&\leq\sup_{h>0}\left\|\frac{f(x+h)-f(x)}{h}\right\|_p\\&\leq\|f_1^{\sim}\|_p<\infty,\end{aligned}$$

i.e., $f\in B_{p,\infty}^{1,1}$.

In the opposite direction $f\in B^{1,1}_{p,\infty}$ implies in case $1<p<\infty$:

$$\|f_1^{\sim}\|_p=\left\| \sup_{t\neq x} \frac{|f(x)-f(t)|}{|x-t|}\right\|_p$$

$$\leq \left\| \sup_{t\neq x} \frac{1}{|x-t|} \int_{\min\{x,t\}}^{\max\{x,t\}} |f'(\xi)|d\xi\right\|_p \leq c_p\|f'\|_p<\infty.$$

The last inequality follows from the fact that the Hardy-Littlewood maximal operator is of type (p,p) for $1<p<\infty$. In conclusion, we have proved $D^1_p=B^{1,1}_{p,\infty}$ in case $1<p<\infty$ and $D^1_1\subset B^{1,1}_{1,\infty}\cap C$ in case $p=1$. □

REMARK. It should be noticed that the inclusion $B^{1,1}_{1,\infty}\cap C\subset D^1_1$ is not valid. For example the function g,

$$g(x):=\left\{\begin{array}{ll} \int_0^x (z\log^2 z)^{-1}dz, & x\in[0,\tfrac{1}{2}] \\ \text{linear} & , x\in[\tfrac{1}{2},2\pi] \\ 0 & , x=2\pi \end{array}\right\},\ g\ 2\pi\text{-periodic},$$

belongs to $AC\subset B^{1,1}_{1,\infty}\cap C$ but

$$\|g_1^{\sim}\|_1\geq \int_0^{\frac{1}{2}} \left\{ \sup_{\substack{t\neq x\\ 0\leq t\leq 1/2}} \frac{1}{|x-t|} \left| \int_t^x (z\log^2 z)^{-1}dz\right|\right\}dx=\infty,$$

i.e., $g\notin D^1_1$ (for details compare [11], p. 33). This example shows that the case $\theta=p=1$ is really difficult and that the characterization of D^1_1 seems to require arguments similar to those used in connection with giving necessary and sufficient conditions for the L_1-boundedness of the Hardy-Littlewood maximal operator (cf. [10]). We conjecture that $f\in D^1_1$ if and only if f is absolutely continuous and f' belongs to the 2π-periodic analogue of the so-called Zygmund class $L\log L$.

Now, we start with the consideration of the non-saturation case. The following fundamental result corresponds to Lemma 6.1 of [4] and covers the case $\theta=1$, too.

THEOREM 3. *For $1\leq p<\infty$ and $0<\theta\leq 1$ the subspaces D^θ_p of L_p are Banach spaces with respect to their corresponding norms*

(2.5) $\|f\|_{p,\theta}:=\|f\|_p+\|f_\theta^{\sim}\|_p.$

PROOF. In the introduction we have already noticed that D^θ_p are normed linear subspaces of L_p. Therefore, we only have to prove that each Cauchy sequence in D^θ_p with respect to $\|\cdot\|_{p,\theta}$ converges in the norm to a function belonging to D^θ_p.

Let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in D_p^θ with respect to $\|\cdot\|_{p,\theta}$. Since L_p is complete there exists a function $f\in L_p$ such that

(2.6) $$\lim_{m\to\infty} \|f-f_m\|_p=0$$

and a suitable subsequence – which we again denote by $(f_m)_{m\in\mathbb{N}}$ – such that

(2.7) $$\lim_{m\to\infty} f_m(x)=f(x)$$

for almost every $x\in[0,2\pi]$.

Now, in a first step we will show that (2.7) is valid for all $x\in[0,2\pi]$ and that f is bounded.

Since f is finite almost everywhere on $[0,2\pi]$ there exists a point $x_0\in[0,2\pi]$ satisfying

(2.8) $$|f(x_0)|<\infty \text{ and } \lim_{m\to\infty} f_m(x_0)=f(x_0).$$

This implies for all $m,n\in\mathbb{N}$ and all $x\in[0,2\pi]$, $x\neq x_0$:

(2.9) $$\left\{\begin{aligned} &|f_m(x)-f_n(x)|\le|(f_m-f_n)(x_0)|+|(f_m-f_n)(x)-(f_m-f_n)(x_0)|\\ &\quad=|(f_m-f_n)(x_0)|\\ &\qquad+\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \frac{|(f_m-f_n)(x)-(f_m-f_n)(x_0)|}{|x-x_0|^\theta}\,d\xi\,|x-x_0|^{\theta-1}\\ &\quad\le|(f_m-f_n)(x_0)|\\ &\qquad+\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \frac{|(f_m-f_n)(x)-(f_m-f_n)(\xi)|}{|x-\xi|^\theta}\,d\xi\,|x-x_0|^{\theta-1}\\ &\qquad+\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \frac{|(f_m-f_n)(x_0)-(f_m-f_n)(\xi)|}{|x_0-\xi|^\theta}\,d\xi\,|x-x_0|^{\theta-1}\\ &\quad\le|(f_m-f_n)(x_0)|+2|x-x_0|^{\theta-1}\|(f_m-f_n)_{\tilde{\theta}}\|_1. \end{aligned}\right.$$

Since $(f_m(x_0))_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{p,\theta}$ (and, therefore, especially with respect to $\|\cdot\|_{1,\theta}$) the right hand side of (2.9) converges to zero for $m,n\to\infty$ and all $x\neq x_0$. This implies that $(f_m(x))_{m\in\mathbb{N}}$ are Cauchy sequences for all $x\in[0,2\pi]$, i.e., (2.7) is valid for all $x\in[0,2\pi]$.

To prove that f is bounded we substitute the point x_0 by another proper point $x_1\in[0,2\pi]$ satisfying $|x_0-x_1|>1$. By the same arguments as used above we get

(2.10) $$|(f_m-f_n)(x)|\le|(f_m-f_n)(x_1)|+2|x-x_1|^{\theta-1}\|(f_m-f_n)_{\tilde{\theta}}\|_1$$

for all $m,n\in\mathbb{N}$ and all $x\in[0,2\pi]$, $x\neq x_1$. By the inverse triangle inequality and the boundedness of f_m for fixed $m\in\mathbb{N}$ (2.9) and (2.10) immediately imply the uniform boundedness of $(f_m)_{m\in\mathbb{N}}$. Together with the validity of (2.7) for all $x\in[0,2\pi]$ we obtain the desired result that f is bounded.

Since f is bounded all differences $f(x)-f(t)$ are well-defined for all $x, t \in [0, 2\pi]$. Moreover, by the above arguments we obtain for all $x \in [0, 2\pi]$ and all $t \neq x$:

$$(2.11) \quad \begin{cases} \dfrac{|f(x)-f(t)|}{|x-t|^{\theta}} = \dfrac{|\lim_{m\to\infty} f_m(x) - \lim_{m\to\infty} f_m(t)|}{|x-t|^{\theta}} \\ \qquad = \lim_{m\to\infty} \dfrac{|f_m(x)-f_m(t)|}{|x-t|^{\theta}} \\ \qquad \le \liminf_{m\to\infty} (f_m)_{\theta}^{\sim}(x). \end{cases}$$

Taking the supremum over all $t \neq x$ on the left side of (2.11) and going over to the p-th power we get by applying Fatou's lemma:

$$(2.12) \quad \|f_{\theta}^{\sim}\|_p \le \liminf_{m\to\infty} \|(f_m)_{\theta}^{\sim}\|_p < \infty,$$

i.e., $f \in D_p^{\theta}$.

Using the same arguments once more but replacing f by $f - f_n$, $n \in \mathbb{N}$ arbitrarily, we get the inequality

$$(2.13) \quad \|(f-f_n)_{\theta}^{\sim}\|_p \le \liminf_{m\to\infty} \|(f_m-f_n)_{\theta}^{\sim}\|_p.$$

Since $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{p,\theta}$ the right hand side of (2.13) converges to zero as $n\to\infty$.

In conclusion, we have shown that $(f_m)_{m\in\mathbb{N}}$ converges to f with respect to $\|\cdot\|_{p,\theta}$. □

The following result gives the embedding theorem for the new smoothness spaces in terms of A-spaces (the corresponding result in case of smoothness spaces embedded by Besov spaces is given in [4], p. 48, Theorem 7.1).

THEOREM 4. *For $1 \le p < \infty$ and $0 < \theta < 1$ we have the embeddings*

$$(2.14) \quad A_{p,p}^{\theta,1} \subset D_p^{\theta} \subset A_{p,\infty}^{\theta,1}.$$

PROOF. The right hand embedding is an easy consequence of the following inequality (the norms again taken with respect to x):

$$\begin{aligned} &\sup_{t>0} (t^{-\theta}\tau_{1,p}(f,t)) \\ &= \sup_{t>0} \left\| \sup \left\{ \frac{|f(a)-f(b)|}{t^{\theta}} : a, b \in \left[x-\frac{t}{2}, x+\frac{t}{2} \right] \right\} \right\|_p \\ &\le \left\| \sup_{a\neq x} \frac{|f(x)-f(a)|}{|x-a|^{\theta}} + \sup_{b\neq x} \frac{|f(x)-f(b)|}{|x-b|^{\theta}} \right\|_p \\ &= 2\|f_{\theta}^{\sim}\|_p. \end{aligned}$$

The left hand embedding follows by

$$\|f_\theta^{\sim}\|_p = \left\| \sup_{t\neq x} \frac{|f(x)-f(t)|}{|x-t|^\theta} \right\|_p$$

$$\leq \left\| \sup_{h>0} \frac{\omega_1(f,x,2h)}{h^\theta} \right\|_p$$

$$\leq \left\{ \left\| \sup_{n\in\mathbb{Z}} \left(\frac{\omega_1(f,x,2\cdot 2^n)}{(2^{n-1})^\theta} \right)^p \right\|_1 \right\}^{1/p}$$

$$\leq \left\{ \left\| \sum_{n\in\mathbb{Z}} \left(\frac{\omega_1(f,x,2^{n+1})}{(2^{n-1})^\theta} \right)^p \right\|_1 \right\}^{1/p}$$

$$\leq \left\{ \sum_{n\in\mathbb{Z}} \left(\frac{\tau_{1,p}(f,2^{n+1})}{(2^{n-1})^\theta} \right)^p \right\}^{1/p}$$

$$\leq \left\{ \sum_{n\in\mathbb{Z}} \int_{2^{n-1}}^{2^n} \left(\frac{\tau_{1,p}(f,4\xi)}{(\xi/2)^\theta} \right)^p \frac{d\xi}{\xi/2} \right\}^{1/p}$$

$$\leq 2^{\theta+1/p} \left\{ \int_0^\infty \left(\frac{\tau_{1,p}(f,4\xi)}{\xi^\theta} \right)^p \frac{d\xi}{\xi} \right\}^{1/p}$$

$$= 2^{3\theta+(1/p)} \left\{ \int_0^\infty \left(\frac{\tau_{1,p}(f,\xi)}{\xi^\theta} \right)^p \frac{d\xi}{\xi} \right\}^{1/p}. \qquad \square$$

REMARK. The embeddings of Theorem 4 are in general not trivial, i.e., we do not have $A_{p,p}^{\theta,1} \supset D_p^\theta$ or $D_p^\theta \supset A_{p,\infty}^{\theta,1}$ at least in the interesting case $\theta p < 1$.

(1) To prove $A_{p,p}^{\theta,1} \not\supset D_p^\theta$ we consider the continuous but nowhere differentiable Weierstraß function W_θ, $0<\theta<1$,

$$W_\theta(x) := \sum_{k=1}^{\infty} 5^{-\theta k} \cos 5^k x, \quad x\in\mathbb{R}. \tag{2.15}$$

In [1], pp. 203, 204, Achieser proves that W_θ satisfies a Lipschitz condition of order θ, i.e., that there exists a constant $M_\theta>0$ such that for all $x, x' \in \mathbb{R}$ we have

$$|W_\theta(x) - W_\theta(x')| \leq M_\theta |x-x'|^\theta. \tag{2.16}$$

This immediately implies $W_\theta \in D_p^\theta$.

On the other hand Achieser shows that there exists a constant $m_\theta>0$ such that in each interval one may find two points x, x' satisfying

$$|W_\theta(x) - W_\theta(x')| \geq m_\theta |x-x'|^\theta. \tag{2.17}$$

Analyzing the proof of (2.17) (cf. [1], pp. 204–206) we see that the inequality may be sharpened in the form

$$\omega_1(W_\theta, x, t) \geq g_\theta t^\theta, \quad x\in[0,2\pi], t>0, \tag{2.18}$$

with a constant $g_\theta > 0$ independent of x and t. This implies

$$\int_0^\infty \left(\frac{\tau_{1,p}(W_\theta, t)}{t^\theta}\right)^p \frac{dt}{t} \geq \int_0^1 \left(\frac{(2\pi)^{1/p} g_\theta t^\theta}{t^\theta}\right)^p \frac{dt}{t}$$

$$= \infty,$$

i.e., $W_\theta \notin A^{\theta,1}_{p,p}$.

(We note that a trigonometric analogue of the example given by DeVore and Sharpley (cf. [4], pp. 51-53) would work, too.)

(2) To prove $D^\theta_p \not\supset A^{\theta,1}_{p,\infty}$, $\theta p < 1$, we consider the continuous piecewise linear function $I_{\theta,p}$ introduced by Ivanov [5]. This function is defined as follows: First of all, we choose $K_{\theta,p} \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K_{\theta,p}$, the points $x_k := k^{-\alpha}$, $\alpha := \theta p/(2-2\theta p)$, and $a_k := e^{-k}$ satisfy

$$x_k - a_k > x_{k+1} + a_{k+1}, \quad k \geq K_{\theta,p}.$$

Now, with $y_k := k^{-\theta/2}$ the function $I_{\theta,p}$ is defined by:

$$(2.19) \quad \begin{cases} I_{\theta,p}(x_k) := y_k, \quad k \geq K_{\theta,p}, \\ I_{\theta,p}(x_k + a_k) := I_{\theta,p}(x_k - a_k) := 0, \quad k \geq K_{\theta,p}, \\ I_{\theta,p} \text{ is linear in } [x_k, x_k + a_k],\ [x_k - a_k, x_k], \quad k \geq K_{\theta,p}, \\ I_{\theta,p} \text{ is linear in } [x_{k+1} + a_{k+1}, x_k - a_k], \quad k \geq K_{\theta,p}, \\ I_{\theta,p}(0) := I_{\theta,p}(\pi) := 0, \\ I_{\theta,p}(x) := 0 \text{ for } x \in [x_{K_{\theta,p}} + a_{K_{\theta,p}}, \pi], \\ I_{\theta,p} \text{ is } \pi\text{-periodic.} \end{cases}$$

In [5] Ivanov shows that $I_{\theta,p}$ belongs to $A^{\theta,1}_{p,\infty}$. On the other hand an easy calculation yields:

$$\|(I_{\theta,p})^{\sim}_\theta\|_p = \left\{ \int_0^{2\pi} \left(\sup_{t \neq x} \frac{|I_{\theta,p}(x) - I_{\theta,p}(t)|}{|x-t|^\theta} \right)^p dx \right\}^{1/p}$$

$$\geq \left\{ \sum_{k=K_{\theta,p}}^\infty \int_{x_{k+1}}^{x_k} \left(\frac{\frac{1}{2}(k+1)^{-\theta/2}}{(k^{-\alpha} - (k+1)^{-\alpha})^\theta} \right)^p dx \right\}^{1/p}$$

$$= \tfrac{1}{2} \left\{ \sum_{k=K_{\theta,p}}^\infty (k^{-\alpha} - (k+1)^{-\alpha})^{1-\theta p} (k+1)^{-\theta p/2} \right\}^{1/p}$$

$$= \tfrac{1}{2} \left\{ \sum_{k=K_{\theta,p}}^\infty \left(1 - \left(1 - \frac{1}{k+1}\right)^\alpha\right)^{1-\theta p} k^{\alpha(\theta p - 1)} (k+1)^{-\theta p/2} \right\}^{1/p}.$$

Since it is well-known that

$$(2.20) \quad \lim_{k \to \infty} (k+1)\left(1 - \left(1 - \frac{1}{k+1}\right)^s\right) = s, \quad s > 0,$$

we finally get with $k_0 \in \mathbb{N}$ sufficiently large:

$$\|(I_{\theta,p})_\theta^\sim\| \geq \tfrac{1}{2}\left(\frac{\alpha}{2}\right)^{1/p-\theta}\left\{\sum_{k=k_0}^{\infty}(k+1)^{\theta p-1}k^{-\theta p/2}(k+1)^{-\theta p/2}\right\}^{1/p}$$

$$\geq \tfrac{1}{2}\left(\frac{\alpha}{2}\right)^{1/p-\theta}\left\{\sum_{k=k_0}^{\infty}(k+1)^{-1}\right\}^{1/p} = \infty,$$

i.e., $I_{\theta,p} \notin D_p^\theta$.

ANNOTATION. An application of the maximal functions $f_\theta^\sim$ in connection with one-sided approximation by algebraic polynomials may be found in [6].

REFERENCES

1. Achieser, N.I. – Vorlesungen über Approximationstheorie, Akademie-Verlag, Berlin, 1967.
2. Butzer, P.L. and H. Berens – Semi-Groups of Operators and Approximation, Springer-Verlag, Berlin/Heidelberg/New York, 1967.
3. Calderón, A.P. and R. Scott – Sobolev type inequalities for $p>0$, Studia Math. **62**, 75–92 (1978).
4. DeVore, R.A. and R.C. Sharpley – Maximal functions measuring smoothness, Memoirs of the American Math. Soc. **47**, No. 293, (1984).
5. Ivanov, K.G. – On the rates of convergence of two moduli of functions, PLISKA Stud. Math. Bulgarica **5**, 97–104 (1983).
6. Lenze, B. – Operators for one-sided approximation by algebraic polynomials, accepted for publication in the Journal of Approximation Theory.
7. Popov, V.A. – Function spaces generated by the averaged moduli of smoothness, PLISKA Stud. Math. Bulgarica **5**, 132–143 (1983).
8. Saks, S. – Theory of the Integral, second revised edition, Hafner Publishing Company, New York, 1937.
9. Sendov, B. and V.A. Popov – Averaged Moduli of Smoothness, Bulgarian Mathematical Monographs Vol. 4, Bulgarian Academy of Science, Sofia, 1983. (in Bulgarian)
10. Stein, E.M. – Note on the class L log L, Studia Math. **32**, 305–310 (1969).
11. Zygmund, A. – Trigonometric Series, Volumes I and II, Cambridge University Press, Cambridge, 1979.